Let $a \in S^m$. Let $\sum_{j=1}^{\infty} \psi_j$ a smooth partition of unity (loca. finite sum).

$$a(x,D) = \sum_{j,k} \psi_j\, a(x,D)\, \psi_k = \sum_{\substack{j,k \\ \operatorname{supp}\psi_j \cap \operatorname{supp}\psi_k \neq \emptyset}} \psi_j\, a(x,D)\, \psi_k + \sum_{\substack{j,k \\ \operatorname{supp}\psi_j \cap \operatorname{supp}\psi_k = \emptyset}} \psi_j\, a(x,D)\, \psi_k .$$

(on $\mathcal{S}'$)

op. with smooth Kernel since the $\operatorname{sing\,supp} K \subset \{x=y\}$ for K kernel of $a(x,D)$.

OP. with Kernel

$K_{jk}(x,y) = \psi_j(x) K(x,y) \psi_k(y)$

having the property:

$\operatorname{supp} K \xrightarrow{\Pi_x} \mathbb{R}^d$, $\operatorname{supp} K \xrightarrow{\Pi_y} \mathbb{R}^d$.

$(x,y) \longmapsto x$ $\quad (x,y) \longmapsto y$.

$\Pi_x^{-1}(\underset{\text{compact}}{C}) \subset\subset \mathbb{R}^{2d}$, $\Pi_y^{-1}(C) \subset\subset \mathbb{R}^{2d}$.

K_{jk} is <u>properly supported</u>.

Thus $\quad a(x,D) = b(x,D) + r(x,D)$ with $r \in S^{-\infty}$

and b has a properly supported Kernel.

Note that, if $a(x,D)$ has a properly supp. kernel, we can <u>extend</u> $a(x,D)$ to $\mathcal{D}'(\mathbb{R}^d)$.

The principal symb. of $a(x,D)$ is $a_0 = \phi(x)\psi(x)\chi(\xi) = \phi(x)\chi(\xi)$.

$a_0(x_0,\xi_0) \neq 0$ and one can find $b_0 \in S^0$; $a_0 b_0 - 1 \in S^{-\infty}$ on $\operatorname{supp}\phi \cap \Gamma'$ $(\xi_0 \in \Gamma' \subset \Gamma)$.

$\Leftarrow$). $\exists V \in \mathcal{V}(x_0)$, Γ conic vicinity of ξ_0 s.t.

$$ab - 1 = r \quad \text{with} \quad r(x,\xi) \in S^{-\infty}(V\times\Gamma).$$

Let $\phi \in C_0^\infty$; $\operatorname{supp}\phi \subset V$, $\chi \in S^0$; $\operatorname{supp}\chi \subset \Gamma' \subset \bar{\Gamma}' \subset \Gamma$.

Then, by composition $\chi(D)\phi(x) r(x,D) = \tilde{r}(x,D)$ with $\tilde{r} \in S^{-\infty}$.

Let $\tau \in C_0^\infty$: $\tau\phi = \phi$. Let $\psi \in C_0^\infty$; $\psi = 1$ in a big enough vicinity of x_0 s.t.

$$(*) \quad \tau K(1-\psi) = 0 \quad \left(\begin{array}{l}\text{where } K \text{ is the kernel} \\ \text{of } a(x,D)\end{array}\right)$$

Now, we can find $b \in S^0$ s.t.

$$b(x,D)\,a(x,D) - I = r(x,D)$$

where $r|_{V\times\Gamma} \in S^{-\infty}$.

Thus
$$b(x,D)\,a(x,D)\psi u = \psi u + r(x,D)\psi u$$
$$\phi\, b(x,D)\,a(x,D)\psi u = \phi u + \phi r(x,D)\psi u$$

$$\hookrightarrow = \underset{\in C_0^\infty}{\phi b(x,D)\tau a(x,D)\psi u} + \underbrace{\underbrace{\phi b(x,D)(1-\tau)}_{\substack{=b'(x,D) \\ \text{with } b' \in S^{-\infty}}}\underbrace{a(x,D)\underset{\in H^{-k}}{\psi u}}_{\in H^{-k}}}$$

$$= \phi b(x,D)\tau a(x,D) u + \underset{\in C_0^\infty}{v} \quad \text{by } (*)$$

Therefore
$$\phi u = \underbrace{\phi b(x,D)\tau \underbrace{a(x,D)u}_{\in C^\infty}}_{\in C_0^\infty} + v - \phi r(x,D)\psi u,$$

and
$$\chi(D)\phi u = \underset{\in C^\infty}{w} - \underbrace{\chi(D)\phi r(x,D)\overset{\in H^{-k}}{\overline{\psi u}}}_{\in C^\infty}.$$

Thus $\chi(\xi)\widehat{\phi u}(\xi)$ is "rapidly decaying".

Th.: Let $a \in S^m$ with $a(x,D)$ properly supported.
Let $u \in \mathcal{D}'$. Then

$$WF(u) \subset \operatorname{Char} a \cup WF(a(x,D)u).$$

Idea of the proof: Take $(x_0,\xi_0) \notin \operatorname{Char} a \cup WF(a(x,D)u)$ and construct a $b(x,D)$ properly supported with $b \in S^0$, $(x_0,\xi_0) \notin \operatorname{Char} b$ and $b(x,D)u \in C^\infty$.

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