(d) See exercise.

1.3. Example: sequence space ℓ^p

Definition 1.15 (ℓ^p-spaces)

Let, for $p \in [1,\infty)$ $(= [1,\infty[)$,

$$\ell^p := \ell^p(\mathbb{N}) := \left\{ x = (x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{C}\ \forall n \text{ and } \|x\|_p := \Big(\sum_{n\in\mathbb{N}} |x_n|^p\Big)^{1/p} < \infty \right\}$$

and $(p=\infty)$

$$\ell^\infty := \ell^\infty(\mathbb{N}) := \left\{ x = (x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{C}\ \forall n \text{ and } \|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n| < \infty \right\}$$

($\|\cdot\|_p$ will be a norm for every $p \in [1,\infty]$, see later)

Lemma 1.16 For every $p \in [1,\infty]$, $d_p(x,y) := \|x-y\|_p$, $x,y \in \ell^p$, defines a metric d_p on ℓ^p

Pf: All properties clear (check!), except for triangle inequality, this follows from Lemma 1.17 (b) below (do!)

Lemma 1.17 (Hölder & Minkowski)

(a) Let $p,q \in [1,\infty]$ be (Hölder) conjugated exponents, i.e., $\frac{1}{p} + \frac{1}{q} = 1$ (convention: "$\frac{1}{\infty} = 0$").
Dual pairing and Hölder inequality: For all $x \in \ell^p$, $y \in \ell^q$:
$\langle x,y\rangle := \sum_{n\in\mathbb{N}} x_n y_n$ is well-defined, and

$$|\langle x,y\rangle| \le \sum_{n\in\mathbb{N}} |x_n y_n| \le \|x\|_p \|y\|_q$$

(b) Minkowski inequality:
For all $x,y \in \ell^p$: $\qquad \|x+y\|_p \le \|x\|_p + \|y\|_p$

Pf: Both (a) & (b) follow from corresponding ineq.'s on $\mathbb{C}^N$, and then passing to the limit; for (a) f.ex.

$$\sum_{n=1}^N |x_n y_n| \le \Big(\sum_{n=1}^N |x_n|^p\Big)^{1/p} \Big(\sum_{n=1}^N |y_n|^q\Big)^{1/q}$$ (see ex. Forster, vol. 1) & carefully (!) take limit

$\boxed{\text{Remark 1.18}}$ (a) Note: alternative notation: l_p

(b) Note: $l^p(\{1,\dots,n\}) := \{x=(x_1,\dots,x_n) \mid x_i \in \mathbb{C},\ \|x\|_p := (\sum_{i=1}^{n} |x_p|^p)^{1/p}\}$ $(n\in\mathbb{N})$ is nothing but $\mathbb{C}^n$ with $\|\cdot\|_p$-norm.

(c) l^p-spaces can also be defined for $p\in(0,1)$ $(=]0,1[)$. But the Minkowski inequality does not remain true in this case. Instead, one has $\|x+y\|_p^p \le \|x\|_p^p + \|y\|_p^p$ $\forall p\in(0,1)$ (quasi-norm). Therefore we consider l^p-spaces for $p\ge 1$ only.

$\boxed{\text{Theorem 1.19}}$ (a) $\forall p\in[1,\infty)$: l^p is separable

(b) l^∞ is not separable

Pf: (a) We use the separability of $\mathbb{C}$. For $n\in\mathbb{N}$, let

$$M_n := \{(x_1,\dots,x_n,0,\dots) \mid x_j \in \mathbb{Q}+i\mathbb{Q},\ j=1,\dots,n\}$$

So M_n is countable and $M := \bigcup_{n\in\mathbb{N}} M_n$ is also countable.

Claim: $\overline{M} = l^p$. Let $y\in l^p$ and $\varepsilon>0$, then there exists $N\in\mathbb{N}$:

$$\sum_{j=N+1}^{\infty} |y_j|^p < \frac{\varepsilon^p}{2} \quad \left(\|y\|_p<\infty \Rightarrow \lim_{L\to\infty}\left(\sum_{j=L}^{\infty}|y_j|^p\right)^{1/p} = 0\right),$$

and since $\mathbb{Q}+i\mathbb{Q}$ is dense in $\mathbb{C}$ there exists $x\in M_N$ such that

$\sum_{j=1}^{N} |x_j-y_j|^p < \frac{\varepsilon^p}{2}$. This implies $(d_p(x,y))^p = \|x-y\|_p^p < \varepsilon^p$

$\blacksquare$

(b) See exercise.

$\boxed{\text{Theorem 1.20}}$ l^p is complete for every $p\in[1,\infty]$

(that is, $(l^p(\mathbb{N}), d_p)$ is a complete metric space)

Pf: (a) Case $p \in [1,\infty)$: Let $(x^{(n)})_{n\in\mathbb{N}} \subseteq \ell^p$ be a Cauchy sequence w.r.t. $\|\cdot\|_p$, where $x^{(n)} := (x_1^{(n)}, x_2^{(n)}, \dots)$. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$(*) \quad \sum_{j=J_1}^{J_2} |x_j^{(n)} - x_j^{(m)}|^p \leq \sum_{j\in\mathbb{N}} |x_j^{(n)} - x_j^{(m)}|^p < \varepsilon^p \quad \forall n,m \geq N, \quad \forall J_1 \leq J_2 \in \mathbb{N}.$$

Step 1: Find a candidate for the limit using the completeness of $\mathbb{C}$.
Let $J_1 = J_2 = J$, then $(*)$ implies $(x_J^{(n)})_n \subseteq \mathbb{C}$ is a Cauchy seq. $(\forall J \in \mathbb{N})$ and, since $\mathbb{C}$ is complete, there is some $x_J \in \mathbb{C}$ s.t

$$(**) \qquad \lim_{n\to\infty} x_J^{(n)} = x_J \qquad \forall J \in \mathbb{N}.$$

so our candidate is $x := (x_1, x_2, \dots)$

Step 2: Prove that the candidate belongs to the space: Use the Minkowski inequality in $\mathbb{C}^{J_2}$, $(**)$, and set $J_1 = 1$ in $(*)$:

$$\Big(\sum_{j=1}^{J_2} |x_j|^p\Big)^{1/p} \leq \lim_{m\to\infty} \Big(\sum_{j=1}^{J_2} |x_j^{(m)} - x_j^{(n)}|^p\Big)^{1/p} + \underbrace{\Big(\sum_{j=1}^{J_2} |x_j^{(n)}|^p\Big)^{1/p}}_{\leq \|x^{(n)}\|_p < \infty}$$

$$\leq \varepsilon + \|x^{(n)}\|_p \quad \forall J_2 \in \mathbb{N}, \forall n \geq N$$

so we get $\|x\|_p \leq \varepsilon + \|x^{(n)}\|_p$ for every $n \geq N$, and hence $x \in \ell^p$

Step 3: Prove that the sequence converges to the candidate, in the norm of the space: Set $J_1 = 1$ in $(*)$ and $m \to \infty$, using $(**)$. Then, for every $n \geq N$,

$$\sum_{j=1}^{J_2} |x_j^{(n)} - x_j|^p \leq \varepsilon^p \qquad \forall J_2 \in \mathbb{N},$$

so by sending $J_2 \to \infty$ in addition, we have

$$d_p(x^{(n)}, x) = \|x^{(n)} - x\|_p \leq \varepsilon, \quad \text{so } x^{(n)} \to x \text{ in } \ell^p.$$

(b) Case $p = \infty$: Replace "$\sum_{j=J_1}^{J_2}$" by "$\sup_{J_1 \leq j \leq J_2}$" and "$|\cdot|^p$" by "$|\cdot|$" ⊠