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This is an overview of material needed for the course. It consists of three parts, material from MEASURE AND INTEGRATION THEORY (Ana3), from FUNCTIONAL ANALYSIS (FA1), and from COMPLEX ANALYSIS (Fkt). We assume knowledge of metric, and more generally, topological spaces. A special case, also assumed known, is that of (semi-)normed vector spaces.

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MEASURE AND INTEGRATION THEORY

Let X be a non-empty set (for example $X = \mathbb{R}^d$ or any subset of $\mathbb{R}^d$), and let $\mathcal{P}(X)$ be the family of all subsets of X (its *power set*).

Definition 1 (**σ-algebra**). *A family of subsets of X, $\mathcal{A} \subset \mathcal{P}(X)$, is called a* σ-algebra *(on X) if and only if ('iff')*

(i) $X \in \mathcal{A}$.

(ii) $A \in \mathcal{A} \Rightarrow X \setminus A \in \mathcal{A}$.

(iii) $\Big(A_j \in \mathcal{A}\ \text{ for all } j \in \mathbb{N}\Big) \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$.

The pair $(X, \mathcal{A})$ is called a measurable space, *and $A \subset X$ is* measurable $:\Leftrightarrow A \in \mathcal{A}$.

Proposition 1 (**Generated σ-algebras; Borel-(σ-)algebra).**

(i) For any family $\mathcal{B} \subset \mathcal{P}(X)$ there exists a smallest σ-algebra $\sigma(\mathcal{B})$ containing $\mathcal{B}$ (that is, $\sigma(\mathcal{B}) \supset \mathcal{B}$, and if $\mathcal{C}$ is a σ-algebra with $\mathcal{C} \supset \mathcal{B}$, then $\mathcal{C} \supset \sigma(\mathcal{B})$), given by

$$\sigma(\mathcal{B}) := \bigcap_{\mathcal{A} \subset \mathcal{P}(X), \mathcal{A}\ \sigma\text{-algebra}, \mathcal{A} \supset \mathcal{B}} \mathcal{A}\,. \tag{1}$$

We call $\sigma(\mathcal{B})$ the σ-algebra generated by $\mathcal{B}$.

(ii) Let $(X, \mathcal{T})$ be a topological space (for example, a metric space (X, d) with the topology $\mathcal{T}_d$ generated by the metric d). The σ-algebra $\sigma(\mathcal{T})$ is called the Borel-σ-algebra *(or* Borel-algebra*) (on $(X, \mathcal{T})$), denoted $\mathcal{B}(X)$ (more correct would be: $\mathcal{B}(X, \mathcal{T})$), and $B \subset X$ is* a Borel-set *(or* Borel *or* Borel-measurable*) $:\Leftrightarrow B \in \mathcal{B}(X)$.*

(iii) For a measurable space $(X, \mathcal{A})$ and a subset $B \subset X$ (not necessarily measurable), the induced σ-algebra *(or* trace-σ-algebra*) on B is defined by $\mathcal{A}_B := \{B \cap A \,|\, A \in \mathcal{A}\}$. If $B \in \mathcal{A}$, then $\mathcal{A}_B \subset \mathcal{A}$.*

Example 1 (**Borel-algebra on** $\mathbb{R}^d$, $\mathbb{R}_{\geq 0}$, $\overline{\mathbb{R}}$, $\overline{\mathbb{R}}_{\geq 0}$). Let $X := \mathbb{R}^d$ with the usual topology $\mathcal{T}_{\text{Eucl}}$, generated by the Euclidean metric $|\cdot|$. We denote $\mathcal{B}^d := \mathcal{B}(\mathbb{R}^d) := \sigma(\mathcal{T}_{\text{Eucl}})$ the Borel-algebra on $\mathbb{R}^d$, and write $\mathcal{B} := \mathcal{B}^1$ when there is no risk of confusion. We denote $\mathcal{B}_{\geq 0} := {\mathcal{B}^1}_{\mathbb{R}_{\geq 0}} := \{\mathbb{R}_{\geq 0} \cap A \,|\, A \in \mathcal{B}^1\}$. It is the Borel-algebra of $\mathbb{R}_{\geq 0}$ (with the topology on $\mathbb{R}_{\geq 0}$ the one induced from $\mathbb{R}$). For $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, we denote $\mathcal{B}(\overline{\mathbb{R}}) := \sigma(\mathcal{B}^1 \cup \{\{-\infty\}\} \cup \{\{+\infty\}\})$. It is the Borel-algebra on $\overline{\mathbb{R}}$ for the usual topology on $\overline{\mathbb{R}}$. Finally, $\overline{\mathcal{B}}_{\geq 0} := \{\overline{\mathbb{R}}_{\geq 0} \cap A \,|, A \in \mathcal{B}(\overline{\mathbb{R}})\}$ $(= \mathcal{B}(\overline{\mathbb{R}}_{\geq 0}))$.

Definition 2 (**(Positive) measure**).

(i) Let $\mathcal{A}$ be a σ-algebra on X. A map $\mu : \mathcal{A} \to [0, \infty]$ is called a (positive) measure (on X, or on $(X, \mathcal{A})$) *iff*

  (i) $\mu(\emptyset) = 0$.

  (ii) For all $A_j \in \mathcal{A}$, $j \in \mathbb{N}$, with $A_j \cap A_k = \emptyset$ for $j \neq k$:

$$\mu\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} \mu(A_j) \quad (\sigma\text{-additivity})\,. \tag{2}$$

  The triple $(X, \mathcal{A}, \mu)$ is called a measure space.

(ii) A measure μ (or, more correctly, a measure space $(X, \mathcal{A}, \mu)$) is called finite *(or* bounded*) iff $\mu(X) < \infty$, and σ-*finite *iff there exists $(X_j)_{j\in\mathbb{N}}$, $X_j \in \mathcal{A}$, $X = \bigcup_{j=1}^{\infty} X_j$, with $\mu(X_j) < \infty$ for all $j \in \mathbb{N}$.*

Example 2 (**Lebesgue-Borel measure on** $\mathbb{R}^d$). There exists a unique measure, called the *Lebesgue-Borel measure* λ^d, on $\mathcal{B}^d$ so that for all rectangles $Q := \times_{j=1}^{d}[a_j, b_j) \subset \mathbb{R}^d$, $-\infty \leq a_j \leq b_j \leq \infty$, holds

$$\lambda^d(Q) = \prod_{j=1}^{d}(b_j - a_j)\,. \tag{3}$$

Furthermore, λ^d is translation and rotation invariant, and σ-finite.

Definition 3 (**(μ-)Null sets; complete measure**). *Let $(X, \mathcal{A}, \mu)$ be a measure space.*

(i) A subset $N \subset X$ is called a (μ-)null set *iff $N \in \mathcal{A}$ and $\mu(N) = 0$.*

(ii) $(X, \mathcal{A}, \mu)$ (or just μ) is called complete *iff all subsets of null sets are null sets.*

Theorem 1 (**Completion of measure**). *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then there exists a smallest complete measure space $(X, \overline{\mathcal{A}}, \overline{\mu})$ (called* the completion of $(X, \mathcal{A}, \mu)$*) containing $(X, \mathcal{A}, \mu)$ (that is, $\overline{\mathcal{A}} \supset \mathcal{A}$, $\overline{\mu}|_{\mathcal{A}} = \mu$, and $\overline{\mu}$ is complete).*

Example 3 (**Lebesgue measure on** $\mathbb{R}^d$). The completion of $(\mathbb{R}^d, \mathcal{B}^d, \lambda^d)$ (which is *not* in itself complete) is denoted $(\mathbb{R}^d, \overline{\mathcal{B}^d}, \overline{\lambda^d})$. Elements of $\overline{\mathcal{B}^d}$ are called *Lebesgue-measurable (subsets of* $\mathbb{R}^d$*)*, and $\overline{\lambda^d}$ is called *(d−dimensional) Lebesgue measure*. One has

$$\overline{\mathcal{B}^d} = \Big\{B \cup \widetilde{N} \,\Big|\, B \in \mathcal{B}^d, \exists N \in \mathcal{B}^d \text{ with } \lambda^d(N) = 0, \widetilde{N} \subset N\Big\}\,. \tag{4}$$

Furthermore, $A \in \overline{\mathcal{B}^d}$ iff for all $\varepsilon > 0$ there exists $U \subset \mathbb{R}^d$ open and $C \subset \mathbb{R}^d$ closed, with $C \subset A \subset U$, such that $\lambda^d(U \setminus C) < \varepsilon$.
Note, in particular, that $\mathcal{B}^d \subsetneq \overline{\mathcal{B}^d} \subsetneq \mathcal{P}(\mathbb{R}^d)$.

Definition 4 (Measurable maps and functions).

(i) *Let* $(X, \mathcal{A})$, $(Y, \mathcal{C})$ *be measurable spaces. A map* $f : X \to Y$ *is called* $(\mathcal{A}$-$\mathcal{C}$-$)$measurable *iff* $f^{-1}(C) \in \mathcal{A}$ *for all* $C \in \mathcal{C}$*. We denote by* $\mathcal{M}(X, Y)$ *the set of all* $\mathcal{A}$-$\mathcal{C}$*-measurable maps. (More correct would be* $\mathcal{M}((X, \mathcal{A}), (Y, \mathcal{C}))$*.)*

(ii) *In the special case* $(Y, \mathcal{C}) = (\mathbb{R}, \mathcal{B}^1)$*, we denote* $\mathcal{M}(X) := \mathcal{M}(X, \mathbb{R})$ *the set of all measurable* functions $f : X \to \mathbb{R}$*, and* $\mathcal{M}_+(X) := \{ f \in \mathcal{M}(X) \mid f \geq 0 \}$*. Note that* $\mathcal{M}_+(X) = \mathcal{M}(X, \mathbb{R}_{\geq 0}) = \mathcal{M}((X, \mathcal{A}), (\mathbb{R}_{\geq 0}, \mathcal{B}_{\geq 0}))$*.*

(iii) *In the special case* $(Y, \mathcal{C}) = (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$*, we denote* $\overline{\mathcal{M}}(X) := \mathcal{M}(X, \overline{\mathbb{R}})$ *the set of all measurable* numerical *(or* extended real-valued*)* functions $f : X \to \overline{\mathbb{R}}$*, and* $\overline{\mathcal{M}}_+(X) := \{ f \in \overline{\mathcal{M}}(X) \mid f \geq 0 \}$*. Again,* $\overline{\mathcal{M}}_+(X) = \mathcal{M}(X, \overline{\mathbb{R}}_{\geq 0})$*.*

(iv) *We denote by* $\mathcal{M}(X, \mathbb{C})$ *the set of all complex functions* $f : X \to \mathbb{C}$ *such that* $\Re(f), \Im(f) \in \mathcal{M}(X, \mathbb{R})$*.*

Remark 1. *One has*

$$\mathcal{M}(X) = \Big\{ f : X \to \mathbb{R} \,\Big|\, f^{-1}\big((-\infty, a)\big) \in \mathcal{A} \text{ for all } a \in \mathbb{R} \Big\} , \tag{5}$$

and similarly with $(-\infty, a)$ *replaced with* $(-\infty, a]$*,* (a, ∞)*, or* $[a, \infty)$*. Analogous statements hold for* $\overline{\mathcal{M}}(X)$*.*
We denote, for $a \in \overline{\mathbb{R}}$*,*

$$\{ f < a \} := f^{-1}\big((-\infty, a)\big) = \{ x \in X \mid f(x) \in (-\infty, a) \} , \tag{6}$$

and similarly for other types of intervals.

Definition 5 (Distribution function).
Let $(X, \mathcal{A}, \mu)$ *be a measure space. For* $f \in \overline{\mathcal{M}}(X)$*, we call the function* $\mu_f : \mathbb{R} \to \mathbb{R}$ *given by*

$$\mu_f(t) := \mu(\{ f > t \}) = \mu\big(\{ x \in X \mid f(x) > t \}\big) \tag{7}$$

the distribution function of f (relative to μ).

Definition 6 (Almost everywhere (a.e.); $f = g$ **a.e.).**
Let $(X, \mathcal{A}, \mu)$ *be a measure space.*

(i) *A mathematical statement* $Q = Q(x)$ *(which is assumed to make sense for all* $x \in X$*) is said to* hold (μ-) almost everywhere (a.e., *or* μ-a.e.) *iff there exists a (*μ*-)null set* N *such that* $Q(x)$ *is true/holds for all* $x \in X \setminus N$*.*

(ii) *Let* $f, g : X \to \mathbb{M}$ *(*$\mathbb{M} \in \{ \mathbb{R}, \mathbb{R}_{\geq 0}, \overline{\mathbb{R}}, \overline{\mathbb{R}}_{\geq 0}, \mathbb{C} \}$*) be measurable, then* $f = g$ a.e. *iff* $f(x) = g(x)$ *a.e. This defines an equivalence relation* $\sim_{a.e.}$ *on* $\mathcal{M}(X, \mathbb{M})$*.*

Definition 7 (Step functions).
Let $(X, \mathcal{A})$ *be a measurable space. A function* $f : X \to \mathbb{R}$ *is called a* step function *iff there exists* $N \in \mathbb{N}$*,* $A_1, \ldots, A_N \in \mathcal{A}$*, and* $a_1, \ldots, a_N \in \mathbb{R}$ *such that*

$$f = \sum_{n=1}^{N} a_n \mathbb{1}_{A_n} . \tag{8}$$

Here, $\mathbb{1}_B$ *is the* characteristic *(or* indicator*)* function of (the set) B*, given by* $\mathbb{1}_B(x) = 1$ *if* $x \in B$*, and equal* 0 *otherwise* $(x \notin B)$*.*
Note that step functions are measurable by definition. We denote the set of all non-negative step functions by

$$E := \left\{ f : X \to \mathbb{R} \,\middle|\, f \geq 0 \,,\, f \text{ step function} \right\}. \tag{9}$$

Theorem 2 (Approximating measurable functions by step functions). *Let* $(X, \mathcal{A})$ *be a measurable space. Then* $f \in \mathcal{M}(X, \overline{\mathbb{R}})$ *iff there exists a sequence* $(f_n)_{n\in\mathbb{N}}$ *of step functions* $f_n : X \to \mathbb{R}$ *with* $f = \lim_{n\to\infty} f_n$ *(pointwise on* X*). If* $f \in E$*, then the sequence can be chosen monotone* $(f_n \nearrow f)$*, and if* f *is a bounded function, then the sequence can be chosen such that the convergence is uniform on* X*.*

Definition 8 (Definition and properties of Lebesgue integral).
Let $(X, \mathcal{A}, \mu)$ *be a measure space.*

1. *Let* $f \in E$ $(f \geq 0,\ f$ *step function), with* $f = \sum_{n=1}^{N} a_n \mathbb{1}_{A_n}$*,* $A_n \in \mathcal{A}, a_n \in \mathbb{R}$*. Then*

$$\int_X f \,\mathrm{d}\mu := \int_X f(x) \,\mathrm{d}\mu(x) := \sum_{n=1}^{N} a_n \mu(A_n) \in [0, \infty] \tag{10}$$

   is the $(\mu$-)integral of f over X*. It is independent of the representation in* (8)*.*

2. *Let* $f \in \overline{\mathcal{M}}_+(X)$ $(f : X \to [0, \infty]$*, measurable), and let* $(f_n)_{n\in\mathbb{N}} \subset E$ *be an approximating sequence as in Theorem 2. Then*

$$\int_X f \,\mathrm{d}\mu := \int_X f(x) \,\mathrm{d}\mu(x) := \lim_{n\to\infty} \left(\int_X f_n(x) \,\mathrm{d}\mu(x) \right) \in [0, \infty] \tag{11}$$

   is the $(\mu$-)integral of f over X*. The limit is well-defined, since the sequence* $(\int_X f_n \,\mathrm{d}\mu)_{n\in\mathbb{N}} \subset [0, \infty]$ *is non-decreasing. The limit is independent of the chosen sequence* $(f_n)_{n\in\mathbb{N}}$*.*

3. *For* $f : X \to \overline{\mathbb{R}}$*, let* $f_\pm := \max\{\pm f, 0\}$ *(so* $f = f_+ - f_-, |f| = f_+ + f_-$*). Then* f *is* $(\mu$-)integrable over $X :\Leftrightarrow f \in \overline{\mathcal{M}}(X)$ *and* $\int_X f_+ \,\mathrm{d}\mu < \infty$*,* $\int_X f_- \,\mathrm{d}\mu < \infty$*. In this case,*

$$\int_X f \,\mathrm{d}\mu := \int_X f(x) \,\mathrm{d}\mu(x) := \int_X f_+ \,\mathrm{d}\mu - \int_X f_- \,\mathrm{d}\mu \in \mathbb{R} \tag{12}$$

   is the $(\mu$-)integral of f over X*. We denote the set of integrable functions by*

$$\mathcal{L}^1 := \mathcal{L}^1(X) := \mathcal{L}^1(\mu) := \mathcal{L}^1(X, \mu) := \mathcal{L}^1(X, \mathcal{A}, \mu) := \left\{ f : X \to \mathbb{R} \,\middle|\, f \ \mu\text{-integrable} \right\},$$
$$\overline{\mathcal{L}^1} := \overline{\mathcal{L}^1}(X) := \overline{\mathcal{L}^1}(\mu) := \overline{\mathcal{L}^1}(X, \mu) := \overline{\mathcal{L}^1}(X, \mathcal{A}, \mu) := \{ f : X \to \overline{\mathbb{R}} \,|\, f \ \mu\text{-integrable} \}.$$

4. *For* $A \in \mathcal{A}$*, and* $f \in \mathcal{L}^1$ *(or* $f \in \overline{\mathcal{L}^1}$*), let* $\int_A f \,\mathrm{d}\mu := \int_X f \mathbb{1}_A \,\mathrm{d}\mu$*.*

5. *Properties of the integral:*

   (a) *For* $f \in \overline{\mathcal{L}^1}$*,* $f \geq 0$*, one has:* $\int_X f \,\mathrm{d}\mu = 0 \Leftrightarrow f = 0$ μ*-a.e.*

   (b) *The map* $f \mapsto \int_X f \,\mathrm{d}\mu$ *from* $\overline{\mathcal{L}^1}$ *to* $\mathbb{R}$ *is linear and monotone.*

(c) *For* $f \in \overline{\mathcal{L}^1}$,

$$\Big|\int_X f\,\mathrm{d}\mu\Big| \leq \int_X |f|\,\mathrm{d}\mu \quad \text{(triangle inequality)}\,. \tag{13}$$

(d) *For* $f \in \overline{\mathcal{L}^1}$, $f \geq 0$, *and all* $\varepsilon > 0$,

$$\mu(\{f \geq \varepsilon\}) \leq \frac{1}{\varepsilon}\int_X f\,\mathrm{d}\mu \quad \text{(Chebyshev's inequality)}\,. \tag{14}$$

Proposition 2 (Riemann versus Lebesgue interal in $\mathbb{R}$).
For $f : [a,b] \to \mathbb{R}$ *(*$a, b \in \mathbb{R}, a < b$*) Riemann-integrable, denote* $\int_a^b f(x)\,dx$ *the Riemann-integral of* f *over* $[a,b]$.

1. *For* $f : [a,b] \to \mathbb{R}$ *Riemann-integrable there exists* $g : \mathbb{R} \to \mathbb{R}$ *measurable, with* $f = g$ *a.e. on* $[a,b]$ *such that*

$$\int_a^b f(x)\,dx = \int_{[a,b]} g(x)\,\mathrm{d}\lambda^1(x)\,. \tag{15}$$

2. *Let* $f : [0,\infty) \to \mathbb{R}$ *be measurable, and continuous on* $(0,\infty)$. *Then*

$$\int_{\mathbb{R}} f\mathbb{1}_{[1,\infty)}\,\mathrm{d}\lambda^1 = \lim_{n\to\infty}\int_1^n f(x)\,dx\,, \tag{16}$$

$$\int_{\mathbb{R}} f\mathbb{1}_{[0,1]}\,\mathrm{d}\lambda^1 = \lim_{n\to\infty}\int_{1/n}^1 f(x)\,dx\,. \tag{17}$$

In particular,

$$\int_{[0,1]} x^a\,\mathrm{d}\lambda^1(x) < \infty \;\Leftrightarrow\; a > -1\,, \tag{18}$$

$$\int_{[1,\infty)} x^b\,\mathrm{d}\lambda^1(x) < \infty \;\Leftrightarrow\; b < -1\,. \tag{19}$$

Also,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} = \lim_{R\to\infty}\Big(\int_{[0,R]}\frac{\sin x}{x}\,\mathrm{d}\lambda^1(x)\Big)\,, \tag{20}$$

$$\int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}\lambda^1(x) = \sqrt{\pi}\,. \tag{21}$$

In what follows, let $(X, \mathcal{A}, \mu)$ be any measure space.

Definition 9 (Essential supremum). *For a measurable function* $f : X \to \overline{\mathbb{R}}$ *the* essential supremum of f *is*

$$\begin{aligned}\operatorname{ess\,sup} f = \operatorname{ess\,sup}_X f &= \inf\{s \in \overline{\mathbb{R}} \,|\, f(x) \leq s \quad \mu\text{-}a.e\ \}\\ &= \inf\Big\{\sup_{x\in X\setminus N} f(x)\,\Big|\, N \subset X, N\ \mu\text{-}null\ set\Big\}\,.\end{aligned} \tag{22}$$

Definition 10 (The semi-normed spaces $\mathcal{L}^p(X)$, $p \in [1,\infty]$).

(i) For $p \in [1,\infty)$, let

$$\begin{aligned} \mathcal{L}^p &:= \mathcal{L}^p(X) := \mathcal{L}^p(\mu) := \mathcal{L}^p(X,\mu) := \mathcal{L}^p(X,\mathcal{A},\mu) \\ &:= \Big\{ f : X \to \mathbb{C} \,\Big|\, f \in \mathcal{M}(X,\mathbb{C})\,,\ \int_X |f|^p \,\mathrm{d}\mu < \infty \Big\} \end{aligned} \tag{23}$$

and, for $f \in \mathcal{L}^p(X)$, let

$$\|f\|_p := \Big(\int_X |f|^p \,\mathrm{d}\mu < \infty \Big)^{1/p} . \tag{24}$$

(ii) For $p = \infty$, let

$$\begin{aligned} \mathcal{L}^\infty &:= \mathcal{L}^\infty(X) := \mathcal{L}^\infty(\mu) := \mathcal{L}^\infty(X,\mu) := \mathcal{L}^\infty(X,\mathcal{A},\mu) \\ &:= \Big\{ f : X \to \mathbb{C} \,\Big|\, f \in \mathcal{M}(X,\mathbb{C})\,,\ \operatorname{ess\,sup}_X |f| < \infty \Big\} , \end{aligned} \tag{25}$$

and, for $f \in \mathcal{L}^\infty(X)$, let

$$\|f\|_\infty := \operatorname{ess\,sup}_X |f| . \tag{26}$$

Then, for all $p \in [1,\infty]$, $\|\cdot\|_p$ is a semi-*norm on $\mathcal{L}^p(X)$: $\|f\|_p = 0 \Leftrightarrow f \sim_{a.e.} 0$ (which does not mean $f = 0$).*

Theorem 3 (**Minkowski and (generalised) Hölder inequalities**).

(i) (Minkowski) *Let $p \in [1,\infty]$, then $\|f+g\|_p \le \|f\|_p + \|g\|_p$ for all $f, g \in \mathcal{L}^p(X)$.*

(ii) (Hölder) *Let $p, q \in [1,\infty]$, with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for all $f \in \mathcal{L}^p(X), g \in \mathcal{L}^q(X)$,*

$$\int_X |fg| \,\mathrm{d}\mu \le \|f\|_p \|g\|_q . \tag{27}$$

(iii) (Generalied Hölder) *Let $n \in \mathbb{N}$ ($n \ge 2$), and let $p_1, \dots, p_n \in [1,\infty]$, and let $p \in [1,\infty]$ satisfy $\frac{1}{p} = \sum_{j=1}^n \frac{1}{p_j}$. Then, for all $f_j \in \mathcal{L}^{p_j}(X)$, $j = 1, \dots, n$,*

$$\Big\| \prod_{j=1}^n f_j \Big\|_p \le \prod_{j=1}^n \|f_j\|_{p_j} . \tag{28}$$

(iv) (Interpolation in $\mathcal{L}^p$-spaces). *Let $1 \le p < r < q \le \infty$, $f \in \mathcal{L}^p(X) \cap \mathcal{L}^q(X)$. Let $\theta \in (0,1)$ with $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$. Then $f \in \mathcal{L}^r(X)$, and*

$$\|f\|_r \le \|f\|_p^\theta \, \|f\|_q^{1-\theta} . \tag{29}$$

Hence, for $f : X \to \mathbb{C}$ measurable, the set

$$\Gamma_f := \{ p \in [1,\infty] \,|\, f \in \mathcal{L}^p(X) \} \subset \mathbb{R} \tag{30}$$

is an interval.

(v) Let $p \in [1,\infty]$, $f \in \mathcal{L}^p(X) \cap \mathcal{L}^\infty(X)$. Then $f \in \cap_{q \ge p} \mathcal{L}^q(X)$, and $\lim_{q\to\infty} \|f\|_q = \|f\|_\infty$.

Theorem 4 (The normed spaces $L^p(X)$, $p \in [1, \infty]$).
For $p \in [1, \infty]$, the relation $\sim_{a.e.}$ defines an equivalence relation on $\mathcal{L}^p(X)$, and $\|\cdot\|_p$ defines a norm on the quotient vector space $L^p(X)$, which makes $(L^p(X), \|\cdot\|_p)$ a Banach space. For $p = 2$, $L^2(X)$ is a Hilbert space, with inner/scalar product $\langle f, g\rangle := \int_X \overline{f(x)} g(x)\,\mathrm{d}\mu(x)$.

Remark 2. *By abuse of notation we will call $f \in L^p(X)$* functions *when we should really be talking about equivalence classes (this abuse of notation/language is well established).*

Theorem 5 (a.e. convergent subsequences).
Let $p \in [1, \infty]$, and assume $(f_j)_{j\in\mathbb{N}} \subset L^p(X), f \in L^p(X)$, satisfy $\lim_{j\to\infty} \|f_j - f\|_p = 0$. Then there exists a subsequence $(f_{j_k})_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} f_{j_k}(x) = f(x)$ a.e., that is, the subsequence $(f_{j_k})_{k\in\mathbb{N}}$ converges pointwise to f for μ-almost every $x \in X$.

Definition 11 (Locally integrable functions). *Let $(X, \mathcal{T})$ be a topological space, and let μ be a measure on $(X, \sigma(\mathcal{T}))$. (Example: $\mathbb{R}^d$ with Lebesque(-Borel) measure.) For $p \in [1, \infty]$, we denote*

$$L^p_{\mathrm{loc}}(X) := \Big\{ f : X \to \mathbb{C} \,\Big|\, f \in \mathcal{M}(X, \mathbb{C})\,,\ f \in L^p(K) \text{ for all } K \subset X \text{ compact} \Big\}\,. \tag{31}$$

Theorem 6 (Monotone convergence / Beppo Levi). *Let $(f_j)_{j\in\mathbb{N}}$, $f_j : X \to \mathbb{R}$, be a sequence of measurable functions with*

$$0 \le f_1 \le f_2 \le f_3 \le \dots\,. \tag{32}$$

Then, with $f(x) := \lim_{j\to\infty} f_j(x)$,

$$\lim_{j\to\infty} \int_X f_j \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu\,. \tag{33}$$

The possibility that both sides are $+\infty$ is included.

Theorem 7 ((Lebesgue) Dominated convergence). *Let $(f_j)_{j\in\mathbb{N}}$, $f_j : X \to \mathbb{R}$, be a sequence of measurable functions. Assume there exists $g \in L^1(X)$ such that $|f_j(x)| \le g(x)$ for a.e. $x \in X$ and all $j \in \mathbb{N}$, and that $f(x) := \lim_{j\to\infty} f_j(x)$ exists a.e. on X.*
Then

$$\lim_{j\to\infty} \int_X f_j \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu\,. \tag{34}$$

In this case both sides are finite.

Theorem 8 (Fatou's Lemma). *Let $(f_j)_{j\in\mathbb{N}}$, $f_j : X \to \mathbb{R}$, be a sequence of measurable functions, with $f_j(x) \ge 0$ a.e. on X for all $j \in \mathbb{N}$. Then*

$$\int_X \Big(\liminf_{j\in\mathbb{N}} f_j \Big) \,\mathrm{d}\mu \le \liminf_{j\in\mathbb{N}} \Big(\int_X f_j \,\mathrm{d}\mu \Big)\,. \tag{35}$$

Theorem 9 (Continuity and differentiability of parameter-dependent integrals).
Let (M, d) be a metric space, $(X, \mathcal{A}, \mu)$ a measure space, and $f : M \times X \to \mathbb{R}$ a map satisfying
(i) *The map $x \mapsto f(t, x)$ is integable for all $t \in M$.*
Let $F : M \to \mathbb{R}$ be given by $F(t) := \int_X f(t, x)\,\mathrm{d}\mu(x)$.

1. *Let $t_0 \in M$, and assume furthermore:*

   (ii) *The map $t \mapsto f(t,x)$ is continuous at t_0 for all $x \in X$.*

   (iii) *There exists integrable an function $g : X \to [0,\infty]$ such that $|f(t,x)| \leq g(x)$ for all $t \in M$ and $x \in X$.*

   Then F is continuous at t_0:

$$\begin{aligned} \lim_{t\to t_0} F(t) = \lim_{t\to t_0} \Big(\int_X f(t,x)\,\mathrm{d}\mu(x) \Big) &= \int_X \Big(\lim_{t\to t_0} f(t,x) \Big)\,\mathrm{d}\mu(x) \\ &= \int_X f(t_0,x)\,\mathrm{d}\mu(x) = F(t_0)\,. \end{aligned} \tag{36}$$

2. *Let $M = I \subset \mathbb{R}$ be an open interval, and assume* (i) *holds. Assume furthermore that*

   (ii') *The map $t \mapsto f(t,x)$ is differentiable on I for all $x \in X$.*

   (iii') *There exists an integrable function $g : X \to [0,\infty]$ such that $|\frac{\partial f}{\partial t}(t,x)| \leq g(x)$ for all $t \in M$ and $x \in X$.*

   Then F is differentiable on I, the map $x \mapsto \frac{\partial f}{\partial t}(t,x)$ is integrable for all $t \in I$, and

$$\frac{d}{dt}\Big(\int_X f(t,x)\,\mathrm{d}\mu(x) \Big) = F'(t) = \frac{dF}{dt}(t) = \int_X \frac{\partial f}{\partial t}(t,x)\,\mathrm{d}\mu(x)\,. \tag{37}$$

Definition 12 (Product-σ-algebra).
Let $(X_j, \mathcal{A}_j)$, $j = 1,\ldots,n$, be measurable spaces. The product-σ-algebra

$$\bigotimes_{j=1}^{n} \mathcal{A}_j := \mathcal{A}_1 \otimes \ldots \otimes \mathcal{A}_n := \sigma(p_1,\ldots,p_n) \tag{38}$$

(on $X := \bigtimes_{j=1}^{n} X_j$) is the smallest σ-algebra such that the projections *$p_j : X \to X_j$, $x = (x_1,\ldots,x_n) \mapsto x_j$, are all measurable.*

Example 4. $\mathcal{B}^d = \mathcal{B}^1 \otimes \ldots \otimes \mathcal{B}^1$ (d times). However, $\overline{\mathcal{B}^d} \supsetneq \overline{\mathcal{B}^1} \otimes \ldots \otimes \overline{\mathcal{B}^1}$.

Theorem 10 (Product measure).
Let $(X_j, \mathcal{A}_j, \mu_j)$, $j = 1,\ldots,n$, be σ-finite (!) measure spaces, and let $X := \bigtimes_{j=1}^{n} X_j$. Then there exists a unique measure $\mu := \otimes_{j=1}^{n} \mu_j := \mu_1 \otimes \ldots \otimes \mu_n$ (called the product measure (of $\mu_1,\ldots,\mu_n$)*) on $\mathcal{A} := \otimes_{j=1}^{n} \mathcal{A}_j$ such that*

$$\Big(\bigotimes_{j=1}^{n} \mu_j \Big)\Big(\bigtimes_{j=1}^{n} A_j \Big) = \prod_{j=1}^{n} \mu_j(A_j) \quad \textit{for all } A_j \in \mathcal{A}_j, j = 1,\ldots,n\,. \tag{39}$$

Furthermore, $(X, \mathcal{A}, \mu)$ is σ-finite.

Theorem 11 (Fubini-Tonelli).
Let $(X_j, \mathcal{A}_j, \mu_j)$, $j = 1,2$, be σ-finite (!) measure spaces, and let $(X, \mathcal{A}, \mu) := (X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mu_1 \otimes \mu_2)$. Let $f : X \to \overline{\mathbb{R}}$ (or $\mathbb{C}$) be $\mathcal{A}$-measurable. Then is, for all $g \in \{\Re(f_+), \Re(f_-), \Im(f_+), \Im(f_-)\}$, the functions

$$X_1 \to [0,\infty]\,,\ x_1 \mapsto \int_{X_2} g(x_1,x_2)\,\mathrm{d}\mu_2(x_2)\,, \tag{40}$$

$$X_2 \to [0,\infty]\,,\ x_2 \mapsto \int_{X_1} g(x_1,x_2)\,\mathrm{d}\mu_1(x_1) \tag{41}$$

$\mathcal{A}_1$-measurable, respectively, $\mathcal{A}_2$-measurable. Furthermore,

1. (Tonelli) *If $f \geq 0$ a.e. (that is, $f(X \setminus N) \subset [0, \infty], \mu(N) = 0$), then*

$$\int_X f(x)\,\mathrm{d}\mu(x) = \int_{X_1} \Big(\int_{X_2} f(x_1, x_2)\,\mathrm{d}\mu_2(x_2) \Big)\,\mathrm{d}\mu_1(x_1)$$
$$= \int_{X_2} \Big(\int_{X_1} f(x_1, x_2)\,\mathrm{d}\mu_1(x_1) \Big)\,\mathrm{d}\mu_2(x_2)\,. \tag{42}$$

   Note: It is possible that all three integrals are $+\infty$.

2. (Fubini) *If* one *of the three integrals*

$$\int_X |f(x)|\,\mathrm{d}\mu(x)\ ,\ \int_{X_1} \Big(\int_{X_2} |f(x_1, x_2)|\,\mathrm{d}\mu_2(x_2) \Big)\,\mathrm{d}\mu_1(x_1)\ ,$$
$$\int_{X_2} \Big(\int_{X_1} |f(x_1, x_2)|\,\mathrm{d}\mu_1(x_1) \Big)\,\mathrm{d}\mu_2(x_2) \tag{43}$$

   is finite*, then they are* all *finite, and* (42) *holds.*

Theorem 12 (Layer Cake Principle). *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and let $\mathcal{B}_{\geq 0}$ be the Borel-algebra of $\mathbb{R}_{\geq 0}$. Let ν be a measure on $\mathcal{B}_{\geq 0}$ such that $\phi(t) := \nu([0, t))$ is finite for all $t > 0$, and let $f : X \to \mathbb{R}_{\geq 0}$ be $\mathcal{A}$-$\mathcal{B}_{\geq 0}$-measurable. Then*

$$\int_X \phi(f(x))\,\mathrm{d}\mu(x) = \int_{\mathbb{R}_{\geq 0}} \mu(\{x \in X \mid f(x) > t\})\,\mathrm{d}\nu(t)\,. \tag{44}$$

Recall that $\mu_f(t) = \mu(\{f > t\})$ is the distribution function of f relative to μ.
In particular, if $f \in \mathcal{L}^p(X)$, then (by choosing $\mathrm{d}\nu(t) = pt^{p-1}\mathrm{d}\lambda^1(t)$)

$$\int_X |f|^p\,\mathrm{d}\mu = p \int_{\mathbb{R}_{\geq 0}} t^{p-1} \mu(\{|f| > t\})\,\mathrm{d}\lambda^1(t)\,, \tag{45}$$

and if $f \in \mathcal{L}^1(X)$ with $f \geq 0$, then (by choosing $p = 1$)

$$\int_X f\,\mathrm{d}\mu = \int_{\mathbb{R}_{\geq 0}} \mu(\{f > t\})\,\mathrm{d}\lambda^1(t)\,. \tag{46}$$

Also (by choosing μ the Dirac measure δ_x at $x \in X$, and $p = 1$),

$$f(x) = \int_{\mathbb{R}_{\geq 0}} \mathbb{1}_{\{f>t\}}(x)\,\mathrm{d}\lambda^1(t) \quad \text{(Layer Cake Representation of } f)\,. \tag{47}$$

Theorem 13 (Transformation formula for λ^d).
Let $U \subset \mathbb{R}^d$ be open, and $\varphi : U \to \varphi(U) \subset \mathbb{R}^d$ a diffeomorphism. Then, for all $f \in \mathcal{L}^1(\varphi(U), \lambda^d)$,

$$\int_{\varphi(U)} f(y)\,\mathrm{d}\lambda^d(y) = \int_U f(\varphi(x)) \Big| \det(D\varphi(x)) \Big|\,\mathrm{d}\lambda^d(x)\,. \tag{48}$$

Lemma 1 (Notation and certain concrete integrals in $\mathbb{R}^d$).

1. *For $x \in \mathbb{R}^d$, $r > 0$, we denote $B^d_r(x) := B_r(x) := \{y \in \mathbb{R}^d \mid |x - y| < r\}$, and $\omega_d := \lambda^d(B_1(0)) = \lambda^d(\overline{B_1(0)})$. One has $\omega_d = \frac{\pi^d/2}{\Gamma(d/2+1)}$, with $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\,dt$, $z > 0$, the* Gamma-function*.*

2. *One has*

$$\int_{B_1(0)} |x|^\alpha \,\mathrm{d}\lambda^d(x) < \infty \;\Leftrightarrow\; \alpha > -d\,, \tag{49}$$

$$\int_{\mathbb{R}^d\setminus B_1(0)} |x|^\alpha \,\mathrm{d}\lambda^d(x) < \infty \;\Leftrightarrow\; \alpha < -d\,, \tag{50}$$

$$\int_{\mathbb{R}^d} \frac{1}{(1+|x|)^\alpha} \,\mathrm{d}\lambda^d(x) < \infty \;\Leftrightarrow\; \alpha > -d\,. \tag{51}$$

Definition 13 (Spaces of differentiable functions on $\mathbb{R}^d$). *Denote, for $k \in \mathbb{N}$,*

$$C^0(\mathbb{R}^d) := C(\mathbb{R}^d) := C(\mathbb{R}^d,\mathbb{C}) := \left\{ f : \mathbb{R}^d \to \mathbb{C} \,\middle|\, f \text{ is continuous} \right\}, \tag{52}$$

$$C^k(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \to \mathbb{C} \,\middle|\, f \text{ is } k \text{ times continuous differentiable} \right\}, \tag{53}$$

$$C^\infty(\mathbb{R}^d) := \bigcap_{k\in\mathbb{N}} C^k(\mathbb{R}^d)\,, \tag{54}$$

and define, for $f \in C(\mathbb{R}^d)$, the support of f *by* $\mathrm{supp}(f) := \overline{\{x \in \mathbb{R}^d \,|\, f(x) \neq 0\}}$. *Denote, for $k \in \mathbb{N}\cup\{\infty\}$,*

$$C_c^k(\mathbb{R}^d) := \left\{ f \in C^k(\mathbb{R}^d) \,\middle|\, \mathrm{supp}(f) \subset \mathbb{R}^d \text{ is compact} \right\}. \tag{55}$$

Theorem 14 (Denseness of $C_c^k(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$).

1. *The set $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ with respect to $\|\cdot\|_p$ for $1 \leqslant p < \infty$.*

   More precisely: For all $f \in L^p(\mathbb{R}^d)$, $1 \leqslant p < \infty$, and all $\varepsilon > 0$ there exists $\phi \in C_c(\mathbb{R}^d)$ with $\|\phi - f\|_p < \varepsilon$. Note: The result fails in $L^\infty(\mathbb{R}^d)$.

2. *The set $C_c^\infty(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ with respect to $\|\cdot\|_p$ for $1 \leqslant p < \infty$.*

   Again, the result fails in $L^\infty(\mathbb{R}^d)$.

3. *As a consequence, $C_c^k(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ with respect to $\|\cdot\|_p$ for $1 \leqslant p < \infty$, and all $k \in \mathbb{N}\cup\{\infty\}$.*

   Again, the result fails in $L^\infty(\mathbb{R}^d)$.

Remark 3 (Notation in $\mathbb{R}^d$). *We will most often write $\int_{\mathbb{R}^d} f(x)\,dx$ or $\int f(x)\,\mathrm{d}x$ or simply $\int f \mathrm{d}x$ instead of $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}\lambda^d(x)$ from now on. Also, we will often use the notation $|A| := \lambda^d(A)$ for the Lebesgue(-Borel) measure of a (measurable) set $A \subset \mathbb{R}^d$. This way, for the distribution function of $f : \mathbb{R}^d \to \mathbb{R}$ (relative to Lebesgue measure λ^d) we have*

$$(\lambda^d)_f(t) = \lambda^d(\{f > t\}) = \lambda^d(\{x \in \mathbb{R}^d \,|\, f(x) > t\}) = |\{f > t\}|\,. \tag{56}$$

Functional Analysis

We assume knowledge of metric, and more generally, topological spaces. A special case, also assumed known, is that of (semi-)normed vector spaces. All vector spaces in the sequel will be over $\mathbb{K}$, $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. Finally, we assume knowledge of scalar (or inner) product spaces. Note that, in complex vector spaces, our scalar products are linear in the *second* entry, and conjugate linear in the *first* one.

Definition 14 (Dense subspaces).
Let $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ *be a normed (vector) space. A (linear) subspace* $V \subset \mathcal{X}$ *is called* dense (in $\mathcal{X}$; more correctly, in $(\mathcal{X}, \|\cdot\|_\mathcal{X})$) *iff* $\overline{V} = \mathcal{X}$ *(closure in the topology generated by the (metric generated by the) norm* $\|\cdot\|_\mathcal{X}$*).*

Definition 15 (Banach and Hilbert spaces).

1. (Banach space) *A normed vector space* $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ *such that the metric space* $(\mathcal{X}, d)$ *with* $d(x,y) := \|x-y\|_\mathcal{X}$, $x, y \in \mathcal{X}$, *is a complete metric space is called a* Banach space.

2. (Hilbert space) *A Banach space* $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ *for which the norm is generated by a scalar product (that is, there exists a scalar product* $\langle\cdot,\cdot\rangle$ *on* $\mathcal{X}$ *such that* $\|x\|^2 = \langle x, x\rangle$ *for all* $x \in \mathcal{X}$*) is called a* Hilbert space.

Definition 16 ((Bounded) linear operators).

1. *Let* $\mathcal{X}, \mathcal{Y}$ *be two vector spaces; a linear map* $T : \mathcal{X} \to \mathcal{Y}$ *is called a* linear operator. *We denote by* $L(\mathcal{X}, \mathcal{Y})$ *the set of all linear operators from* $\mathcal{X}$ *to* $\mathcal{Y}$*. It is a vector space. If* $\mathcal{X} = \mathcal{Y}$*, we write* $L(\mathcal{X})$*.*

2. *If* $(\mathcal{X}, \|\cdot\|_\mathcal{X})$, $(\mathcal{Y}, \|\cdot\|_\mathcal{Y})$ *are both normed spaces, and if there exists a constant* $C > 0$ *such that (*$Tx := T(x)$*)*

$$\|Tx\|_\mathcal{Y} \le C\,\|x\|_\mathcal{X} \quad \textit{for all } x \in \mathcal{X}\,, \tag{57}$$

then T *is called a* bounded (linear) operator *(or simply* bounded*). We denote the vector space of all bounded operators from* $\mathcal{X}$ *to* $\mathcal{Y}$ *(more correctly, from* $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ *to* $(\mathcal{Y}, \|\cdot\|_\mathcal{Y})$*) by* $B(\mathcal{X}, \mathcal{Y})$*. If* $\mathcal{X} = \mathcal{Y}$ *(*and $\|\cdot\|_\mathcal{X} = \|\cdot\|_\mathcal{Y}$ *(!)), we write* $B(\mathcal{X})$*.*

3. *We denote by*

$$\|T\| := \|T\|_{B(\mathcal{X},\mathcal{Y})} := \sup_{x \in \mathcal{X}\setminus\{0\}} \frac{\|Tx\|_\mathcal{Y}}{\|x\|_\mathcal{X}} = \sup_{x\in\mathcal{X}, \|x\|_\mathcal{X}=1} \|Tx\|_\mathcal{Y} \ (< \infty) \tag{58}$$

the operator norm *of* $T \in B(\mathcal{X}, \mathcal{Y})$*.*

Theorem 15 (Continuous linear maps and the space $(B(\mathcal{X}, \mathcal{Y}), \|\cdot\|_{B(\mathcal{X},\mathcal{Y})})$**).**
Let $(\mathcal{X}, \|\cdot\|_\mathcal{X})$, $(\mathcal{Y}, \|\cdot\|_\mathcal{Y})$ *be normed spaces.*

1. *A linear operator* $T \in L(\mathcal{X}, \mathcal{Y})$ *is continuous iff it is a bounded linear operator (that is, iff* $T \in B(\mathcal{X}, \mathcal{Y})$*).*

2. *The operator norm* $\|\cdot\|_{B(\mathcal{X},\mathcal{Y})}$ *is a norm on* $B(\mathcal{X}, \mathcal{Y})$*.*

3. *If* $(\mathcal{Y}, \|\cdot\|_\mathcal{Y})$ *is a Banach space, then so is* $(B(\mathcal{X}, \mathcal{Y}), \|\cdot\|_{B(\mathcal{X},\mathcal{Y})})$*.*

4. *In particular, the* (topological) dual of $\mathcal{X}$ *(more correctly,* of $(\mathcal{X}, \|\cdot\|_\mathcal{X})$*), defined by* $\mathcal{X}' := B(\mathcal{X}, \mathbb{K})$, $\|\cdot\|_{\mathcal{X}'} := \|\cdot\|_{B(\mathcal{X},\mathbb{K})}$*, is* always *a Banach space.*

Definition 17 (Bi-dual and reflexive space).
Let $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ *be a normed space.*

1. *The* bi-dual (of $\mathcal{X}$) *is the space* $\mathcal{X}'' := (\mathcal{X}')'$ *(with the norm defined accordingly). It is always a Banach space.*

2. *The* natural injection *is the (linear) map* $J : \mathcal{X} \to \mathcal{X}''$*,* $x \mapsto J(x)$*, given by* $J(x)(\varphi) := \varphi(x)$*,* $\varphi \in \mathcal{X}'$*. (By Hahn-Banach below,* J *is injective.)*

3. *If the natural injection* $J : \mathcal{X} \to \mathcal{X}''$ *is surjectiv, then* $\mathcal{X}$ *is called* reflexive*. Note that any reflexive space is necessarily a Banach space.*

Example 5 (**Dual of** $L^p(X)$)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. The spaces $(L^p(X), \|\cdot\|_p)$ are reflexive for $p \in (1, \infty)$: $(L^p(X))' = L^{p'}(X)$ (or rather, isometrically ismorphic) with $1/p + 1/p' = 1$.

2. If μ is σ-finite, then $(L^1(X))' = L^\infty(X)$ (again, isometrically ismorphic).

3. The dual of $L^\infty(X)$ is more complicated to express.

Theorem 16 (**Hahn-Banach and consequences**)**.**

1. *Let* $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ *be a normed space, and* $U \subset \mathcal{X}$ *a linear subspace. If* $\varphi : U \to \mathbb{K}$ *is linear and continuous (that is, bounded on* U*), then there exists an extension (not necessarily unique!)* $\tilde{\varphi} : \mathcal{X} \to \mathbb{K}$ *of* φ *which is also continuous and linear (hence,* $\tilde{\varphi} \in \mathcal{X}'$*), and which has the same (operator/dual) norm as* φ*,* $\|\tilde{\varphi}\|_{\mathcal{X}'} = \|\varphi\|_{B(U,\mathbb{K})}$*.*

2. *Let* $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ *be a normed space, and* $U \subset \mathcal{X}$ *a linear subspace. Assume* $z \in \mathcal{X} \setminus \overline{U}$*. Then there exists* $\psi \in \mathcal{X}'$ *with* $\psi(x) = 0$ *for all* $x \in U$*,* $\psi(z) = 1$*, and* $\|\psi\|_{\mathcal{X}'} = \operatorname{dist}(z, U)^{-1}$*.*

3. *In particular, let* $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ *be a normed space and* $z \in \mathcal{X}$*. Then there exists* $\psi \in \mathcal{X}'$ *with* $\psi(z) = \|z\|_{\mathcal{X}}$*, and* $\|\psi\|_{\mathcal{X}'} \leq 1$*. In particular, for all* $x \in \mathcal{X}$*,*

$$\|x\|_{\mathcal{X}} = \max\left\{|\varphi(x)| \,\middle|\, \varphi \in \mathcal{X}', \|\varphi\|_{\mathcal{X}'} = 1\right\}. \tag{59}$$

As a result, the natural injection $J : \mathcal{X} \to \mathcal{X}''$ *is an* isometry*:* $\|J(x)\|_{\mathcal{X}''} = \|x\|_{\mathcal{X}}$ *for all* $x \in \mathcal{X}$*.*

Example 6 (**Bounded operators between** L^p**-spaces**)**.**
Let $(X, \mathcal{A}, \mu)$ be a measure space.

1. One has, by (59) and $(L^p(X))' = L^{p'}(X)$, that for all $p \in [1, \infty)$,

$$\|f\|_p = \max\left\{|\langle g, f\rangle| \,\middle|\, g \in L^{p'}(X), \|g\|_{p'} = 1\right\}, \tag{60}$$

with the notation that, for $f, g \in \mathcal{M}(X, \mathbb{C})$,

$$\langle g, f\rangle := \int_X g(x) f(x) \, \mathrm{d}\mu(x). \tag{61}$$

With the convention that $1' := \infty$, (60) also holds for $p = \infty$, using that $f \in L^\infty(X) = (L^1(X))'$ and the definition of the norm in the dual.

2. It follows that, for $T : L^p(X) \to L^q(X)$ bounded, $p, q \in [1, \infty]$, one has

$$\begin{aligned}\|T\| = \|T\|_{B(L^p(X), L^q(X))} = \|T\|_{B(L^p, L^q)} &= \sup_{f \in L^p(X), \|f\|_p = 1} \|Tf\|_{L^q(X)} \\ &= \sup\left\{|\langle Tf, g\rangle| \,\middle|\, f \in L^p(X), \|f\|_p = 1, g \in L^{q'}(X), \|g\|_{L^{q'}(X)} = 1\right\}. \end{aligned} \tag{62}$$

Theorem 17 (**Riesz' representation theorem (Hilbert space)**).
Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, and let $\varphi \in H'$. Then there exists a unique $x = x(\varphi) \in H$ such that $\varphi(y) = \langle x, y\rangle$ for all $y \in H$. The map $\varphi \mapsto x(\varphi)$ (from H' to H) is linear if H is real, and conjugate linear if H is complex.

Theorem 18 (**The BLT-Theorem**).
Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed spaces, let $V \subset \mathcal{X}$ be a dense (!) subspace, and assume $T : V \to \mathcal{Y}$ is a bounded operator (with $\|\cdot\|_{\mathcal{X}}$ restricted to V the norm on V). If $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is a Banach space (!), then there exists a unique (!) bounded operator $\tilde{T} \in B(\mathcal{X}, \mathcal{Y})$ extending T (that is, $\tilde{T}|_V = T$).

Theorem 19 (**Open mapping theorem**).
Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and assume $T \in B(\mathcal{X}, \mathcal{Y})$ is surjective. Then T is an open mapping*, that is, if $U \subset \mathcal{X}$ is open, then $T(U) \subset \mathcal{Y}$ is open.*

Corollary 1 (**Bounded inverse theorem**). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and assume $T \in B(\mathcal{X}, \mathcal{Y})$ is bijective. Then the inverse operator $T^{-1} : \mathcal{Y} \to \mathcal{X}$ is also bounded (hence, continuous).*

Corollary 2 (**Closed graph theorem**). *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and assume $T \in L(\mathcal{X}, \mathcal{Y})$ (!) satisfies: For every sequence $(x_n)_{n\in\mathbb{N}} \subset \mathcal{X}$ with both $x_n \to 0 (\in \mathcal{X}), n \to \infty$, and $Tx_n \to y (\in \mathcal{Y}), n \to \infty$, follows that $y = 0$. Then $T \in B(\mathcal{X}, \mathcal{Y})$ (that is, T is continuous/bounded).*

Theorem 20 (**Uniform boundedness principle / Banach-Steinhaus**). *Let $\mathcal{X}$ be a Banach space (!) and $\mathcal{Y}$ a normed (!) vector space. Suppose that $F \subset B(\mathcal{X}, \mathcal{Y})$ is a collection of continuous linear operators (that is, bounded operators) from $\mathcal{X}$ to $\mathcal{Y}$. If for all $x \in \mathcal{X}$ one has*

$$\sup_{T\in F} \|Tx\|_{\mathcal{Y}} < \infty\,, \tag{63}$$

then

$$\sup_{T\in F, \|x\|_{\mathcal{X}}=1} \|Tx\|_{\mathcal{Y}} = \sup_{T\in F} \|T\|_{B(\mathcal{X},\mathcal{Y})} < \infty\,. \tag{64}$$

Corollary 3 (**Useful consequence of Banach-Steinhaus**). *Let $\mathcal{X}$ be a Banach space (!) and $\mathcal{Y}$ a normed (!) vector space, and let $(T_n)_{n\in\mathbb{N}} \subset B(\mathcal{X}, \mathcal{Y})$. Assume $\lim_{n\to\infty} T_n x =: Tx$ exists for all $x \in \mathcal{X}$. Then $T \in B(\mathcal{X}, \mathcal{Y})$. (Importance: pointwise limits of continuous maps are not, in general, continuous.)*

Complex Analysis

Let $U \subset \mathbb{C}$ be open.

Definition 18 (**Holomorphic function**).

1. *A function $f : U \to \mathbb{C}$ is called* complex differentiable at $z_0 \in U$ *iff*

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \; (=: f'(z_0) \in \mathbb{C}\,) \tag{65}$$

exists. The (complex) number $f'(z_0)$ is called the (complex) derivative of f at z_0.

2. *The function f is called* holomorphic (in U) *iff it is complex differentiable at z for all $z \in U$.*

Definition 19 (**(Complex) analytic function).**

1. *A function $f : U \to \mathbb{C}$ is said to* have a (convergent) power series expansion at $z_0 \in U$ *iff there exist a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ and $r > 0$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \textit{for all } z \in B_r(z_0) \subset U\,. \tag{66}$$

2. *The function f is called* (complex) analytic (in U) *iff it has a (convergent) power series expansion at z for all $z \in U$.*

Theorem 21 (**Holomorphic equals (complex) analytic).** *A function $f : U \to \mathbb{C}$ is holomorphic iff it is (complex) analytic. In this case, for all $z_0 \in U$ there exists $r = r(z_0) > 0$ such that*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n\,,\ z \in B_r(z_0) \subset U \quad \text{(Taylor-series)}\,, \tag{67}$$

with

$$f^{(0)}(z) := f(z)\,,\ f^{(1)}(z) := f'(z)\,,\ f^{(n+1)}(z) := (f^{(n)})'(z)\,,\ n \geq 1\,. \tag{68}$$

Let $D \subset \mathbb{C}$ be connected and open.

Theorem 22 (**Maximum Modulus Principle).** *Let $f : D \to \mathbb{C}$ be holomorphic. If there exists $z_0 \in D$ and $r > 0$ such that*

$$|f(z_0)| \geq |f(z)| \quad \textit{for all } z \in B_r(z_0) \subset D\,, \tag{69}$$

then there exists $c \in \mathbb{C}$ such that $f(z) = c$ for all $z \in D$ (that is, f is constant*).*