1.4. Compactness

Definition 1.21 Let X be a topological space, and $A \subseteq X$

(a) A is *compact* $:\Leftrightarrow$ For every *open cover* $\bigcup_{\alpha\in I} B_\alpha \supseteq A$ with an index set $I \neq \emptyset$, and $B_\alpha \subseteq X$ open for all $\alpha \in I$, there exists a *finite open subcover*, i.e., $\exists N \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_N \in I$ such that $\bigcup_{n=1}^{N} B_{\alpha_n} \supseteq A$ (*Heine-Borel-property*).

(b) A is *sequentially compact* $:\Leftrightarrow$ Every sequence in A has a convergent subsequence with limit in A (*Bolzano-Weierstraß property*).

(c) A is *relatively (seq.) compact* $:\Leftrightarrow \bar{A}$ is (seq.) compact.

Remark 1.22 (a) Def. applies in particular to X. In this case we have "$=$" instead of "$\supseteq$".

(b) Some books (e.g. Bourbaki) use compactness only for Hausdorff spaces; they call our property *quasi-compact*.

Theorem 1.23 Let X be a topological space

(a) Assume X is 1^{st} countable. Then:

$$X \text{ compact} \Rightarrow X \text{ sequentially compact}$$

(b) Assume X is 2^{nd} countable. Then:

$$X \text{ compact} \Leftrightarrow X \text{ sequentially compact}.$$

In the proof we will use/need:

Theorem 1.24 (*Lindelöf's Lemma* (!)) Let X be a 2^{nd} countable top. space, and let $\bigcup_{\alpha\in I} A_\alpha = X$ be an open cover. Then there exists $(\alpha_k)_{k\in\mathbb{N}} \subseteq I$ such that $\bigcup_{k\in\mathbb{N}} A_{\alpha_k} = X$, i.e., there exists a *countable (!) open subcover*

Pf: Let $\{B_k\}_{k\in\mathbb{N}}$ be a countable base of the topology. Let $K := \{k \in \mathbb{N} \mid \exists \alpha \equiv \alpha_k \in I \text{ with } B_k \subseteq A_{\alpha_k}\}$. Then $\bigcup_{k\in K} B_k \subseteq \bigcup_{\alpha\in I} A_\alpha$ (1)

Now, every A_α is a union of certain B_k's, and the corresponding k's necessarily belong to K. Thus $\bigcup_{\alpha \in I} A_\alpha \subseteq \bigcup_{k \in K} B_k$ (ΔΔ)

(Δ) & (ΔΔ) imply $X = \bigcup_{\alpha \in I} A_\alpha = \bigcup_{k \in K} B_k = \bigcup_{k \in K} A_{\alpha_k}$ ∎

Pf. of Thm 1.23: (a) By contradiction: Assume X is compact, but that there exists a sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ without convergent subsequence.

Claim: $\forall x \in X$ there exists a neighbourhood $U(x)$ such that

$x_n \in U(x)$ for at most finitely many n.

Pf. of claim: Suppose claim false. Then $\exists x \in X$ and a countable neighbourhood base $\{V_k\}_{k \in \mathbb{N}}$ of x with $V_k \supseteq V_{k+1}$ $\forall k \in \mathbb{N}$ (holds wlog. (o.B.d.A), see pf. Thm. 1.6 (b)) such that

(*) $$\forall k \in \mathbb{N}: x_n \in V_k \text{ for infinitely many } n$$

So, for all $k \in \mathbb{N}$, define $n_k \in \mathbb{N}$ such that $x_{n_k} \in V_k$. Since (*) holds for infinitely many n, we can choose the n_k s.t. $n_k < n_{k+1}$ $\forall k \in \mathbb{N}$

But then $(x_{n_k})_{k \in \mathbb{N}}$ is a subsequence of $(x_n)_{n \in \mathbb{N}}$ and $\lim_{k \to \infty} x_{n_k} = x$ ↯

Wlog., the neighbourhoods $U(x)$ from the claim can be assumed to be open. (otherwise shrink $U(x)$ to the open set contained in it which itself contains x). Now, $X = \bigcup_{y \in X} U(y) = \bigcup_{j=1}^{n} U(y_j)$ for some $n \in \mathbb{N}$ and some $y_1, \dots, y_n \in X$ because X is compact. The claim implies that each $U(y_j)$ contains at most finitely many members of the sequence $(x_n)_{n \in \mathbb{N}}$, so X contains at most finitely many members of the sequence $(x_n)_{n \in \mathbb{N}}$ ↯

(b) "⟹" follows from Theorem 1.2 and (a). We prove "⟸" by contradiction:

Assume every sequence has a convergent subsequence, but there exists an open cover of X without finite subcover. As X is 2^{nd} countable, there exists a countable subcover $X = \bigcup_{j \in \mathbb{N}} C_j$ of this cover (by Thm. 1.24). For every $n \in \mathbb{N}$ pick a point

(**) $$x_n \in X \setminus \left(\bigcup_{j=1}^{n} C_j \right)$$

(possible $\forall n \in \mathbb{N}$, since there exists no finite subcover)

By hypothesis, $(x_n)_n \subseteq X$ has a convergent subsequence $x_{n_k} \xrightarrow{k\to\infty} x \in X$. Then exists $N \in \mathbb{N}$ such that $x \in C_N^{\circ}$, so C_N° is a neighbourhood of x. Now, $(x_{n_k})_{k\in\mathbb{N}}$ being convergent means $x_{n_k} \in C_N$ for finally all k (i.e. $\exists K: k \geq K \Rightarrow x_{n_k} \in C_N$). On the other hand, $n_k \geq N$ for finally all k, hence $x_{n_k} \notin C_N$ for finally all k by (**) $\lightning$ ∎

Theorem 1.25 Let X be a topological space and $A \subseteq X$. Then:

(a) X compact and A closed $\Rightarrow$ A compact

(b) X Hausdorff and A compact $\Rightarrow$ A closed.

Pf. (a) Let $\bigcup_{\alpha\in I} U_\alpha \supseteq A$ be an open cover. Since A is closed, A^c is open, and
$$X = A^c \cup \left(\bigcup_{\alpha\in I} U_\alpha\right)$$
is an open cover. Since X is compact, there exists $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n \in I$ such that
$$X = A^c \cup \left(\bigcup_{i=1}^{n} U_{\alpha_i}\right)$$
and so $\bigcup_{i=1}^{n} U_{\alpha_i} \supseteq A$ is a finite subcover.

(b) See Exercise.

Warning 1.26 Bounded and closed do NOT imply compact in general!

Example: ℓ^p, $p \in [1,\infty]$, and (the closed unit ball)
$$\overline{B_1(0)} := \{x \in \ell^p \mid \|x\|_p \leq 1\} = \overline{\{x \in \ell^p \mid \|x\|_p < 1\}} = \overline{B_1(0)}$$
is bounded and closed but: consider $e^{(n)} := (\dots, 0, 1, 0, \dots) \in \overline{B_1(0)}$ (with 1 at the n-th position), $n \in \mathbb{N}$. Then
$$d_p(e^{(n)}, e^{(m)}) = \|e^{(n)} - e^{(m)}\|_p = \begin{cases} 2^{1/p}, & p < \infty \\ 1, & p = \infty \end{cases} \quad \forall n, m \in \mathbb{N},\ n \neq m,$$
so there exists no convergent subsequence, and $\overline{B_1(0)}$ is not seq. compact. Hence, by Thm. 1.23 (a), $\overline{B_1(0)}$ is not compact.

Theorem 1.27 Let X be a metric space. Then

$$\begin{array}{ccc} X \text{ is compact} & \overset{(a)}{\Longleftrightarrow} & X \text{ is sequentially compact} \\ & & \Downarrow (c) \\ X \text{ is } 2^{nd} \text{ countable} & \underset{(b)}{\Longleftrightarrow} & X \text{ is separable} \end{array}$$

(Recall that X is 1^{st} countable & Hausdorff (1.7))

Pf: (b) was proved in Thm. 1.9.

(a) "⟹": This is Thm. 1.23 (a) (since X is 1^{st} countable).

"⟸": Follows from pf. of (c) (see below), "⟸" of (b), and Thm. 1.23 (b).

(c): We prove that sequential compactness implies separability by constructing a countable set M with $\overline{M} = X$. Fix $n \in \mathbb{N}$ and use the following algorithm to define points $x_k^{(n)}$:

(i) Choose an arbitrary $x_1^{(n)} \in X$; set $k := 1$

(ii) WHILE $R_k^{(n)} := X \setminus \left(\bigcup_{j=1}^{k} B_{\frac{1}{n}}(x_j^{(n)})\right) \neq \emptyset$ DO

$$\{\text{pick } x_{k+1}^{(n)} \in R_k^{(n)} \text{ and increment } k \to k+1\}$$

Claim: This algorithm stops after finitely many steps.

True, because for $k \neq l$, we have $d(x_k^{(n)}, x_l^{(n)}) \geq \frac{1}{n}$. Hence, if the algorithm did not stop after finitely many steps, we would have an infinite sequence $(x_k^{(n)})_{k \in \mathbb{N}} \subseteq X$ without a convergent subseq., in contradiction to X being sequentially compact. ✓

The claim now implies that: $\forall n \in \mathbb{N}\ \exists K_n \in \mathbb{N}$ such that

$$(*) \qquad X = \bigcup_{j=1}^{K_n} B_{\frac{1}{n}}(x_j^{(n)})$$

Set $M_n := \{x_j^{(n)} \mid j = 1, \dots, K_n\}$ and $M := \bigcup_{n \in \mathbb{N}} M_n$. Then M is countable.

To prove denseness ($\overline{M} = X$), let $x \in X$ and $\varepsilon > 0$. Choose $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$.

Then $(*) \Rightarrow \exists l \in \{1, \dots, K_n\}$ such that $x \in B_{\frac{1}{n}}(x_l^{(n)})$. Hence,

$d(x, M) \leq d(x, x_l^{(n)}) < \frac{1}{n} < \varepsilon$, so $\overline{M} = X$ ∎

Theorem 1.28 (Tychonoff's Theorem - or Tikhonov)

Let $J \neq \emptyset$ be an index set, and X_α a compact topological space for all $\alpha \in J$. Then

$$\bigtimes_{\alpha \in J} X_\alpha = \{ f : J \to \bigcup_{\alpha \in J} X_\alpha \mid f(\alpha) \in X_\alpha \}$$

is compact in the product topology.

Pf: See any textbook on topology (Kelley, Munkres, v. Querenburg f.ex.)

Definition 1.29 Let X, Y be topological spaces. Define

(i) $C(X,Y) := \{ f : X \to Y \mid f \text{ is continuous} \}$

In particular, for $Y = \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, set $C(X) := C(X, \mathbb{K})$

(ii) $C_b(X) := \{ f \in C(X) \mid \|f\|_\infty < \infty \}$ (bounded continuous functions)

where $\|f\|_\infty := \sup_{x \in X} |f(x)| = \sup \{ |f(x)| \mid x \in X \}$

Theorem 1.30 Let X, Y be topological spaces, X compact, $f \in C(X,Y)$.

Then: (a) $f(X)$ is compact

(b) Assume: Y is Hausdorff and f is a bijection.

Then f is a homeomorphism (i.e., f^{-1} is continuous)

(c) If X, Y are metric spaces, then f is uniformly continuous:

$$\forall \varepsilon > 0 \ \exists \delta = \delta_\varepsilon > 0 : \forall x \in X : f(B_\delta(x; d_X)) \subseteq B_\varepsilon(f(x); d_Y)$$

(equivalently: $\forall \varepsilon > 0 \ \exists \delta > 0 : d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$)

(d) Assume $Y = \mathbb{R}$ (i.e. $f : X \to \mathbb{R}$). Then f takes on its maximum and minimum: $\exists x_1, x_2 \in X$:

$$\forall x \in X : \ f(x_1) \leq f(x) \leq f(x_2)$$

Pf: (a) Let $\bigcup_{\alpha \in J} V_\alpha \supseteq f(X)$ be an open cover. Then

$$(*) \qquad X \subseteq f^{-1}\Big(\bigcup_{\alpha \in J} V_\alpha\Big) = \bigcup_{\alpha \in J} f^{-1}(V_\alpha) \qquad (\text{"="} \text{ in fact})$$

Because f is continuous & V_α open, $f^{-1}(V_\alpha)$ is open $\forall \alpha \in J$, hence (*) gives an open cover of X – which is compact. Hence, $\exists N \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_N$ s.t. $X \subseteq \bigcup_{n=1}^{N} f^{-1}(V_\alpha)$, and therefore,

$$f(X) \subseteq \bigcup_{n=1}^{N} V_{\alpha_n}$$

(b), (c), (d): see Exercises.

1.5. Example: Spaces of continuous functions

General assumptions in this section:

(i) X is a compact Hausdorff space

(ii) $C(X)$ is equipped with the uniform (supremum) metric:

$$d_\infty(f,g) := \|f-g\|_\infty := \sup_{x \in X} |f(x)-g(x)|$$

(Note: sup = max is finite $\forall f,g \in C(X)$ by Thm. 1.30 (d))

Theorem 1.31 $C(X)$ is complete.

Pf: Follows from completeness of $C_b(X)$ (bounded cont. functions; see exercise), and that $C(X) = C_b(X)$, which follows from compactness of X, and Thm. 1.30 (d). ∎

Theorem 1.32 X is metrisable $\iff$ $C(X)$ is separable

(A topological space (X, J) is metrisable $:\iff$ $\exists$ metric d on X that generates the topology J)

Pf: For "$\Leftarrow$": See (f.ex) Bourbaki, "Elements of Mathematics, General Topology", Part 2, Sect. X 3.3, Thm. 1.1.

Here, we only prove:

"$\Rightarrow$": Fix any metric that is compatible with the topology.

For $m, n \in \mathbb{N}$, define

$$G_{m,n} := \left\{ f \in C(X) \;\middle|\; f(B_{\frac{1}{m}}(x)) \subseteq B_{\frac{1}{n}}(f(x)) \quad \forall x \in X \right\}$$

By compactness of X we get

(i) Any $f \in C(X)$ is even uniformly continuous, by Thm. 1.30(c).