Example: $(2\pi)^d \delta(x) = \int_{\mathbb{R}^d} e^{i\langle x,\theta\rangle} d\theta.$

We have to check that

$\forall u \in C_0^\infty$, $\quad (2\pi)^d u(0) = \int e^{i\langle x,\theta\rangle} u(x)\,dx\,d\theta.$ (oscillatory integral)

By the Th.,

$$\int e^{i\langle x,\theta\rangle} u(x)\,dx\,d\theta = \lim_{\varepsilon\to 0} \int e^{i\langle x,\theta\rangle} \chi(\varepsilon\theta) u(x)\,dx\,d\theta.$$

But $I_\varepsilon = \int e^{i\langle x,\theta\rangle} \chi(\varepsilon\theta) u(x)\,dx\,d\theta \underset{\text{ch. var.}}{=} \varepsilon^{-d} \int e^{i\langle x,\frac{\theta'}{\varepsilon}\rangle} \chi(\theta') u(x)\,dx\,d\theta'$

$$= \varepsilon^{-d} \int \hat{\chi}\left(-\tfrac{1}{\varepsilon}x\right) u(x)\,dx \underset{\text{ch. var.}}{=} \int \hat{\chi}(-y) u(\varepsilon y)\,dy$$

$\varepsilon \to 0 \downarrow$ dominated cv.

$$u(0) \int \hat{\chi}(-y)\,dy$$

Thus $= \int e^{i\langle x,\theta\rangle} u(x)\,dx\,d\theta = u(0)\,(2\pi)^d \chi(0) = (2\pi)^d u(0).$

Properties of oscill. integ.: one can

* integrate by parts
* differentiate under the int. sign.
* interchange order of integration.

Wave front set of an oscill. integ.:

Th. Let $A = \int e^{i\phi(\cdot,\theta)} a(\cdot,\theta)\,d\theta.$

Let $S = \{x \in \mathbb{R}^d,\ \exists\,\theta$ satisfying $(x,\theta) \in F$ and $\nabla_\theta \phi(x,\theta) = 0\}$.

A restricted to $\mathbb{R}^d \setminus S$ is equal to T_f with

$$f : x \longmapsto \int e^{i\phi(x,\theta)} a(x,\theta)\,d\theta \quad C^\infty.$$

In particular, $\operatorname{sing\,supp} A \subset S$.

"Proof": For $x \notin S$ fixed, $\int e^{i\phi(x,\theta)} a(x,\theta)\, d\theta$

is a oscillary inte. Since $d_\theta \phi \neq 0$.

So f is well-defined (and continuous).

Using $f(x) = \lim_{\varepsilon \to 0} \int e^{i\phi(x,\theta)} \chi(\varepsilon\theta)\, a(x,\theta)\, d\theta$

one can show that $f \in C^\infty$ (derivation under the inte. sign).

For $u \in C_0^\infty$ with $\operatorname{supp} u \subset \mathbb{R}^d \setminus S$,

$$A(u) = \lim_{\varepsilon \to 0} \int e^{i\phi(x,\theta)} \chi(\varepsilon\theta)\, a(x,\theta)\, u(x)\, dx\, d\theta$$

$$= \int \left(\lim_{\varepsilon \to 0} \int e^{i\phi(x,\theta)} \chi(\varepsilon\theta)\, a(x,\theta)\, d\theta \right) u(x)\, dx$$

$$= \int f u\, dx. \quad \square$$

<u>Th.</u>: With the same notation,

$$WF(A) \subset \left\{ (x, \nabla_x \phi(x,\theta)) ;\ \exists \theta \in \mathbb{R}^N \setminus 0 ;\ (x,\theta) \in F \text{ and } \nabla_\theta \phi(x,\theta) = 0 \right\}$$

<u>Idea of the proof</u>:

$\psi \in C_0^\infty$. $\quad \widehat{\psi A}(\xi) = \langle A, \psi e^{-i\langle \cdot, \xi \rangle} \rangle$

$$= \int e^{i[\phi(x,\theta) - \langle x,\xi \rangle]} \psi(x)\, a(x,\theta)\, dx\, d\theta$$

oscillatory int.

Let V a cone in $\mathbb{R}^d$ s.t.

$$V \cap \left\{ \nabla_x \phi(x,\theta) ;\ x \in \operatorname{supp} \psi \text{ and } \exists \theta ;\ \begin{array}{l} (x,\theta) \in F \\ \nabla_\theta \phi(x,\theta) = 0 \end{array} \right\}.$$

We need to show that $\widehat{\psi A}(\xi)$ "decays rapidly" in V. Consider the new phase function

$$\tilde{\Phi}(x,\theta) = \phi(x,\theta) - \langle x, \xi \rangle.$$

→

$\nabla_x \tilde{\phi} = \nabla_x \phi - \xi \ , \ \nabla_\theta \tilde{\phi} = \nabla_\theta \phi .$

It turns out that

$\exists c>0 ; \ \forall \xi \in V, \ \forall (x,\theta) \in F$ with $x \in \operatorname{supp}\psi$, $|\xi - \nabla_x \phi| + |\theta|\,|\nabla_\theta \phi| \geq c(|\theta| + |\xi|)$.

Now we can "win decay in $|\xi|$" "$|\theta|$" using

$$L = \frac{1}{|\xi - \nabla_x\phi|^2 + |\nabla_\theta \phi|^2}\left(\langle \xi - \nabla_x \phi , D_x\rangle + \langle \nabla_\theta \phi , D_\theta \rangle\right). \ \Box$$

Application: $(2\pi)^d \delta = \int e^{i\langle x,\theta\rangle}\, d\theta$, $\quad \phi(x,\theta) = \langle x,\theta\rangle$

$$WF(\delta) \subset \underbrace{\{ (x, \nabla_x \phi(x,\theta)) ; \ \exists \theta \in \mathbb{R}^d \setminus 0 ; \ \nabla_\theta \phi(x,\theta) = 0 \}}_{= \{(0;\theta) ; \ \theta \in \mathbb{R}^d \setminus 0\} = \{0\} \times (\mathbb{R}^d \setminus 0)}.$$

* Ex. 7.8.4. (Hö. 1).

Pseudodifferential operators

Op. of the form

$$(a(x,D)u)(x) = \int e^{i\langle x,\xi\rangle} a(x,\xi)\hat{u}(\xi)\, d\xi = \int e^{i\langle x-y,\xi\rangle} a(x,\xi)u(y)\, dy\, d\xi$$

ok if $a \in C_0^\infty$ and $u \in$

Motivations:

* Let $u \in \mathcal{S}$ et $a>0$. Let $v = (-\Delta + a)u \in \mathcal{S}$.

Then $\hat{v} = (|\xi|^2 + a)\hat{u}$ and $\hat{u} = \frac{1}{|\xi|^2+a}\hat{v}$.

Thus $v \mapsto u$ extend to a bounded op. on L^2.

This means that $(-\Delta + a)^{-1} \in \mathcal{L}(L^2)$. Note that

$$u = \int e^{i\langle x,\xi\rangle}\hat{v}(\xi)\frac{d\xi}{(2\pi)^d} = \underbrace{\int e^{i\langle x,\xi\rangle}\frac{1}{|\xi|^2+a}\hat{v}(\xi)\frac{d\xi}{(2\pi)^d}}_{\text{This is a pseudodiff. op. acting on } v} \text{"}=\text{"} \int e^{i\langle x-y,\xi\rangle}\frac{1}{|\xi|^2+a} v(y)\, dy\, d\xi$$

* If $P = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha$, we saw that, for $u \in \mathcal{S}$,

$$Pu(x) = \int e^{i\langle x,\xi\rangle} \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \, \hat{u}(\xi) \frac{d\xi}{(2\pi)^d}.$$

We have the same formula

(it is also a pseudodiff. op.)

So we plan to extend the class of differential operators in a way s.t. their resolvents are in the extension.

* This extension will be useful to understand the elliptic regularity.

Weyl calculus:

For $u \in \mathcal{S}(\mathbb{R}^d)$ and $a \in \mathcal{S}(\mathbb{R}^{2d})$, define

$$(a(x,D)u)(x) \stackrel{\text{def}}{=} (2\pi)^{-d} \int e^{i\langle x-y,\xi\rangle} a(x,\xi) u(y)\, dy\, d\xi = (2\pi)^{-d}\int e^{i\langle x,\xi\rangle} a(x,\xi)\hat{u}(\xi)$$

(abs. cv.).

$$= \int \underbrace{\left((2\pi)^{-d} \int e^{i\langle x-y,\xi\rangle} a(x,\xi)\, d\xi \right)}_{\mathcal{S}(\mathbb{R}^{2d}) \ni K(x,y) \text{ Kernel of } a(x,D).} u(y)\, dy$$

We have

$$(*)\begin{cases} K(x,y) = (2\pi)^{-d} \left(\mathcal{F}^{-1}_{\xi\to x'} a(x,\cdot) \right)_{|x'=y-x} \text{ and} \\ a(x,\xi) = \mathcal{F}_{y\to\xi}\left(K(x; x-y) \right) \quad (\text{Fourier inv. formula}) \end{cases}$$

Note that if $v \in \mathcal{S}$, $\langle a(x,D)u, v\rangle = \int (a(x,D)u)\, \bar{v}\, dx$

$$= \int (2\pi)^{-d} e^{i\langle x,\xi\rangle} \hat{u}(\xi)\bar{v}(x)\, a(x,\xi)\, dx\, d\xi.$$

By duality, we extend the calculus to $a, K \in \mathcal{S}'(\mathbb{R}^{2d})$:

Let $a \in \mathcal{S}'(\mathbb{R}^{2d})$ and $u, v \in \mathcal{S}(\mathbb{R}^d)$. We set

→

$$\langle a(x,D)u, v\rangle_{\mathcal{S}'(\mathbb{R}^d),\mathcal{S}(\mathbb{R}^d)} := \langle a, (2\pi)^{-d} e^{i\langle x,\xi\rangle} \hat{u}(\xi)\bar{v}(x)\rangle_{\mathcal{S}'(\mathbb{R}^{2d}),\mathcal{S}(\mathbb{R}^{2d})}.$$

It turns out $a(x,D)$ is continuous from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$.

Note that the formulae $(*)$

$$K(x,y) = (2\pi)^{-d}\left(\mathcal{F}_{\xi\to x'} a(x,\cdot)\right)\Big|_{x'=y-x}$$

and $$a(x,\xi) = \mathcal{F}_{y\to\xi}\left(K(x,y-x)\right)$$

extend to $\mathcal{S}'(\mathbb{R}^{2d})$.

Now we consider regular symbols to have better properties.

<u>Composition</u>: Let $a, b \in \mathcal{S}(\mathbb{R}^{2d})$. For $u\in\mathcal{S}$,

$$a(x,D)\, b(x,D)u = (2\pi)^{-d}\int e^{i\langle x-y,\eta\rangle} a(x,\eta)\,(b(x,D)u)(y)\, d\eta\, dy$$

$$= (2\pi)^{-2d}\int e^{i\langle x-y,\eta\rangle} a(x,\eta)\, e^{i\langle y,\xi\rangle} b(y,\xi)\,\hat{u}(\xi)\, d\xi\, d\eta\, dy$$

$$= (2\pi)^{-d}\int e^{i\langle x,\xi\rangle}\underbrace{\left((2\pi)^{-d}\int e^{i\langle x-y,\eta-\xi\rangle} a(x,\eta) b(y,\eta)\, d\eta\, dy\right)}_{c(x,\xi)}\hat{u}(\xi)\, d\xi$$

$$= c(x,D)u.$$

We have

$$c(x,\xi) = \left(a(x,\eta)b(y,\xi) *_{(y,\eta)} (2\pi)^{-d} e^{-i\langle y,\eta\rangle}\right)(x,\xi).$$

Let $\mathcal{F} = \mathcal{F}_{(y,\eta)\to(\hat{y},\hat{\eta})}$. Then $\mathcal{F}\left((2\pi)^{-d} e^{-i\langle y,\eta\rangle}\right)(\hat{y},\hat{\eta}) = e^{i\langle\hat{y},\hat{\eta}\rangle}$

(cf. Fourier transf. of exp. functions). Thus

$$c(x,\xi) = \left[\mathcal{F}^{-1}\left(e^{i\langle\hat{y},\hat{\eta}\rangle}\mathcal{F}(a(x,\cdot)b(\cdot,\xi))\right)\right]_{(y,\eta)=(x,\xi)}$$

$$=: \left[e^{i\langle D_y, D_\eta\rangle} a(x,\eta)\, b(y,\xi)\right]_{(y,\eta)=(x,\xi)}.$$