Similarly we find that $a(x,D)^* = b(x,D)$

with $b = e^{i\langle D_x, D_\xi\rangle} \bar{a}(x,\xi)$.

For $m \in \mathbb{R}$, let $S^m = \{a \in C^\infty(\mathbb{R}^{2d}) ; \forall (\alpha,\beta) \in \mathbb{N}^{2d}, \sup_{(x,\xi)} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \langle \xi \rangle^{|\beta|-m} < \infty\}$

Fréchet space.

For $a \in S^m$, we can defined $a(x,D)u$ $(u \in \mathcal{S})$ by an oscillatory int. $\int e^{i\langle x-y,\xi\rangle} a(x,\xi) u(y) \frac{dy\,d\xi}{(2\pi)}$ $a(x,D)$ extends to a conti. map. from $\mathcal{S}'$ to $\mathcal{S}'$

It turns out that:

$*$ $\mathcal{S} \ni a \longmapsto e^{i\langle D_x, D_\xi\rangle} \bar{a}(x,\xi) \in \mathcal{S}$

is conti. for the topo. of S^m.

$*$ $\mathcal{S} \times \mathcal{S} \ni (a;b) \mapsto \left[e^{i\langle D_y, D_\eta\rangle} a(x,\eta) b(y,\xi)\right]_{(y;\eta)=(x,\xi)} \in \mathcal{S}$

is conti. on $\mathcal{S}\times\mathcal{S}$ with topo. of $S^m \times S^{m'}$ to $\mathcal{S}$ with the topo. of $S^{m+m'}$.

By density of $\mathcal{S}$, these operations extend conti. on S^m (resp. from $S^m \times S^{m'}$ to $S^{m+m'}$).

Furthermore we have the expansions:

$$e^{i\langle D_x, D_\xi\rangle} \bar{a}(x,\xi) = \bar{a}(x,\xi) + b_1(x,\xi)$$

with $b \in S^{m-1}$

Longer expansion possible. Local expressions

$$a\#b(x,\xi) = \left[e^{i\langle D_y, D_\eta\rangle} a(x,\eta) b(y;\xi)\right]_{(y,\eta)=(x,\xi)} = (ab)(x,\xi) + b_2(x,\xi)$$

with $b_2 \in S^{m+m'-1}$.

Elliptic symb.: Let $a \in S^m$ $(m \geq 0)$ s.t.

$\exists\, C, R > 0, \forall (x,\xi), |\xi| \geq R \Rightarrow |a(x,\xi)| \geq c|\xi|^m$.

We say that a is elliptic.

If $a(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$, $a \Leftrightarrow$ ~~$\forall (x,\xi \neq 0)$~~, $a_m(x,\xi) \neq 0$.

For a elliptic in S^m $(m \geq 0)$, we can find $\chi \in C_0^\infty(\mathbb{R}^d)$ s.t. $\chi = 1$ in a large enough ball and $b = \frac{1-\chi}{a} \in S^{-m}$. Now

$$a \# b = 1 - \chi + r_0 \quad \text{with } r_0 \in S^{m-m-1} = S^{-1}.$$

$$= 1 + r_1 \quad \text{with } r_1 \in S^{-1}.$$

Take $b_1 = b - b \# r_1$. Then, since $\#$ is associative and 1 neutral ele.

$$a \# b_1 = a \# b - (a \# b) \# r_1 = 1 + r_1 - (1 + r_1) \# r_1$$

$$= 1 - \underbrace{r_1 \# r_1}_{\in S^{-2}}.$$

Using a ~~truncated~~ Neumann Serie, we can get

$$a \# b_N = 1 + r_N \quad \text{with } r_N \in S^{-N}.$$

L^2-continuity:

Th.: If $a \in S^0$, then $a(x,D)$ is conti. on L^2 and $S^0 \ni a \mapsto a(x,D) \in \mathcal{L}(L^2)$ is conti.

Note that $m < 0 \Rightarrow S^m \subset S^0$.

Cor.: If $a \in S^m$, then $a(x,D)$ is conti. from the Sobolev space H^{m+s} to H^s for all s.

Note that, for $m < 0$, $a(x,D)$ improves the regularity.

Proof of Cor.: $\langle D \rangle^s$ isom. from H^s to L^2. $\{H^s = \{u;\ \langle\xi\rangle^s \hat{u} \in L^2\}\}$

By comp. $\langle D \rangle^s a(x,D) \langle D \rangle^{-s-m} = b(x,D)$ with $b \in S^0$

Thus, by Th., $\langle D \rangle^s a(x,D) \langle D \rangle^{-s-m} \in \mathcal{L}(L^2)$.

This means that $a(x,D) \in \mathcal{L}(H^{s+m}, H^s)$. [parer par 3 pour jus]

Proof of the theorem. (later?)

Lemma (Schur): Let $K \in C^{0}(\mathbb{R}^{2d})$ s.t.
$\exists C>0$; $\sup_x \int |K(x,y)| dy \le C$ and $\sup_y \int |K(x,y)| dx$.
Then $L^2 \ni u \xrightarrow{K} \int K(x,y)u(y)dy \in L^2$ is well-defined, conti.
with norm $\le C$.

Proof (lemma): Let $\mathcal{S}$. $|Ku(x)|^2 = \left| \int K(x,y)u(y)dy \right|^2$

$$\le \left(\int |K(x,y)|\,|u(y)|\,dy \right)^2 \underset{C.S.}{\le} \int (|K(x,y)|^{1/2}|u(y)|)^2 dy \cdot \int (|K(x,y)|^{1/2})^2$$

$$\le \int |K(x,y)|\,|u(y)|^2 dy \cdot \underbrace{\int |K(x,y)|\,dy}_{\le C}.$$

Thus $\int |Ku(x)|^2 dx \le C \int \left(\sup_y \int |K(x,y)| dx \right) |u(y)|^2 dy$

$$\le C^2 \|u\|^2. \quad \square$$

Proof (Th.): Let $a \in S^{-d-1}$. Then $K(x,y) = (2\pi)^{-d} \left(\mathcal{F}_{\xi \to x'} a(x,\cdot) \right)_{|x'=y}$
is continuous and bounded. For $u \in \mathcal{S}$,

$[x_j,$ $x_j K u - K x_j u = \int (x_j - y_j) K(x,y) u(y) dy.$

Thus $(x-y)^\alpha K(x,y)$ is the kernel of the op.

$$[x_1^{\alpha_1}, \cdots, [x_d^{\alpha_d}, a(x,D)] \cdots] \underset{\substack{\uparrow \\ \text{direct.} \\ \text{computation}}}{=} i^{\alpha} (\partial_\xi^\alpha a)(x,D).$$

Since $\partial_\xi^\alpha a \in S^{-m-1}$, $(x-y)^\alpha K$ is also conti. and bounded.

Now take $a \in S^{-\frac{m+1}{2}}$. For $u \in \mathcal{S}$,
$$\|a(x,D)u\|^2 = \langle a(x,D)^* a(x,D)u, u \rangle = \langle b(x,D)u, u \rangle$$
with $b \in S^{-m-1}$. Thus
$$\|a(x,D)u\|^2 \le \underset{\text{indep. of } u}{C} \|u\|^2.$$

So, by induction, we get the conti. on L^2 for all $a \in S^h$ with $h \le -1$.

Let now $a \in S^0$. Choose $M>0$ s.t. $M \geq 2 \sup |a(x,\xi)|$.

Then $c = \left(M - |a(x,\xi)|^2\right)^{1/2} \in S^0$. By composition,

$$C(x,D)^* C(x,D) = (c^2)(x,D) + r_1(x,D) \quad \text{with } r_1 \in S^{-1}$$
$$= M - a(x,D)^* a(x,D) + r_2(x,D) \quad - r_2 \in S^{-1}.$$

Hence, for $u \in \mathcal{S}$,

$$\|a(x,D)u\|^2 = M\|u\|^2 - \|c(x,D)u\|^2 + \|r_2(x,D)u\|^2$$
$$\leq M\|u\|^2 + \|r_2(x,D)u\|^2 \leq M\|u\|^2 + C\|u\|^2$$ since $r_2 \in S^{-1}$. $\square$

Elliptic regularity:

Def: An ope. $a(x,D)$ with $a \in S^m$ is elliptic if so is a.

Parametrix:

Th: Let $a \in S^m$ elliptic. Then, there exists $b \in S^{-m}$ s.t. $b(x,D)\, a(x,D) = I + R$

where R is an integral op. with C^∞ kernel.

Proof (idea): use a Neumann serie.

Note that R send all H^s_{loc} to $\bigcap_{r \in \mathbb{R}} H^r_{loc}$.

Th: Let $a \in S^m$ elliptic and $P = a(x,D)$. Then, for all $u \in \mathcal{E}'$,

$$\text{sing supp } u \subset \text{sing supp } Pu.$$

Proof: We have $u \in \mathcal{E}'$. On some K, u is of order k.

Thus $u \in H^{-k}_{loc}(\Omega)$ $(\Omega \subset K)$.

But $u = b(x,D)\, Pu - \underbrace{Ru}_{\in C^\infty(\Omega)}$;

Now, if $Pu \in C^\infty(\Omega)$, $b(x,D)Pu \in C^\infty$ and $u \in C^\infty$.

Microlocalization :

Def. : * Let $a_0 \in S^m$ and $A = a(x,D)$ with $a \in S^m$.
a_0 is a principal symb. of A if $a_0(x,D) - A = b(x,D)$ with $b \in S^{m-1}$;

* $A = a(x,D)$ with $a \in S^m$. Let a_0 a principal symb. of A.
A is not characteristic at $(x_0,\xi_0) \in \mathbb{R}^d \times \mathbb{R}^d_0$ if, for some $b_0 \in S^{-m}$, $a_0 b_0 - 1 \in S^{-\infty}$ $(\bigcap_m S^m)$ in a conic neighborhood of (x_0,ξ_0). We denote by char A the set of characteristic points.

* In that case, one can find $b \in S^{-m}$; $ab - 1 \in S^{-\infty}$ in a conic neigh. of (x_0,ξ_0)

(def. indep. of choice of $\underline{a_0}$).

* properly supported

see p. 23'

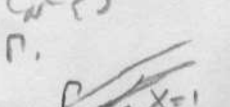
One can show that
$a_0 b_0 - 1 \in S^{-\infty}$ in a conic neighborhood Γ of (x_0,ξ_0)
$\iff \exists c, \exists R$; $|a(x,\xi)| \geq c|\xi|^m$ in Γ and $|\xi| \geq R$.

If $\Gamma = \mathbb{R}^d$, $a_0 b_0 - 1 \in S^{-\infty} \iff \exists b$; $b(x,D)a(x,D) - I = r(x,D)$ with $r \in S^{-\infty}$

Characterization of the wave front set.

Th. : Let $u \in \mathcal{D}'(\mathbb{R}^d)$.
$(x_0,\xi_0) \notin WF(u) \iff \exists\ A = a(x,D)$ with $a \in S^0$, A properly supported, and $(x_0,\xi_0) \notin \operatorname{char} A$ s.t. $Au \in C^\infty$.

$\Rightarrow$) $\exists \phi \in C_0^\infty$; $\phi(x_0) \neq 0$. $\exists \Gamma$; $|\widehat{\phi u}(\xi)| \leq C_N \langle\xi\rangle^{-N}$ in Γ.



Let $\chi \in S^0$, Let $\psi \in C_0^\infty$; $\psi\phi = \phi$. Let $a(x,D) = \psi(x)\chi(D)\phi(x)$. ($a \in S^0$, a has a properly supp. kernel), $a(x,D)u = \psi(x)\underline{\chi(D)(\phi u)} \in C^\infty$.
$\hookrightarrow \mathcal{F}^{-1}(\underline{\chi(\xi)\widehat{\phi u}(\xi)}) \in C^\infty$ rap.