Fourier transf. :

for $f \in L^1$, $\hat{f}(\xi) = \int e^{-i\langle x,\xi\rangle} f(x)\,dx \in L^\infty$

A set $\mathcal{S}$ between C_0^∞ and C^∞ is well adapted to Fourier transf. since $\hat{\mathcal{S}} = \mathcal{S}$ et ^ is continuous. We have

$C_0^\infty \subset \mathcal{S} = \{ f \in C^\infty ;\ \forall k \in \mathbb{N}, \forall \alpha \in \mathbb{N}^d, \sup_x |x^k D_x^\alpha f| < \infty \} \subset C^\infty$.

equiped with the semi-norms ↑, (Fréchet space).

Let $\mathcal{S}'$ the dual. $\mathcal{S}'$ can be viewed as a subset of $\mathcal{D}'$ (if $T \in \mathcal{S}'$ and $\phi \in C_0^\infty$, $T(\phi)$ makes sense since $C_0^\infty \subset \mathcal{S}$).

If $T \in \mathcal{E}'$, T can be viewed as an elem. of $\mathcal{S}'$ si $\mathcal{S} \subset C^\infty$. So $\mathcal{E}' \subset \mathcal{S}' \subset \mathcal{D}'$.

For $T \in \mathcal{S}'$, we define $\hat{T} \in \mathcal{S}'$ by :

$\forall \phi \in \mathcal{S}$, $\hat{T}(\phi) = T(\hat{\phi})$.

* If $T \in \mathcal{E}'$, $\hat{T} = T_f$ where $f(\xi) = T(e^{-i\langle \cdot,\xi\rangle})$ ext. to $\mathbb{C}^d$ analytically

Example : $\theta \in \mathbb{R}^d$ $\widehat{e^{i\langle \cdot,\theta\rangle}} = (2\pi)^d \delta_{(\cdot-\theta)}$

$x \overset{f}{\mapsto} e^{i\langle x,\theta\rangle}$ is in C_b^∞ so $T_f \in \mathcal{S}'$.

For $\phi \in \mathcal{S}$, $\phi(x) = (2\pi)^{-d} \int e^{i\langle \xi,x\rangle} \hat{\phi}(\xi)\,d\xi$

Fourier inversion formula.

$\phi \in \mathcal{S}$, $(2\pi)^d \delta_{(\cdot-\theta)}(\phi) = (2\pi)^d \phi(\theta) = \int e^{i\langle\theta,\xi\rangle} \hat{\phi}(\xi)\,d\xi$

$= \widehat{e^{i\langle\theta,\cdot\rangle}}(\phi)$.

Convolution :

Let $u \in C_0^0(\mathbb{R}^d)$ and $v \in C^0$, Define

$\forall x$, $\underset{\text{Convolution product}}{\underline{u * v}}(x) = \int u(x-y)\, v(y)\,dy = \int u(y)\, v(x-y)\,dy$.

(compact supp.) $= v * u\,(x)$

It can be extended to $u \in \mathcal{E}'$ and $v \in \mathcal{D}'$.

It turns out that, for $v \in \mathcal{D}'$, $\delta * v = v * \delta = v$.

Application : Let $P(D)$ a differential op. with $\underline{\text{constant}}$ coeff.

One can find $E \in \mathcal{D}'$ s.t. $P(D)E = \delta$.

Let $f \in \mathcal{D}'$. Then $u = E * f$ satisfies $P(D)u = f$

$\Big(P(D)(E * f) = (P(D)E) * f = E * f = f \Big)$.

Rk $\quad$ * $\{$sol. of $P(D)u = f\} = E * f + \{$sol. of $P(D)u = 0\}$.

* works only for ct. coeff.

Example : $\Delta u = f$. $\quad E = -\dfrac{|x|^{2-d}}{(d-2)C_d}$ for $d \geq 3$. $\quad C_d = \text{Area}(S^{d-1})$

$\Delta E = \delta \qquad\qquad = (2\pi)^{-1} \log|x|$ for $d = 2$.

Since sing supp $E = \{0\}$, one can show that

sing supp $(Eu) =$ sing supp (Pu)

II Wave front set

To study the regularity of sol. u of $Pu = f$,

it turns out that it is useful to "see the singularities of u in the phase space $\mathbb{R}^d_x \times \mathbb{R}^d_\xi \ (= T^*\mathbb{R}^d)$.

This leads to the notion of wave front set.

Before defining it, let us give the

Payley-Wiener-Schwartz theorem.

Let K be a compact set in $\mathbb{R}^d$.

Let $H: \mathbb{R}^d \longrightarrow \mathbb{R}$

$$\xi \longmapsto \sup_{x\in K} \langle x,\xi\rangle = \sup_{x\in Ch(K)} \langle x,\xi\rangle .$$

(convex hull)

H is the supporting funct. of K.

And $Ch(K)=\{x;\ \forall \xi,\ \langle x,\xi\rangle \le H(\xi)\}$.

Th.: Let K be convex compact with supp. funct. H.

If $u\in\mathcal{D}'^N$ with supp $u\subset K$ then

$\exists C>0;\ \forall \xi\in\mathbb{C}^d,\ |\hat{u}(\xi)| \le C(1+|\xi|)^N e^{H(\Im\xi)}$.

Conversely a function satisfying this bound is the Fourier (-Laplace) transf. of some $u\in\mathcal{D}'$ with supp $u\subset K$.

If $u\in C_0^\infty$ with supp $u\subset K$ then

$\forall N,\ \exists C_N>0;\ \forall\xi\in\mathbb{C}^d,\ |\hat{u}(\xi)| \le C_N(1+|\xi|)^{-N} e^{H(\Im\xi)}$.

Conversely a function is $\hat{u}$, $u\in C_0^\infty$ with supp $u\subset K$.

From this theorem, we learn that the regularity of u can be seen in the asympt. $|\xi|\to\infty$ of $\hat{u}$.

To see if $u\in\mathcal{D}'$ is smooth on $\Omega\subset\bar{\Omega}\subset\subset\mathbb{R}^d$, it suffices to show that $\underbrace{\psi}_{\substack{1\\ \Omega}} u$ is smooth. This can be checked on the behaviour of $\widehat{\psi u}$. ✓

Def: Let $u \in \mathcal{D}'$. The wave front set $WF(u)$ of u is the subset of $\mathbb{R}^d_x \times (\mathbb{R}^d_\xi \setminus \{0\})$ defined by the following rule:

2 def. $\Leftrightarrow$!!

$(x_0;\xi_0) \notin WF(u)$ if $\exists\, \phi \in C_0^\infty$ ($\exists$ neigh. of x_0: V, supp. in V) (correct) with $\phi(x_0) \neq 0$ and $\underset{\neq 0}{\xi_0}$, $\exists\, \Gamma$ a conic neighborhood of ξ_0, s.t.

$$\forall N, \exists C_N > 0, \forall \xi \in \Gamma, \ |\widehat{\phi u}(\xi)| \leq C_N (1+|\xi|)^{-N}.$$

$WF(u)$ is closed

Th: $\Pi_x WF(u) = \text{sing supp } u$ **Proof**

$\left(\Pi_x : \mathbb{R}^d_x \times \mathbb{R}^d_\xi \to \mathbb{R}^d_x,\ (x,\xi) \mapsto x\right)$.

Examples: $WF(\delta) = \{(x;\xi_{\neq 0});\ x = 0\}$.

$M = \{x_1 = 0\}$, $WF(\delta_M) = \{(x;\underset{\neq 0}{\xi});\ x_1 = 0 \text{ and } \xi' = 0\}$ (i.e. $x \in M$ and $\xi \in M^\perp$)

Since sing supp $\delta_M \subset \{x_1 = 0\}$, $WF(\delta_M) \subset \{0\} \times (\mathbb{R}^d_\xi \setminus \{0\})$.

Proof: Let $x^0 \in M$ and $\phi \in C_0^\infty$, $\phi(x_0) \neq 0$.

$$\widehat{\phi\delta_M}(\xi) = \phi\delta_M(e^{-i\langle \cdot, \xi\rangle}) = \int_M \phi(x) e^{-i\langle x,\xi\rangle} d\sigma_M(x)$$

$$= \int_{\mathbb{R}^{d-1}} \phi(x_1, y) e^{-i\langle y, \xi'\rangle} dy$$

$\widehat{\phi\delta_M}(\xi_1, 0)$ does not "decay rapidly" for all ϕ.

details

If $|\xi'| \geq c|\xi_1|$, $\quad |\xi|^k |\widehat{\phi\delta_M}(\xi)| \leq C\, \big| |\xi'|^k \widehat{\phi\delta_M}(\xi)\big| \leq C_k$

iterate $\Big[\ j \neq 1 \quad \xi_j \int_{\mathbb{R}^{d-1}} \phi(x_1,y) e^{-i\langle y,\xi'\rangle} dy = \int_{\mathbb{R}^{d-1}} \phi(x_1,y)\, D_{x_j} e^{-i\langle y,\xi'\rangle} dy$ int. by parts.

Orsay : Topologie Dynamique

Nantes $\to$ EDP et Physique mathématique

$\to$ Géométrie et Analyse globale.

Proof of $\Pi_x WF(u) = \mathrm{Sing\,supp}\, u$. | 9 bis

We prove : $x_0 \notin \mathrm{Singsupp}\, u \iff x_0 \notin \Pi_x WF(u)$.

$\Rightarrow$). u is C^∞ on a neighborhood of x_0. $\exists\, \phi \in C_0^\infty$ with small enough support s.t. $\phi(x_0) \neq 0$ and $\phi u \in C_0^\infty$.

By Paley-Wiener-Schwartz theorem,

$\forall N, \exists C_N, \forall \xi \in \mathbb{C}^d, \ |\widehat{\phi u}(\xi)| \leq C_N \langle \xi \rangle^{-N}$. By def.,

Thus, for all $\xi \neq 0$, $(x_0, \xi) \notin WF(u)$.

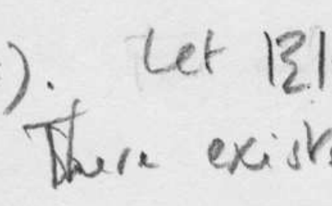
this gives $x_0 \notin \Pi_x WF(u)$.

$\Leftarrow$). Let $|\xi| = 1$. By assumption, $(x_0, \xi) \notin WF(u)$.

There exists Γ_ξ a conic open neighborhood of ξ (in $\mathbb{R}^d \setminus \{0\}$) and $\phi_\xi \in C_0^\infty$ with $\phi_\xi(x_0) \neq 0$

s.t. $\forall N, \exists C_{N,\xi}$; $\forall \zeta \in \Gamma_\xi$, $|\widehat{\phi_\xi u}(\zeta)| \leq C_{N,\xi} \langle \zeta \rangle^{-N}$.

For each $\xi \neq 0$, let Γ'_ξ a smaller conic neigh. of ξ $(\overline{\Gamma'_\xi} \subset \mathring{\Gamma}_\xi)$.



Since $\mathbb{S}^{d-1} = \bigcup_{\xi \in \mathbb{S}^{d-1}} (\mathbb{S}^{d-1} \cap \Gamma'_\xi)$ and since $\mathbb{S}^{d-1}$ is compact

$\mathbb{S}^{d-1} = \bigcup_{j=1}^{M} (\mathbb{S}^{d-1} \cap \Gamma'_{\xi_j})$. Take $\phi = \phi_{\xi_1} \cdots \phi_{\xi_M} \in C_0^\infty$.

On $\bigcup_j \mathrm{supp}\, \phi_{\xi_j}$ (compact) $\supset \mathrm{supp}\, \phi$, u has finite order k. $\longrightarrow$