De Finetti theorems, mean-field limits and Bose-Einstein condensation

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Plan for the introduction

1. **Bose-Einstein Condensation**
2. Classical statistical mechanics
3. Mean-field approximation and de Finetti theorems

Bose-Einstein Condensation in a nutshell

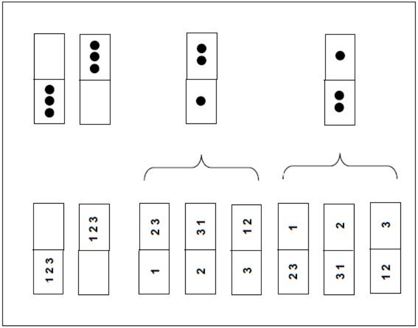
- Many particles (bosons) in the same quantum state
- Bosons = indistinguishable quantum particles that do not satisfy Pauli's exclusion principle
- Phenomenon due to bosonic statistics, requires very low temperature
- Theoretical prediction: Bose 1924, Einstein 1925.
- For free bosonic particles, below a certain critical temperature, the lowest energy state is macroscopically occupied.
- First experimental realization: 1995, Boulder (Colorado) and MIT (Nobel prize in physics 2001: Cornell, Wieman and Ketterle).
- Nowadays observed and studied in many laboratories, mostly in metastable states of dilute alkali gases
- BECs = macroscopic quantum objects ($\sim 10\mu m$), offer remarkable possibilities for "directly" observing quantum phenomena.

Bose and Einstein's argument

- N non interacting particles to be distributed in discrete energy levels.
- Thermostat, temperature T. Free energy = energy - $T\times$ entropy.
- Computation of entropy at fixed energy, basic statistical mechanics: how many possible distributions of particles correspond to a given energy ?
- For indistinguishable particles, one obtains the Bose-Einstein distribution

$$n_i = \frac{1}{\exp\left(\beta(E_i - \mu)\right) - 1}$$

- Einstein $\rightsquigarrow$ for large β, lowest energy state macroscopically occupied



Natural objections

Einstein → Ehrenfest :
"**This is a beautiful idea, but does it contain a part of truth ?**"

First note how revolutionary the idea was: 1925 is the pre-dawn of quantum mechanics, Schrödinger's equation is from 1926 !

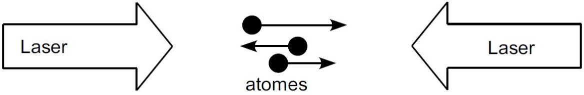
Moreover, three serious objections sprang to the mind of contemporaries:

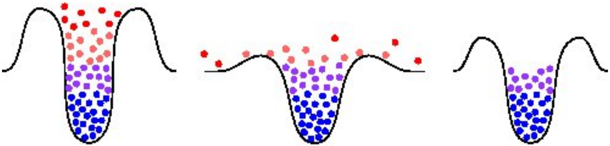
1. Critical temperature for BEC is extremely low, unrealistically so, as it seemed in the 1920's.
2. At such temperatures, all known materials should be in a solid phase, not gaseous as assumed in Einstein's paper.
3. How would interactions affect the phenomenon ?

However, BEC was finally observed in the 90's, 70 years later ...
... in experiments where interactions DO matter.

Experimental advances: 70 years of hard work

- Laser cooling (Nobel prize in physics 1997: Chu, Phillips and Cohen-Tannoudji). Slow down atoms using matter-light interaction.
- Radiative pressure, Doppler effect ... $\rightsquigarrow$ m-Kelvin
- Magneto-optic traps: separate gas from any material wall.
- Evaporative cooling: play with the trap's potential barrier $\rightsquigarrow$ μ-Kelvin

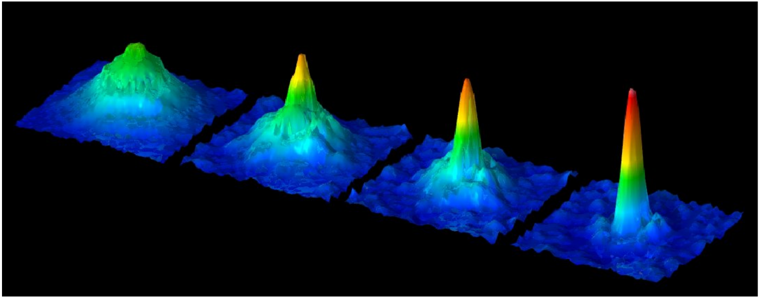




At the end of the day ... very cold and dilute metastable gases in which one is able to detect BEC.

Experimental evidence (1): macroscopic occupancy

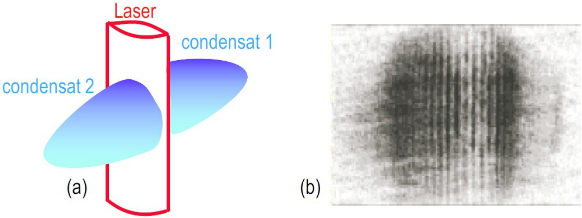
- Image a condensate after ballistic expansion
- Trapped particles in a potential well, at equilibrium
- Switch off potential and image the expanding cloud
- Reconstruct the initial distribution in momentum/energy space



Energy distribution of a could of trapped atoms (Sodium), decreasing temperature from left to right.
Davis-Mewes-Andrews-van Druten-Durfee-Kurn-Ketterle, PRL, 95.

Experimental evidence (2): interfering condensates

- Test the wave nature of the condensate
- Create two condensates in the two wells of a symmetric potential
- Switch the barrier off: condensates overlap and interfere
- Coherent matter waves have indeed been created



Interference pattern created by the overlap of two BECs.
Andrews-Townsend-Miesner-Durfee-Kurn-Ketterle, Science, 97.

Model for interacting trapped BEC: Gross-Pitaevskii theory

- N particles in same quantum state: single wave function $\psi \in L^2(\mathbb{R}^d)$
- $N|\psi|^2$ = matter density
- Interactions dealt with in a mean-field-like approximation
- Trapping potential $V : \mathbb{R}^d \to \mathbb{R}$, interaction potential $w : \mathbb{R}^d \to \mathbb{R}$
- Units: mass $= \hbar = 1$, coupling constant g
- Energy functional:

$$\mathcal{E}[\psi] = \int_{\mathbb{R}^d} \left(|\nabla \psi|^2 + V(\mathbf{x})|\psi|^2 + \frac{g}{2}(w * |\psi|^2)|\psi|^2 \right)$$

- Ground state $\to$ minimize under mass constraint

$$\int_{\mathbb{R}^d} |\psi|^2 = 1.$$

- For dilute gases, take contact interactions, $w = \delta_0$

$$\mathcal{E}^{\mathrm{GP}}[\psi] = \int_{\mathbb{R}^d} \left(|\nabla \psi|^2 + V(\mathbf{x})|\psi|^2 + \frac{g}{2}|\psi|^4 \right)$$

A look at the density: interactions do matter !

- Small interactions, single particle physics:

$$\int_{\mathbb{R}^d} \left(|\nabla \psi|^2 + V(\mathbf{x})|\psi|^2 \right), \quad -\Delta\psi + V\psi = \mu\psi$$

e.g. $V(x) = |x|^2$, harmonic oscillator, exactly soluble $\rightsquigarrow$ gaussian

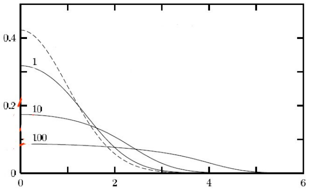
- Large interactions, "Thomas-Fermi regime" solve for $\rho = |\psi|^2$

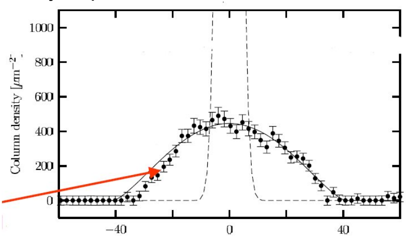
$$\int_{\mathbb{R}^d} \left(V(\mathbf{x})\rho + \frac{g}{2}\rho^2 \right), \quad g\rho + V = \mu$$

- Thomas-Fermi profile, e.g. $V(x) = |x|^2 \rightsquigarrow$ inverted parabola

$$\rho^{\mathrm{TF}} = g^{-1}\left[\mu - V\right]_+$$

much better approximation in many experiments.

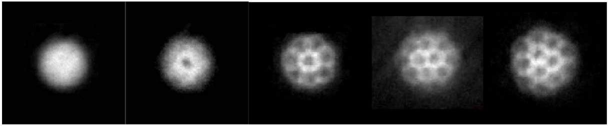




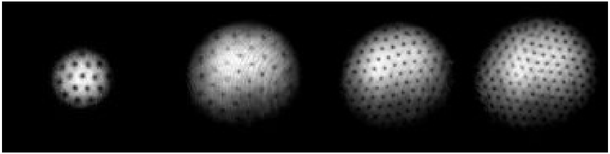
Hau, Busch, Liu, Dutton, Burns, Golovchenko, PRA, 1998

Superfluidity and quantized vortices

- A BEC is a superfluid, and thus responds to rotation by the nucleation of quantized vortices
- Vortices organize in triangular lattices, cf mixed phase of type II superconductors (Abrikosov lattices)
- Rotational symmetry breaking: consequence of interactions



First few vortices appearing in a rotating BEC.
Jean Dalibard's group, Laboratoire Kastler Brossel.



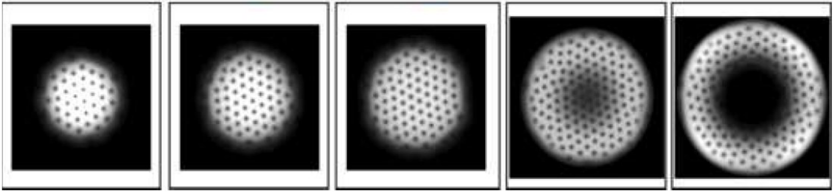
Vortex lattices.
W. Ketterle's group, MIT.

Model for rotating trapped BEC

- Look for equilibrium state in the rotating frame. Energy functional:

$$\mathcal{E}[\psi] = \int_{\mathbb{R}^2} \left(|\nabla - i\Omega \mathbf{x}^\perp \psi|^2 + \left(V(\mathbf{x}) - \Omega^2 |\mathbf{x}|^2 \right) |\psi|^2 + \frac{g}{2} |\psi|^4 \right)$$

- Non interacting case $\rightsquigarrow$ no vortex lattice ! Single-particle physics. Hamiltonian commutes with angular momentum. Ground state's density is rotationally symmetric.
- Interacting case: there is a vast literature about vortex nucleation in rotating superfluids, including mathematical theorems.
- Rigorous theorems establish that vortices tend to be uniformly distributed, repelling each other via 2D Coulomb-like forces.
- **Open problem**: the hexagonal lattice $\Leftrightarrow$ crystallization for the 2D Jellium.



Numerically minimizing the GP energy, Ionut Danaila 2005.

Summary so far, and the questions it suggests

- Existence of BEC can be guessed by simple statistical mechanics considerations.
- For free indistinguishable particles, only need the "statistical enhancement" of condensed configurations.
- Experimental requirements seemed unrealistic in the 20's, now met in many labs worldwide.
- Effective models based on assuming BEC efficient to describe experiments.
- In many experiments, interactions between particles DO matter.

However, the case for **BEC in presence of interactions is theoretically unclear**.

1. Is it true that the ground state of an interacting dilute Bose system shows BEC ?
2. Is it true that BEC is preserved by the dynamical evolution along the N-body Schrödinger flow ?
3. Is it true that the thermal state of an interacting dilute Bose system shows Bose condensation below a critical temperature ?

Summary so far, and the questions it suggests

- Existence of BEC can be guessed by simple statistical mechanics considerations.
- For free indistinguishable particles, only need the "statistical enhancement" of condensed configurations.
- Experimental requirements seemed unrealistic in the 20's, now met in many labs worldwide.
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However, the case for BEC in presence of interactions is theoretically unclear.

1. Is it true that the ground state of a dilute Bose system shows Bose condensation ? ✓YES !
2. Is it true that BEC is preserved by the dynamical evolution along the N-body Schrödinger flow ? ✓YES !
3. Is it true that the thermal state of a dilute Bose system shows Bose condensation below a critical temperature ? × OPEN PROBLEM.

We believe these questions to be of such importance that it is worth looking for mathematically rigorous proofs.

Main theme of the lectures

Link the fundamental, microscopic, many-body Schrödinger description with the effective, macroscopic, non-linear Schrödinger theory.

(We shall focus on the ground state to keep things within bounds.)

One may think of two kind of approaches:

- Methods based on properties of the Hamiltonian. Estimate in terms of auxiliary one-body Hamiltonian. Depends on the physics at hand.
- Methods based on the structure of bosonic states. Enhanced role of condensed states. Very general.

Program

- Derive NLS description in a mean-field limit: large N, weak interactions.
- Other, more realistic limits: dilute gases, strong but rare interactions.
- Main tools: quantum de Finetti theorems.
- Use as little as possible the properties of the Hamiltonian.
- Very much in the spirit of the pioneers: what matters is the statistics of the particles.

For symmetry reasons, *any bosonic N-body state looks like a superposition of condensates for large N.*