Let $\xi \neq 0$. There exists j; $\xi \in \Gamma'_{\xi_j}$. Then

$$\widehat{\Phi u}(\xi) = \widehat{\left(\prod_{k \neq j} \phi_{\xi_k}\right) \phi_{\xi_j} u}(\xi) = \widehat{\left(\prod_{k \neq j} \phi_{\xi_k}\right)} * \widehat{\phi_{\xi_j} u}(\xi).$$

Now, the lemma below gives that

$$\forall N, \quad |\widehat{\Phi u}(\xi)| \leq \underset{\text{indep. of } j, \xi}{D_N} \langle \xi \rangle^{-N}.$$

Thus $\Phi u \in C_0^\infty$ and $x_0 \notin \operatorname{sing\,supp} u$. $\square$

<u>Lemma</u>: Let $k \in \mathbb{N}$. Let Γ, Γ' open cones in $\mathbb{R}^d \setminus 0$ s.t. $\Gamma' \subset \overline{\Gamma'} \subset \Gamma$. Let $\psi, \varphi : \mathbb{R}^d \to \mathbb{C}$ s.t.

$$\forall N, \exists C_N; \quad \begin{aligned} &\forall \xi \in \mathbb{R}^d \setminus 0, \quad |\psi(\xi)| \leq C_N \langle \xi \rangle^{-N} \\ &\forall \xi \in \Gamma, \quad |\varphi(\xi)| \leq C_N \langle \xi \rangle^{-N}. \\ &\forall \xi \in \mathbb{R}^d \setminus 0, \quad |\varphi(\xi)| \leq C \langle \xi \rangle^{k}. \end{aligned}$$

Then

$$\forall N, \exists C'_N; \forall \xi \in \Gamma', \quad |\psi * \varphi(\xi)| \leq C'_N \langle \xi \rangle^{-N},$$

(C'_N depends on k, Γ, Γ', C and the C_p)

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Proof of the lemma: $\xi \in \Gamma'$,

$$\langle \xi \rangle^N (\psi * \varphi)(\xi) = \int_{|\xi - \zeta| \geq c_{\Gamma,\Gamma'} |\xi|} \langle \xi \rangle^N \psi(\xi - \zeta) \varphi(\zeta) d\zeta + \int_{|\xi - \zeta| \leq c_{\Gamma,\Gamma'} |\xi|} \langle \xi \rangle^N \psi(\xi - \zeta) \varphi(\zeta) d\zeta$$

$\langle \zeta \rangle^{k+d+1} \langle \zeta \rangle^{-k} \langle \zeta \rangle^{-d-1}$

$\langle \zeta \rangle^{d+1} \langle \zeta \rangle^{-d-1}$

here $|\xi| \leq c'_{\Gamma,\Gamma'} |\xi - \zeta|$

$\langle \xi - \zeta \rangle^{N+k+d+1} |\psi(\xi - \zeta)| \leq C_{N+k+d+1}$

here $|\xi| \leq c'_{\Gamma,\Gamma'} |\zeta|$

$\zeta \in \Gamma$

$\langle \zeta \rangle^{N+d+1} |\varphi(\zeta)| \leq C_{N+d+1}$

Main result:

Th.: Let $P = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha$. Then, for all $u \in \mathcal{D}'$,

$$WF(u) \subset \mathrm{Char}(P) \cup WF(Pu).$$

Here $\mathrm{Char}\, P = \left\{ (x, \xi_{\neq 0}) ;\ \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha = 0 \right\}$.

characteristic set of P

Rk.: * P is elliptic if its characteristic set is empty. ~~details~~ If P is elliptic and f smooth then any sol. u of $Pu = f$ is also smooth.

* In general, one says that P is elliptic away from $\mathrm{Char}(P)$. The result is called elliptic regularity.

* One can prove this theorem without using the pseudo-differential calculus but, to understand the idea behind it, it is useful to apply this calculus.

Examples: * $\Delta u = f$. $\Delta = \sum_{|\alpha| = 2} D^\alpha$. So Δ is elliptic

$\sum_{|\alpha|=2} \xi^\alpha = |\xi|^2 > 0$ if $\xi \neq 0$.

and $WF(u) \subset WF(\Delta u)$.

* $P = x \cdot \nabla$. $P = \sum_{|\alpha|=1} i x^\alpha D^\alpha$.

$\mathrm{Char}\, P = \{ (x ; \xi_{\neq 0}) ;\ \langle x, \xi \rangle = 0 \}$.

$P u_1 = 0$ $\quad \phi \in C_0^\infty$

$$2 x \cdot \nabla u_1(\phi) = -2 u_1(\nabla \cdot x \phi) = -\int_{x_1 > 0} dx\, (\nabla \cdot x \phi) + \int_{x_1 < 0} dx\, (\nabla \cdot x \phi)$$

$$= -\int dx' \int_{x_1 > 0} \partial_{x_1}(x_1 \phi)\, dx_1 - \sum_{j > 1} \int_{x_1 > 0} d\hat{x}_j \int \cancel{\partial_{x_j}(x_j \phi)}\, dx_j + \int dx' \int_{x_1 < 0} \partial_{x_1}(x_1 \phi)\, dx_1 + 0$$

$$2\pi \cdot Du_1(\phi) = -\int dx' \left[x_1\phi\right]_0^{+\infty} + \int dx' \left[x_1\phi\right]_{-\infty}^{0} = 0$$

Result gives: $\quad WF(u_1) \subset \mathrm{Char}\,P \cup \underbrace{WF(Pu_1)}_{\phi}$

$$WF(u_1) \subset \{(x;\xi_{\neq 0}),\ \langle x,\xi\rangle = 0\}.$$

This is quite imprecise. Indeed, since $\partial_{x_1} u_1 = \delta_M$ with $M = \{x_1 = 0\}$, the same result applied to ∂_{x_1} gives

details $\Big($ $WF(u_1) \subset \{(x;\xi_{\neq 0});\ \underline{x_1 = 0} \text{ and } (\xi_1 = 0 \text{ or } \xi' = 0)\}$.

One can check that $WF(u_1) = \{(x;\xi_{\neq 0});\ x_1 = 0 \text{ and } \xi' = 0\}$.

Oscillatory integrals:

Motivations:

* We saw
$$(2\pi)^d \delta_{(\cdot-\theta)}(\phi) = \int e^{i\langle\theta,\xi\rangle}\hat{\phi}(\xi)\,d\xi = \int e^{i\langle\theta,\xi\rangle}\left(\int e^{-i\langle x,\xi\rangle}\phi(x)\,dx\right)d\xi \overset{?}{=} \int \underbrace{\left(\int e^{i\langle\theta - x,\xi\rangle}\,d\xi\right)}_{??}\phi(x)\,dx$$

* $u \in C_0^\infty$
$$P(x,D)u(x) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha u(x) = \sum_{|\alpha|\le m} a_\alpha(x)\int e^{i\langle x,\xi\rangle}\xi^\alpha \hat{u}(\xi)\frac{d\xi}{(2\pi)^d}$$
$$= \int e^{i\langle x,\xi\rangle} P(x,\xi)\hat{u}(\xi)\frac{d\xi}{(2\pi)^d} = \int e^{i\langle x,\xi\rangle} P(x,\xi)\int e^{-i\langle y,\xi\rangle}u(y)\,dy\,\frac{d\xi}{(2\pi)^d}$$
$$\overset{?}{=} \int \underbrace{\left(\int e^{i\langle x-y,\xi\rangle} P(x,\xi)\frac{d\xi}{(2\pi)^d}\right)}_{??} u(y)\,dy.$$

Classical oscillatory integrals

* $\int_1^{+\infty} e^{it^2}\,dt$ is semi-convergent.

$$\int_1^T e^{it^2}dt = \int_1^T \frac{1}{2ti}\, 2ti\, e^{it^2} dt = \left[\frac{e^{it^2}}{2it}\right]_1^T - \underbrace{\int_1^T \frac{e^{it^2}}{2i}\frac{-1}{t^2}dt}_{\text{abs. cv.}}$$

cv. as $T\to\infty$

* $\int_1^{+\infty} \frac{e^{it}}{t}dt$ is semi-convergent.

$$\int_1^T \frac{e^{it}}{t} = \left(\frac{e^{it}}{it}\right)_1^T - \int_1^T \frac{e^{it}}{i}\frac{-1}{t^2}dt \qquad \text{cv. as } T\to\infty.$$

Generalization:

We want to give a meaning as distribution to integrals of the form

$$\int_{\mathbb{R}^N} e^{i\phi(x,\theta)} a(x,\theta)\,d\theta\,,\quad x\in\mathbb{R}^d$$

where ϕ is real.

Def.: Let Γ a open cone in $\mathbb{R}^n\times(\mathbb{R}^N\setminus 0)$ ($(x,\theta)\in\Gamma \Rightarrow (x,t\theta)\in\Gamma, \forall t>0$).

A phase function ϕ in Γ is a C^∞ function on Γ s.t. $\forall (x,\theta)\in\Gamma, \forall t>0$, $\phi(x,t\theta)=t\,\phi(x,\theta)$

$\forall (x,\theta)\in\Gamma$, $d\phi_{(x,\theta)}\neq 0$.

Ex.: $d=N$ and $\phi(x,\theta)=\langle x,\theta\rangle$.

Def.: A symbol a is a funct. in $C^\infty(\mathbb{R}^n\times\mathbb{R}^N)$ s.t. $\exists m\in\mathbb{R}$; $\forall \alpha,\beta\in\mathbb{N}^{d+N}$, $\forall K\subset\subset\mathbb{R}^n$, $\exists C_{\alpha\beta K}>0$, $|\partial_x^\alpha\partial_\theta^\beta a(x,\theta)|\le C_{\alpha\beta K}\langle\theta\rangle^{m-|\beta|}$, $x\in K, \theta\in\mathbb{R}^N$

$\langle\theta\rangle = (1+|\theta|^2)^{1/2}$, we note $a \in S^m$.

Th. : Let F a closed cone $\subset \Gamma \cup (\mathbb{R}^d \times \{0\})$.

Let $a \in S^m$ with $m < -N$. Then

$$C_0^\infty \ni u \xrightarrow{I_{\phi,a}} I_\phi(au) := \int e^{i\phi(x,\theta)} a(x,\theta)\, u(x)\, dx\, d\theta$$) oscillatory integr.

is well defined (abs. cv. int.) as a distribution.

It can be extended uniquely to $a \in \bigcup_m S^m$ with $\operatorname{supp} a \subset F$. We have:

→ $I_{\phi,a} \in \mathcal{D}'^{l}$ with $l \leq k$ if $a \in S^m$ with $m - k < -N$.

→ For fixed $u \in C_0^\infty$, $a \mapsto I_\phi(au)$ is conti. on S^m (equipped with semi-norms).

→ $\chi \in C_0^\infty$, $\chi(0) = 1$.

For $a \in S^m$ and $u \in C_0^\infty$,

$$I_{\phi,a} \underset{\text{in } \mathcal{D}'}{=} \lim_{\varepsilon \to 0} \int e^{i\phi(x,\theta)} \chi(\varepsilon\theta)\, a(x,\theta)\, d\theta$$

✓

$$\left(I_\phi(au) = \lim_{\varepsilon \to 0} \int e^{i\phi(x,\theta)} \chi(\varepsilon\theta)\, a(x,\theta)\, u(x)\, dx\, d\theta \right)$$

This gives a meaning to $\int e^{i\phi(x,\theta)} a(x,\theta)\, d\theta$ as a distribution (oscillatory integral)

Idea of the proof. in the case: $d = N$ and $\phi(x,\theta) = \langle x, \theta\rangle$.

$$I_\varepsilon^a = \int e^{i\langle x,\theta\rangle} \chi(\varepsilon\theta)\, a(x,\theta)\, u(x)\, dx\, d\theta = \int L\left(e^{i\langle x,\theta\rangle}\right) \chi(\varepsilon\theta)\, a(x,\theta)\, u(x)\, dx\, d\theta$$

where $L = \langle\theta\rangle^{-2}\left(1 + \langle\theta, D_x\rangle\right)$, $\left({}^tL = \langle\theta\rangle^{-2}(1 - \langle\theta, D_x\rangle)\right)$

By inte. by parts,

$$I_\varepsilon = \int e^{i\langle x,\theta\rangle} \chi(\varepsilon\theta)\, {}^tL\left(a(x,\theta)\, u(x)\right) dx\, d\theta.$$

If $|a(x,\theta)| \leq C\langle\theta\rangle^{-N}$, $\left|{}^tL\left(a(x,\theta)u(x)\right)\right| \leq C'\langle\theta\rangle^{-N-1}$ integrable.

in general, $D_x\phi$ can vanish for non zero θ → we need to derivate in θ

→