



Prof. Dr. Thomas Ø. Sørensen
Version 0.2

FA 1
HANDOUT

SoSe 2022
26. April 2022

This is an overview of material on METRIC AND TOPOLOGICAL SPACES (Ana2).

TO BE UPDATED - CHECK DATE AND VERSION

METRIC AND TOPOLOGICAL SPACES

Let X be a (non-empty) set (for example $X = \mathbb{R}^d$ or any subset of \mathbb{R}^d), and let $\mathcal{P}(X)$ be the family of all subsets of X (its power set).

Definition 1 (Topology; topological space). A family of subsets of X , $\mathcal{T} \subseteq \mathcal{P}(X)$, is called a topology (on X) if and only if ('iff')

(i) $\emptyset, X \in \mathcal{T}$.

(ii) $A_1, A_2 \in \mathcal{T} \Rightarrow A_1 \cap A_2 \in \mathcal{T}$.

(iii) For any index set I : $(A_j \in \mathcal{T} \text{ for all } j \in I) \Rightarrow \bigcup_{j \in I} A_j \in \mathcal{T}$.

The pair (X, \mathcal{T}) is called a topological space (often just X), and $A \subseteq X$ is open $:\Leftrightarrow A \in \mathcal{T}$.

Remark 1. By induction, (ii) is equivalent to:

(ii') For any $n \in \mathbb{N}$: $(A_j \in \mathcal{T} \text{ for all } j = 1, \dots, n) \Rightarrow \bigcap_{j=1}^n A_j \in \mathcal{T}$.

Definition 2 (Coarser/finer topologies). Let $\mathcal{T}_1, \mathcal{T}_2$ be topologies on X . We say that \mathcal{T}_1 is finer than \mathcal{T}_2 , and \mathcal{T}_2 is coarser than \mathcal{T}_1 , iff $\mathcal{T}_2 \subset \mathcal{T}_1$.

Example 1. 1. Indiscrete topology: $\mathcal{T} := \{\emptyset, X\}$.

2. Discrete topology: $\mathcal{T} := \mathcal{P}(X)$.

3. All topologies are finer than the indiscrete, and coarser than the discrete topology.

4. Euclidean (or standard) topology \mathcal{T}_E on \mathbb{R}^d , $d \in \mathbb{N}$: $A \subseteq \mathbb{R}^d$ is open (in $(\mathbb{R}^d, \mathcal{T}_E)$) iff $\forall x \in A \exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^d \mid |x - y| < \varepsilon\}$ is the Euclidean ball of radius $\varepsilon > 0$ about $x \in \mathbb{R}^d$.

Definition 3 (Induced / relative topology). Let (X, \mathcal{T}) be a topological space, and let $A \subseteq X$ (not necessarily open!). The relative (or induced) topology \mathcal{T}_A on A is

$$\mathcal{T}_A := \{B \subseteq A \mid \exists C \in \mathcal{T} \text{ with } B = C \cap A\} \subseteq \mathcal{P}(A).$$

Remark 2. 1. \mathcal{T}_A is a topology on A .

2. If $A \notin \mathcal{T}$ and $B \in \mathcal{T}_A$, then it may happen that $B \notin \mathcal{T}$. Example: Let $X = \mathbb{R}$ with the standard topology \mathcal{T}_E , $A = [0, 1]$. Then $B := [0, \frac{1}{2}[\in \mathcal{T}_A$ but $B \notin \mathcal{T}_E$.

Definition 4 (Closed set; neighbourhood; Hausdorff space; limit/accumulation point; closure; interior point; interior; boundary point; boundary; dense).

Let (X, \mathcal{T}) be a topological space, $A \subseteq X$, $x \in X$.

- (i) The set A is called closed (in X , or, correctly, in (X, \mathcal{T})) iff $A^c := X \setminus A \in \mathcal{T}$ (that is, the complement A^c of A is open (in X)).
- (ii) A set $U \subseteq X$ (not necessarily open) is called a neighbourhood of x iff $\exists A \in \mathcal{T}$ such that $x \in A$ and $A \subseteq U$.
- (iii) The (topological) space X (or, correctly, (X, \mathcal{T})) is called a Hausdorff space iff For all $x, y \in X, x \neq y$, there exist neighbourhoods U_x of x and U_y of y such that $U_x \cap U_y = \emptyset$.
- (iv) The point x is called a limit point of A (or accumulation point) iff For all neighbourhoods U of x : $U \cap A \neq \emptyset$.
- (v) The closure \overline{A} of A is $\overline{A} := \{x \in X \mid x \text{ limit point of } A\}$.
- (vi) The point x is called a boundary point of A iff For all neighbourhoods U of x : $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$.
- (vii) The boundary ∂A of A is $\partial A := \{x \in X \mid x \text{ boundary point of } A\}$.
- (viii) The point x is called an interior point of A iff There exists a neighbourhood U of x such that $U \subseteq A$.
- (ix) The interior A° of A is $A^\circ := \{x \in X \mid x \text{ interior point of } A\}$.
- (x) The set A is dense in X (correctly, in (X, \mathcal{T})) iff $\overline{A} = X$.

Remark 3.

- 1. According to this definition, every point of A is a limit point of A . Hence, $A \subseteq \overline{A}$.
- 2. Note that $\overline{A} = A \cup \partial A$ and $A^\circ = A \setminus \partial A$.

Lemma 1. Let X be a topological space, $A \subseteq X$.

- (i) A is open iff $\forall x \in A : x$ is an interior point of A .
- (ii) A is closed iff $A = \overline{A}$.
- (iii) \overline{A} and ∂A are closed.

Definition 5 (Base; subbase; neighbourhood base).

Let (X, \mathcal{T}) be a topological space.

- (i) A family $\mathcal{B} \subseteq \mathcal{T}$ is called a base (or basis) for (the topology) \mathcal{T} iff \mathcal{T} consists of unions of sets from \mathcal{B} .

- (ii) A family $\mathcal{S} \subseteq \mathcal{T}$ is called a *subbase* (or *subbasis*) for (the topology) \mathcal{T} iff finite intersections of sets from \mathcal{S} form a base for \mathcal{T} .
- (iii) A family $\mathcal{N} \subseteq \mathcal{T}$ is called a *neighbourhood base* (or *neighbourhood basis*) at $x \in X$ iff Every $N \in \mathcal{N}$ is a neighbourhood of x and for every neighbourhood U of x there exists $N \in \mathcal{N}$ with $N \subseteq U$.

Example 2. Consider \mathbb{R}^d with the standard topology. Let $x \in \mathbb{R}^d$.

1. The family $\{B_{1/n}(x) \mid n \in \mathbb{N}\}$ is a neighbourhood base at x .
2. The family $\{B_{1/n}(q) \mid n \in \mathbb{N}, q \in \mathbb{Q}^d\}$ is a base for the standard topology.

Lemma 2. Let $\mathcal{S} \subseteq \mathcal{P}(X)$. Then there exists a topology $\mathcal{T}(\mathcal{S})$ on X such that

- (i) \mathcal{S} is a subbase for $\mathcal{T}(\mathcal{S})$.
- (ii) $\mathcal{T}(\mathcal{S})$ is the coarsest topology containing \mathcal{S} . That is,

$$\mathcal{T}(\mathcal{S}) = \bigcap_{\mathcal{T} \supseteq \mathcal{S}; \mathcal{T} \text{ topology}} \mathcal{T}.$$

The topology $\mathcal{T}(\mathcal{S})$ is called the topology generated by \mathcal{S} .

Definition 6 (Product topology). Let $I \neq \emptyset$ be an arbitrary (!) index set. For every $\alpha \in I$, let $(X_\alpha, \mathcal{T}_\alpha)$ be a topological space. The product topology on the Cartesian product space,

$$\bigtimes_{\alpha \in I} X_\alpha := \left\{ f : I \rightarrow \bigcup_{\alpha \in I} X_\alpha \mid \forall \alpha \in I : f(\alpha) \in X_\alpha \right\} \quad (1)$$

is the one generated by the family

$$\left\{ \bigtimes_{\alpha \in I} A_\alpha \mid \forall \alpha \in I : A_\alpha \in \mathcal{T}_\alpha, \text{ and } A_\alpha \neq X_\alpha \text{ for at most finitely many } \alpha \text{'s} \right\}.$$

Remark 4. If I is finite, then the condition ‘ $A_\alpha \neq X_\alpha$ for at most finitely many α ’s’ is always fulfilled.

Definition 7 (Metric; metric space). A map $d : X \times X \rightarrow [0, \infty[$ is called a *metric* (on X) iff

- (i) $d(x, y) \geq 0 \quad \forall x, y \in X$, with $d(x, y) = 0$ iff $x = y$.
- (ii) $d(x, y) = d(y, x) \quad \forall x, y \in X$.
- (iii) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$.

If d is a metric on X , then (X, d) is called a *metric space* (often just X).

Definition 8 (Induced metric; open (metric) ball; open set; Cauchy sequence; convergent sequence; limit; complete (metric) space).

Let (X, d) be a metric space.

- (i) Let $Y \subseteq X$. The map $d_Y := d|_{Y \times Y} : Y \times Y \rightarrow [0, \infty[$ is called the *induced metric* (on Y).

- (ii) The set $B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$ is called the open (metric) ball of radius $\varepsilon > 0$ about x . (More correctly: $B_\varepsilon(x; d)$.)
- (iii) A set $A \subseteq X$ is called open (in the metric space (X, d)) iff For every $x \in A$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$.
- (iv) A sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ is called a Cauchy sequence (or, is said to be Cauchy (!)) (in X , or, in (X, d)) iff

For every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n, m \geq N : d(x_n, x_m) < \varepsilon$.

- (v) A sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ is called convergent (or, is said to converge) (in X , or, in (X, d)) iff There exists $x \in X$ such that $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ (which is equivalent to: $\forall \varepsilon > 0 \exists n_0 \in \mathbb{N} : x_n \in B_\varepsilon(x)$). In this case, $(x_n)_{n \in \mathbb{N}}$ is said to converge to x , and x is called its limit. We write $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x, n \rightarrow \infty$.
- (vi) The space X (more correctly, (X, d)) is called complete (is a complete metric space) iff every Cauchy sequence in X converges. That is: For all Cauchy sequences $(x_n)_{n \in \mathbb{N}} \subseteq X$ there exists $x \in X$ (!) such that $\lim_{n \rightarrow \infty} x_n = x$.

Remark 5. Completeness is not a topological notion! See Exercise (??)

Definition 9. Let (X, d) be a metric space, let $A \subseteq X, x \in X$.

- (i) The (extended) number $\text{diam}(A) := \sup_{a, b \in A} d(a, b) \in [0, \infty]$ is called the diameter of A .
- (ii) The number $\text{dist}(x, A) := \inf_{a \in A} d(x, a)$ is called the distance of x to A .

Lemma 3. Let X be a complete metric space, and let $A \subseteq X$. Then A is closed iff A is complete.

(More precisely/correctly: Assume (X, d) is a complete metric space, $A \subseteq X$. Then A is closed in (X, d) iff (A, d_Y) is a complete metric space.