Pf: (a) Let $x_n \xrightarrow{n\to\infty} x$ in X. Let $V_0 \subseteq Y$ be a neighbourhood of $f(x)$. Then $\exists\, V \subseteq V_0$ open with $f(x) \in V$. Set $U := f^{-1}(V)$. U is open (because f is continuous and V is open), and $x \in U$, so U is a neighbourhood of x. Hence, we can apply the def. of convergence of $(x_n)_n$ to x:
$\exists\, n_0 \in \mathbb{N}: \forall n \geq n_0: x_n \in U$. But this means: $\forall n \geq n_0$:
$f(x_n) \in V \subseteq V_0$. So $f(x_n) \xrightarrow{n\to\infty} f(x)$ in Y.

(b) By contradiction: Suppose f not continuous, i.e., there exists open subset $V \subseteq Y$ such that (s.t.)
$U := f^{-1}(V)$ is not open, i.e.,

(*) $\exists\, x \in U: \forall$ neighbourhood N of x: $N \cap U^c \neq \emptyset$

Let $\{N_k\}_{k\in\mathbb{N}}$ be a countable neighbourhood base at x.
Consider $\tilde{N}_k := \bigcap_{j=1}^{k} N_j$ for $k \in \mathbb{N}$. Then $\{\tilde{N}_k\}_{k\in\mathbb{N}}$ is a countable neighbourhood base of x, with $\tilde{N}_{k+1} \subseteq \tilde{N}_k\ \forall k$.
$\forall k \in \mathbb{N}: \tilde{N}_k$ is a neighbourhood of $x \overset{(*)}{\Longrightarrow} \exists\, x_k \in \tilde{N}_k \cap U^c$.
Thus (1) $\forall k \in \mathbb{N}\ \forall l \geq k: x_l \in \tilde{N}_k$, so: $x_l \xrightarrow{l\to\infty} x$

(2) $\forall k \in \mathbb{N}: f(x_k) \in f(U^c) \subseteq V^c$. But $f(x) \in V$, so $\{f(x_k)\}_k$ cannot converge to $f(x)$ ↯ (contradiction) ☒

1.2. Metric spaces

Lemma 1.7 Let X be a metric space. Then, the open sets of X are precisely the ones of the metric topology, i.e., the topology generated by the base $\{B_{1/n}(x)\}_{n\in\mathbb{N},\, x\in X}$.
Moreover, X is 1^{st} countable and Hausdorff with respect to (w.r.t.) the metric top.

Pf: (i) Open sets coincide: Let $A \subseteq X$.

Claim: A open in metric space $\iff$ A open in metric top.
(see Handout)

Pf claim: "$\Rightarrow$" Let A be open according to Def. 8(iii) in Handout.

Then: $\forall x \in A\ \exists \varepsilon_x > 0$: $B_{\varepsilon_x}(x) \subseteq A$, so $A = \bigcup_{x \in A} B_{\varepsilon_x}(x)$.

Now, for all $x \in A$ choose $\mathbb{N} \ni n_x > \frac{1}{\varepsilon_x}$, then $A = \bigcup_{x \in A} B_{\frac{1}{n_x}}(x)$

so A is open in the metric topology.

"$\Leftarrow$": Exercise!

(ii) 1^{st} countable follows from:

Claim: $\forall x \in X$: $\{B_{1/n}(x)\}_{n \in \mathbb{N}}$ is a neighbourhood base at x.

Pf claim: (a) $B_{1/n}(x)$ is a neighbourhood of x for every $n \in \mathbb{N}$

(b) Let N be a neighbourhood of $x \Rightarrow \exists G$ open s.t. $x \in G \subseteq N$.

By (i), use Def. 8(iii) in Handout for openness: $\exists \varepsilon > 0$: $B_\varepsilon(x) \subseteq G$.

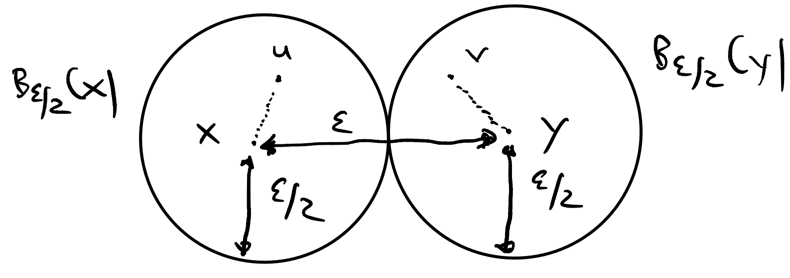
Let $\mathbb{N} \ni n > \frac{1}{\varepsilon} \Rightarrow B_{1/n}(x) \subseteq N$.

(iii) X is Hausdorff: Let $x, y \in X$ with $x \neq y$.

Then $\varepsilon := d(x,y) > 0$, and for all $u \in B_{\varepsilon/2}(x)$, $v \in B_{\frac{\varepsilon}{2}}(y)$, the triangle inequality yields:

$$\varepsilon = d(x,y) \leq d(x,u) + d(u,v) + d(v,y) < \frac{\varepsilon}{2} + d(u,v) + \frac{\varepsilon}{2}.$$

Hence $d(u,v) > 0 \Rightarrow u \neq v \Rightarrow B_{\varepsilon/2}(x) \cap B_{\varepsilon/2}(y) = \emptyset \Rightarrow X$ is Hausdorff



Corollary 1.8 All topological notions are available in a metric space X. In particular, for $(x_n)_n \subseteq X$, $x \in X$, $A \subseteq X$:

(a) $\lim_{n\to\infty} x_n = x$ (as in Def. 1.3 wrt. metric topology)

$$\iff \lim_{n\to\infty} d(x_n, x) = 0 \quad \text{(in } \mathbb{R}\text{, with 1.1)}$$

(b) $x \in \overline{A} \iff \exists (y_n)_n \subseteq A : \lim_{n\to\infty} y_n = x$.

(c) If Y is a topological space, and $f: X \to Y$, then

f is continuous $\iff$ f is sequentially continuous

Pf: (a), (b): simple exercises (do!) (c): See Theorem 1.6 ☒

Theorem 1.9 Let X be a metric space.

Then: X separable $\Rightarrow$ X 2^{nd} countable

(and, hence, $\iff$ by Theorem 1.2)

Pf: See exercises. Strategy: Let $A \subseteq X$ be countable and dense. Prove that for every open subset $G \subseteq X$ we have the representation

$$G = \bigcup_{x \in G} B_{\frac{1}{n(x)}}(a(x)) \quad \text{with } n(x) \in \mathbb{N} \text{ and } a(x) \in A \ \forall x \in G,$$

that is, $\{B_{\frac{1}{n}}(a)\}_{\substack{n\in\mathbb{N}\\ a\in A}}$ is a countable base ☒

Example 1.10 The above proof establishes the claim of the Handout (Ex. 2(2)): $\{B_{1/n}(q)\}_{\substack{n\in\mathbb{N}\\ q\in\mathbb{Q}^d}}$ is a base of the Euclidean topology on $\mathbb{R}^d$.

Example 1.11 Consider the space of all continuous (cont.) functions $f:[0,1]\to\mathbb{C}$:

$$C([0,1];\mathbb{C}) := C([0,1]) := \{f:[0,1]\to\mathbb{C} \mid f \text{ is continuous}\}$$

with two different metrics:

(a) Supremum metric:

$$d_\infty(f,g) := \|f-g\|_\infty, \quad \text{where } \|f\|_\infty := \sup_{x\in[0,1]} |f(x)|.$$

The metric space $(C([0,1]),d_\infty)$ is complete and separable (see Thms. 1.31 & 1.32 later)

(b) 1-metric:

$$d_1(f,g) := \|f-g\|_1, \quad \text{where } \|f\|_1 := \int_0^1 |f(x)|\,dx.$$

Then: (i) d_1 defines a metric on $C([0,1])$ (see Exercise)

(ii) $(C([0,1]),d_1)$ is separable because $d_1(f,g)\le d_\infty(f,g)$ and $(C([0,1]),d_\infty)$ is separable

(iii) $(C([0,1]),d_1)$ is not complete (see Exercise)

Definition 1.12 Let (X,d_X), (Y,d_Y) be metric spaces, $T:X\to Y$ map

(i) T is called an isometry $:\Leftrightarrow$ $\forall x,x'\in X$: $d_X(x,x') = d_Y(T(x),T(y))$.

(ii) X, Y are isometric $:\Leftrightarrow$ there exists a bijective (correctly, (X,d_X) & (Y,d_Y)) isometry $T:X\to Y$

Remark 1.13 Let $T:X\to Y$ be an isometry. Then

(a) T is injective and continuous

(b) X and the range of T (or image; see 1.5(c))

$$\operatorname{ran} T := T(X) = \{y\in Y \mid \exists x\in X \text{ with } y = T(x)\}$$ are isometric

$\boxed{\text{Theorem 1.14}}$ Let X be a metric space. Then there exists a complete metric space $\tilde{X}$, and an isometry $i : X \to \hat{X}$ such that $i(X)$ is dense in $\tilde{X}$. The space $\tilde{X}$ is unique up to isometric spaces.

Pf: Consists of 4 steps:

(a) Construct $\tilde{X}$

(b) Construct the isometry i, and $W := i(X)$ is dense in $\hat{X}$

(c) $\hat{X}$ is complete

(d) Uniqueness

(a): Define an equivalence relation $\sim$ on the set of all Cauchy sequences $(x_n)_n \subseteq X$:

$$(x_n)_n \sim (y_n)_n :\Leftrightarrow \lim_{n\to\infty} d(x_n, y_n) = 0$$

($\sim$ is clearly reflexive, symmetric, and transitive). Consider the equivalence class $[(x_n)_n] := \{(y_n)_n \subseteq X \text{ Cauchy} \mid (y_n)_n \sim (x_n)_n\}$ we obtain this way, and let

$$\tilde{X} := \{[(x_n)_n] \mid (x_n)_n \subseteq X \text{ Cauchy}\}.$$

Given $\tilde{x} \in \tilde{X}$, we write $(x_n)_n \in \tilde{x}$ for a representative $(x_n)_n$ of the equivalence class $\tilde{x}$. Define a metric on $\tilde{X}$:

$$\tilde{d}(\tilde{x}, \tilde{y}) := \lim_{n\to\infty} d(x_n, y_n), \quad \tilde{x}, \tilde{y} \in \tilde{X}, (x_n)_n \in \tilde{x}, (y_n)_n \in \tilde{y}$$

We have to check that:

(1) $\tilde{d}$ is well-defined:

(i) Existence of the limit:

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n)$$

$\Rightarrow \quad d(x_n, y_n) - d(x_m, y_m) \leq d(x_n, x_m) + d(y_m, y_n),$

and the same inequality holds with m & n interchanged.

Thus we have:

$$(*) \qquad \Big| \underbrace{d(x_n, y_n)}_{=:\alpha_n} - \underbrace{d(x_m, y_m)}_{=:\alpha_m} \Big| \leq d(x_n, x_m) + d(y_m, y_n).$$

Since $(x_n)_n$ and $(y_n)_n$ are Cauchy sequences, we have:

$\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ such that $\forall n, m \geq N$:

$$d(x_n, x_m) < \frac{\varepsilon}{2} \text{ and } d(y_n, y_m) < \frac{\varepsilon}{2}$$

So $|\alpha_n - \alpha_m| < \varepsilon$ for $n, m \geq N$ and so $(\alpha_n)_n \subseteq \mathbb{R}$ is a Cauchy sequence, thus convergent (because $\mathbb{R}$ is complete)

(ii) Independence of representatives: Let $(x_n)_n \sim (x_n')_n$ and $(y_n)_n \sim (y_n')_n$. We have to prove that

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n', y_n')$$

Repeat the derivation of $(*)$ above, interchanging $x_m \leftrightarrow x_n'$ and $y_m \leftrightarrow y_n'$, and obtain

$$\big| d(x_n, y_n) - d(x_n', y_n') \big| \leq d(x_n, x_n') + d(y_n, y_n') \xrightarrow{n\to\infty} 0$$

by definition of $\sim$.

(2) $\tilde{d}$ fulfills the axioms of a metric:

(i) $\tilde{d} \geq 0$: $\quad \tilde{d}(\tilde{x}, \tilde{y}) = 0 \Leftrightarrow \lim_{n\to\infty} d(x_n, y_n) = 0$ for some representatives

$\Leftrightarrow (x_n)_n \sim (y_n)_n \Leftrightarrow \tilde{x} = \tilde{y}.$

(ii) Symmetry

and (iii) Triangle inequality

are clear from the properties of d (do!).

(b): Define

$$i : X \to \tilde{X}$$
$$b \mapsto \tilde{b} := [(b, b, b, \dots)]$$

Set $W := i(X)$. The map i is an isometry since

$$\tilde{d}(i(a), i(b)) = \tilde{d}(\tilde{a}, \tilde{b}) = \lim_{n\to\infty} d(a,b) = d(a,b).$$

It remains to show that W is dense in $\tilde{X}$ with respect to $\tilde{d}$. Let $\tilde{x} \in \tilde{X}$ and $\varepsilon > 0$. Pick any representative $(x_n)_n \in \tilde{x}$. As $(x_n)_n$ is a Cauchy sequence (in X) we have: $\exists N \in \mathbb{N}\ \forall n, m \geq N$: $d(x_n, x_m) < \frac{\varepsilon}{2}$. Let $\tilde{b} := [(x_N, x_N, x_N, \dots)] = i(x_N) \in W$. Then

$$\tilde{d}(\tilde{b}, \tilde{x}) = \lim_{n\to\infty} d(x_N, x_n) \leq \frac{\varepsilon}{2} < \varepsilon.$$

So W is dense in $\tilde{X}$.

(c): Let $(\tilde{x}^{(k)})_{k\in\mathbb{N}} \subseteq \tilde{X}$ be a Cauchy seq. in $\tilde{X}$; W is dense in $\tilde{X}$: $\forall k \in \mathbb{N}\ \exists\, \tilde{z}^{(k)} := [(z_k, z_k, \dots)] \in W$, $z_k \in X$, s.t.

$$\tilde{d}(\tilde{x}^{(k)}, \tilde{z}^{(k)}) < \frac{1}{k}$$

For every $k, l \in \mathbb{N}$ we get (due to the isometry property of i):

$$(**)\quad d(z_k, z_l) = \tilde{d}(\underbrace{i(z_k)}_{\tilde{z}^{(k)}}, \underbrace{i(z_l)}_{\tilde{z}^{(l)}}) \leq \underbrace{\tilde{d}(\tilde{z}^{(k)}, \tilde{x}^{(k)})}_{< \frac{1}{k}} + \tilde{d}(\tilde{x}^{(k)}, \tilde{x}^{(l)}) + \underbrace{\tilde{d}(\tilde{x}^{(l)}, \tilde{z}^{(l)})}_{< \frac{1}{l}}$$

Let $\varepsilon > 0$. Since $(\tilde{x}^{(k)})_{k\in\mathbb{N}} \subseteq \tilde{X}$ is Cauchy (wrt. $\tilde{d}$), there exists $K \in \mathbb{N}$ such that $\tilde{d}(\tilde{x}^{(k)}, \tilde{x}^{(l)}) < \frac{\varepsilon}{3}$ for all $k, l \geq K$. Hence, $(**)$ implies that $d(z_k, z_l) < \varepsilon$ for all $k, l \geq \max\{\frac{3}{\varepsilon}, K\}$, that is, $(z_n)_{n\in\mathbb{N}} \subseteq X$ is Cauchy. Let $\tilde{x} := [(z_n)_n] \in \tilde{X}$.

We prove now that

$$\lim_{k\to\infty} \tilde{d}(\tilde{x}^{(k)}, \tilde{x}) = 0.$$

Indeed, let $\varepsilon > 0$. Then, as $(z_n)_n$ is Cauchy, $\exists N \in \mathbb{N}$ s.t. $d(z_k, z_n) < \frac{\varepsilon}{2}$ for all $k, n \geq N$. Thus, for all $k > \max\{\frac{2}{\varepsilon}, N\}$,

$$\tilde{d}(\tilde{x}^{(k)}, \tilde{x}) \leq \underbrace{\tilde{d}(\tilde{x}^{(k)}, \tilde{z}^{(k)})}_{< \frac{1}{k}} + \underbrace{\tilde{d}(\tilde{z}^{(k)}, \tilde{x})}_{= \lim_{n\to\infty} d(z_k, z_n)} < \varepsilon \qquad \square$$

(d) See exercise.

1.3. Example: sequence space ℓ^p

$\boxed{\text{Definition 1.15}}$ (ℓ^p-spaces)

Let, for $p \in [1,\infty)$ $(= [1,\infty[)$,

$$\ell^p := \ell^p(\mathbb{N}) := \left\{ x = (x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{C}\ \forall n \text{ and } \|x\|_p := \Big(\sum_{n\in\mathbb{N}} |x_n|^p\Big)^{1/p} < \infty \right\}$$

and $(p = \infty)$

$$\ell^\infty := \ell^\infty(\mathbb{N}) := \left\{ x = (x_n)_{n\in\mathbb{N}} \mid x_n \in \mathbb{C}\ \forall n \text{ and } \|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n| < \infty \right\}$$

($\|\cdot\|_p$ will be a norm for every $p \in [1,\infty]$, see later)

$\boxed{\text{Lemma 1.16}}$ For every $p \in [1,\infty]$, $d_p(x,y) := \|x-y\|_p$, $x,y \in \ell^p$, defines a metric d_p on ℓ^p

Pf: All properties clear (check!), except for triangle inequality, this follows from Lemma 1.17 (b) below (do!)

$\boxed{\text{Lemma 1.17}}$ (Hölder & Minkowski)

(a) Let $p,q \in [1,\infty]$ be (Hölder) conjugated exponents, i.e., $\frac{1}{p} + \frac{1}{q} = 1$ (convention: "$\frac{1}{\infty} = 0$").
Dual pairing and Hölder inequality: For all $x \in \ell^p$, $y \in \ell^q$:
$\langle x,y\rangle := \sum_{n\in\mathbb{N}} x_n y_n$ is well-defined, and

$$|\langle x,y\rangle| \le \sum_{n\in\mathbb{N}} |x_n y_n| \le \|x\|_p \|y\|_q$$

(b) Minkowski inequality:
For all $x,y \in \ell^p$: $\quad \|x+y\|_p \le \|x\|_p + \|y\|_p$

Pf: Both (a) & (b) follow from corresponding ineq.'s on $\mathbb{C}^N$, and then passing to the limit; for (a) f.ex.

$$\sum_{n=1}^N |x_n y_n| \le \Big(\sum_{n=1}^N |x_n|^p\Big)^{1/p} \Big(\sum_{n=1}^N |y_n|^q\Big)^{1/q}$$ (see ex. Forster, vol. 1) & carefully (!) take limit $\blacksquare$