Chapter I: Topological and metric spaces

1.1. Limits and continuity

Definition 1.1 Let X be a topological space.

(a) X is called *separable* $:\Leftrightarrow \exists A \subseteq X$ countable with $\overline{A} = X$

(b) X is called *1^{st} (first) countable* $:\Leftrightarrow$ Every $x \in X$ has a countable neighbourhood base.

(c) X is called *2^{nd} (second) countable* $:\Leftrightarrow$ there exists a countable (sub)base for the topology.

[Note: countable base $\Leftrightarrow$ countable subbase (see exercise)]

Theorem 1.2 Let X be a topological space.
Then: X is 2^{nd} countable $\Rightarrow$ X is 1^{st} countable and separable.

Pf (proof): Let $\mathcal{B}$ be a countable base for the topology $\mathcal{T}$ on X.

1^{st} countable: Let $x \in X$ and $\mathcal{N}_x := \{B \in \mathcal{B} \mid x \in B\}$. Then $\mathcal{N}_x$ is countable (clear (why?)) and a neighbourhood base at x:
Indeed, (i) every element of $\mathcal{N}_x$ is a neighbourhood of x, and
(ii) Let N be any neighbourhood of x. Then $\exists C \in \mathcal{T}$ with $x \in C \subseteq N$. By the definition of a base, $C = \bigcup_{\alpha \in I} B_\alpha$, where I is an index set and $B_\alpha \in \mathcal{B}$ $\forall \alpha \in I$. Hence,
$\exists \alpha_x \in I : x \in B_{\alpha_x}$, i.e., $B_{\alpha_x} \in \mathcal{N}_x$ and $B_{\alpha_x} \subseteq N$.

Separable: $\forall \emptyset \neq B \in \mathcal{B}$ choose $x_B \in B$, and let $A := \{x_B \mid \emptyset \neq B \in \mathcal{B}\}$.
We claim A is countable (trivial (why?)) and $\overline{A} = X$.
For all $x \in X$ and neighbourhoods U of x there exists $G \in \mathcal{T}$ such that $x \in G \subseteq U$. But $G \neq \emptyset$ is a union of sets in $\mathcal{B}$, so $\exists x_B \in G$. Thus, $A \cap U \neq \emptyset$.
Since U was arbitrary, x is an adherent point of A ∎

Definition 1.3 Let X be a topological space and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence. We say that $(x_n)_n$ converges to $x \in X$:$\Leftrightarrow$ for every neighbourhood U of x there exists $n_0 \in \mathbb{N}$ so for all $n \geq n_0$: $x_n \in U$. We write: $\lim_{n\to\infty} x_n = x$, or $x_n \xrightarrow{n\to\infty} x$ or $x_n \to x$, $n \to \infty$.

Remark 1.4

(a) Convergence is harder for finer topologies.

(b) X Hausdorff $\Rightarrow$ limits are unique (see exercise)

Definition 1.5 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be top. spaces and let $f: X \to Y$.

(a) f is sequentially (seq.) continuous :$\Leftrightarrow$
$x_n \xrightarrow{n\to\infty} x$ (in X) implies ($\Rightarrow$) $f(x_n) \xrightarrow{n\to\infty} f(x)$ (in Y)

(b) f is continuous :$\Leftrightarrow$ for every $A \in \mathcal{T}_Y$: $f^{-1}(A) \in \mathcal{T}_X$.
($f^{-1}(A) := \{x \in X \mid \exists y \in A : f(x) = y\}$ is the inverse image or pre-image of A)

(c) f is open (open map) :$\Leftrightarrow$ $\forall A \in \mathcal{T}_X$: $f(A) \in \mathcal{T}_Y$
($f(A) := \{y \in Y \mid \exists x \in A : y = f(x)\}$ is the image of A (under f))

(d) f is a homeomorphism :$\Leftrightarrow$ f is bijective, open, and continuous
(i.e., a bijection compatible with topological structure).

Theorem 1.6 Let X, Y be topological spaces, $f: X \to Y$ a map.
Then:

(a) f is continuous $\Rightarrow$ f is seq. continuous

(b) f is seq. continuous & X is 1^{st} countable
$\Rightarrow$ f is continuous

Pf: (a) Let $x_n \xrightarrow{n\to\infty} x$ in X. Let $V_0 \subseteq Y$ be a neighbourhood of $f(x)$. Then $\exists\, V \subseteq V_0$ open with $f(x) \in V$. Set $U := f^{-1}(V)$. U is open (because f is continuous and V is open), and $x \in U$, so U is a neighbourhood of x. Hence, we can apply the def. of convergence of $(x_n)_n$ to x:

$\exists\, n_0 \in \mathbb{N} : \forall n \geq n_0 : x_n \in U$. But this means: $\forall n \geq n_0$:

$f(x_n) \in V \subseteq V_0$. So $f(x_n) \xrightarrow{n\to\infty} f(x)$ in Y.

(b) By contradiction: Suppose f not continuous, i.e., there exists open subset $V \subseteq Y$ such that (s.t.)

$U := f^{-1}(V)$ is not open, i.e.,

(*) $\exists\, x \in U : \forall$ neighbourhood N of x : $N \cap U^c \neq \emptyset$

Let $\{N_k\}_{k\in\mathbb{N}}$ be a countable neighbourhood base at x

Consider $\tilde{N}_k := \bigcap_{j=1}^{k} N_j$ for $k \in \mathbb{N}$. Then $\{\tilde{N}_k\}_{k\in\mathbb{N}}$ is a countable neighbourhood base of x, with $\tilde{N}_{k+1} \subseteq \tilde{N}_k$ $\forall k$.

$\forall k \in \mathbb{N}$: $\tilde{N}_k$ is a neighbourhood of x $\overset{(*)}{\Longrightarrow} \exists\, x_k \in \tilde{N}_k \cap U^c$.

Thus (1) $\forall k \in \mathbb{N}\ \forall l \geq k : x_l \in \tilde{N}_k$, so: $x_l \xrightarrow{l\to\infty} x$

(2) $\forall k \in \mathbb{N} : f(x_k) \in f(U^c) \subseteq V^c$. But $f(x) \in V$, so $\{f(x_k)\}_k$ cannot converge to $f(x)$ $\lightning$ (contradiction) $\blacksquare$

1.2. Metric spaces

Lemma 1.7 Let X be a metric space. Then, the open sets of X are precisely the ones of the metric topology, i.e., the topology generated by the base $\{B_{1/n}(x)\}_{n\in\mathbb{N},\, x\in X}$.

Moreover, X is 1^{st} countable and Hausdorff with respect to (w.r.t.) the metric top.