Recall the Fourier transform:

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx$$

and the "inversion formula"

(*) $$f(x) = c\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi = (2\pi)^{-n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} f(y)\,dy\,d\xi$$

Some notation: multiindices:

For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_i \in \mathbb{N}_0$ ($\alpha_i \geq 0$, integer),

the length of α is $|\alpha| = \sum_{i=1}^{n} \alpha_i$

and we write

$$\partial^\alpha f = \frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial^{\alpha_1+\alpha_2+\dots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}} f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$$

We denote $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$

(so $+\Delta u = +\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$

Note: using (*), if everything ok, if $\hat{f}$ has compact support. ($\operatorname{supp} g = \overline{\{x \in \mathbb{R}^n \mid g(x) \neq 0\}}$ - so $g = 0$ on $\mathbb{R}^n \setminus \operatorname{supp} g$)

we have

$$\partial^\alpha f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} (i\xi)^\alpha e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi$$

when y^α, for $y \in \mathbb{R}^n$, α multiindex, is

$$y^\alpha = y_1^{\alpha_1}\cdot y_2^{\alpha_2}\cdots y_n^{\alpha_n}$$

The function (!) $(i\xi)^\alpha e^{ix\cdot\xi} \hat{f}(\xi)$ still has compact support in ξ, so integral ok (if no singularities...)

Hence, f can be differentiated as much as we want, so $f \in C^\infty$. This even works if $\hat{f}(\xi)$ does not have compact support, but decays faster than any polynomial at $+\infty$.

So: (decay) or compact support of $\hat{f}$ ensures smoothness of f

(shall see more, and much more detailed / sofisticated results on this in the Payley - Wiener - Schwartz Theorem).

Let us look at the eq.

$(**)$ $\boxed{\Delta u = f}$ for f given, $f \in C_0^\infty(\mathbb{R}^n)$

(f smooth, with compact support).

Questions: existence, uniqueness - and regularity. - What can we say about u / how can we estimate u in terms of f?

Taking the Fourier transform we (all formally!) get $\widehat{(\Delta u)} = (\Delta u)^\wedge = \hat{f}$

or (see last time)

$$-\sum_{j=1}^{n} |\xi_j|^2 \hat{u}(\xi) = \hat{f}(\xi)$$

$$\left(\text{since } (\partial_{x_j} g)(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} (i\xi_j)\hat{g}(\xi)\, d\xi \right.$$

$$\equiv (2\pi)^{-n} \int e^{ix\cdot\xi}\, \widehat{(\partial_{x_j} g)}(\xi)\, d\xi$$

$$\left. \text{so } \widehat{\partial_{x_j} g}(\xi) = (i\xi_j)\hat{g}(\xi) \right).$$

Formally, then

$$\hat{u}(\xi) = -\frac{1}{|\xi|^2}\hat{f}(\xi)$$

$$\xi = (\xi_1, \dots, \xi_n), \quad |\xi|^2 = \sum_{j=1}^{n} |\xi_j|^2$$

IF $n=2$ then $-\frac{1}{|\xi|^2}$ is not integrable at 0:

??

$$\int_{B_1(0)} \frac{1}{|\xi|^2}\, d\xi = 2\pi \int_0^1 \frac{1}{|\xi|^2} |\xi|^2\, d|\xi| = 2\pi \int_0^1 \frac{1}{|\xi|^2}\, d|\xi| = +\infty$$

– otherwise, we would find

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \left(-\frac{1}{|\xi|^2} \hat{f}(\xi) \right) d\xi \qquad (*)$$

In the 1950's people used this idea to study existence & regularity for linear partial diff. eq.'s with constant coefficients:

$$L = \sum_{|\alpha|\le m} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha} = \sum_{|\alpha|\le m} a_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha, \quad a_\alpha \in \mathbb{C}$$

Ehrenpreiss wrote like $(\square)$ above and then using Cauchy-theory (Cauchy's integral thm./formula) he related (!) this to

$$\int -\frac{1}{|\xi+i\eta|^2}\,\hat{f}(\xi+i\eta)\,e^{-ix\cdot(\xi+i\eta)}\,d\xi$$

– this way, there is no singularity in the integrand at $\xi=0$

Malgrange, on the other hand, first studied $(**)$ for f's such that $\hat{f}(\xi)=0$, and $\hat{f}$ vanishes to some order at $\xi=0$, to get convergence in $(\square)$. Then he used some functional analysis to get to the general case

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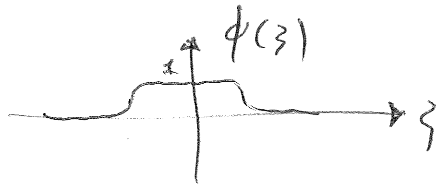
The idea of ΨDO's is to replace the "Fourier multiplier" $\frac{1}{|\xi|^2}$ above with the multiplier $\frac{1-\phi(\xi)}{|\xi|^2}$ where $\phi \in C_0^\infty(\mathbb{R}^n)$ and $\phi \equiv 1$ near $\xi = 0$:



Then define

$$(\widehat{Pg})(\xi) = -\frac{1-\phi(\xi)}{|\xi|^2}\,\hat{g}(\xi)$$

Problem

for $g \in C_0^\infty(\mathbb{R}^n)$

or equivalently

$$(Pg)(x) = \left(- \frac{1-\phi(\xi)}{|\xi|^2} \hat{g}(\xi) \right)^{\vee}$$

with $\vee$ the <u>inverse</u> of the Fourier transform:

$$f = (\hat{f})^{\vee}.$$

Now look at $\quad u - Pf \qquad (\Delta u = f)$

$$(u - Pf)^{\wedge} = \hat{u} - \widehat{Pf}$$

$$= - \frac{1}{|\xi|^2} \hat{f} + \frac{1-\phi(\xi)}{|\xi|^2} \hat{f}$$

$$= - \frac{\phi(\xi)}{|\xi|^2} \hat{f}$$

Now $-\dfrac{\phi(\xi)}{|\xi|^2} \hat{f}$ has <u>compact</u> support in ξ

- so, it is (!!) the Fourier transform of a C^∞-function, as seen above.

<u>I.e.</u> $\quad u - Pf$ is C^∞. So, studying the regu<u>larity</u> of u is the <u>same</u> (the diff. is C^∞) as studying the regularity of Pf. We call P a <u>parametrix</u> for the partial diff. op. Δ

Problem

<u>And</u>: the important part is

that P has <u>symbol</u> $-\dfrac{1-\phi(\xi)}{|\xi|^2}$ which has <u>no</u> singularity at $\xi = 0$ (remember that $\phi(\xi) = 1$ for $|\xi| < 1$, say).

Now look at the PDO $\qquad Lu = f$

$$L = \sum_{|\alpha| \le m} a_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha$$ with <u>const.</u> coeff.'s

Then for, say $f \in C_0^\infty(\mathbb{R}^n)$,

$$(Lf)(x) = (2\pi)^{-n} \int e^{ix\cdot\xi}\, l(\xi) \hat{f}(\xi)\, d\xi$$

with $\qquad l(\xi) = \sum_{|\alpha| \le m} a_\alpha (i\xi)^\alpha$

(before, $l(\xi) = -|\xi|^2$)

One might again hope ~~that~~ to do something like above: Assume for simplicity that

$l(\xi)$ again <u>only</u> vanishes when $\xi = 0$ (*)

- take $\Phi \in C_0^\infty$ with $\phi(\xi) = 1$ for $|\xi| \le 1$ and $\Phi(\xi) \equiv 0$ for $|\xi| \ge 2$, and let

$$m(\xi) = (1 - \Phi(\xi)) \frac{1}{l(\xi)}$$

This is, in fact, exactly what happens in the <u>elliptic</u> case, as we shall see later.

Problem

Problem

As before, define a map

$$T : g \longmapsto \left((1-\phi(\xi)) \cdot \frac{1}{l(\xi)} \, \hat{g}(\xi) \right)^{\vee}$$

Since, as above, composition of the two operators L and T corresponds to multiplying their symbols ($l(\xi)$ and $(1-\phi(\xi)) \frac{1}{l(\xi)}$) we get

$$\left((L \circ T) f \right)^{\wedge} = \left(l(\xi) \cdot \frac{1-\phi(\xi)}{l(\xi)} \right) \hat{f}(\xi)$$

$$= (1-\phi(\xi)) \, \hat{f}(\xi)$$

and so $L \circ T = Id +$ (negligible error term)

in the sense that $L \circ T = Id + R_1$

where Rf is always a C^∞ - function

(similarly, $T \circ L = Id + R_2$)

For L with non-constant coefficients one does the same thing:

$$L = \sum_{|\alpha| \leq m} a_\alpha(x) \left(\frac{\partial}{\partial x} \right)^\alpha \qquad a_\alpha : \mathbb{R}^n \to \mathbb{C}$$

Problem $\boxed{}$ Again, one hopes (!) that a parametrix is given by something like the map

$$g \longmapsto \left(\frac{1}{\ell(x,\xi)} \hat{\phi} \right)^{\vee} \qquad (\Delta)$$

where now the symbol of L is

$$\ell(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x)(i\xi)^\alpha$$

Again, one treats the case where ℓ only vanishes at $\xi = 0$ (elliptic !)

and set $\quad m(x,\xi) = (1-\phi(\xi)) \dfrac{1}{\ell(x,\xi)}$

(Δ) is

$$(Kg)(x) = (2\pi)^{-n} \int e^{ix\xi} \frac{1}{\ell(x,\xi)} \hat{\phi}(\xi)\, d\xi$$

$$= (2\pi)^{n} \int e^{ix\xi}\, a(x,\xi)\, \hat{\phi}(\xi)\, d\xi,$$

$$a(x,\xi) \equiv \frac{1}{\ell(x,\xi)}$$

Problem One then hopes (!) that the map

$$T : f \mapsto \left((1-\phi(\xi)) \frac{1}{\ell(x,\xi)} \hat{f}(\xi) \right)^{\vee}$$

i.e.

$$(Tf)(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} \left(\frac{1-\phi(\xi)}{\ell(x,\xi)} \right) \hat{f}(\xi)\, d\xi$$

$$= (2\pi)^{-n} \int e^{ix\xi} \, b(x,\xi)\, \hat{f}(\xi)\, d\xi$$

gives an approximate right / left inverse to L:

$$T \circ L = Id + (\text{negligible err. terms})$$

$$L \circ T = Id + (\text{negligible err terms})$$

However: - Finding the symbol of the composed operators $T \circ L$, $L \circ T$ is more complicated because of the "x": before, the symbol was the product of two symbols - this will not be the case anymore - one needs to figure out how to compute the symbol

Problem ~~A~~ of a composition

What one builds is what is called a "calculus of ΨDO's" - it is a collection of operators ("integral operators") which

- contains all PDO's
- all their elliptic parametrices
- is closed under composition of operators
- closed under taking adjoints

(before the development of ΨDO's, by people like Mikhlin, Calderón - Zygmund, Kohn - Nirenberg, Hörmander, people had tried other types of operators (to build a calculus of other types of op.'s), namely so-called "singular integral op.'s")