Lecture in Munich

Aim: – explain the notion of elliptic regularity for pdes.

– introduce powerful tools to show this regularity: pseudodifferential calculus and wave front set.

Rk: – to win time, I will focus on basic examples (as guideline) and I will skip almost all proofs.

Ref: 2 books by Hörmander.
1 book by Lerner

Intro.: Given a differential op. P on $\mathbb{R}^d$, we want to know the regularity of a sol. u of $Pu = f$, knowing the regularity of f.

Examples: $\Delta u = f$, $\Delta_g u = f$, $(-\Delta + |x|^2)u = f$
$\partial_{x_1} u = f$, $x \cdot Du = f$, $x_1 u = f$.

Notation: $P = \sum_{\substack{|\alpha| \le m \\ m \in \mathbb{N}}} a_\alpha(x) D^\alpha$, $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, $|\alpha| = \sum \alpha_j$, $D^\alpha = \left(\frac{1}{i}\partial_{x_1}\right)^{\alpha_1} \cdots \left(\frac{1}{i}\partial_{x_d}\right)^{\alpha_d}$.
$\underline{a_\alpha \in C^\infty}$

Let us look at the case $\Delta u = f$ with $f(x) = e^{-|x|^2}$ and
We consider a C^2 funct. u s.t. $u, \partial_{x_j} u, \partial^2_{x_j} u \in L^2$ and

$$\Delta u = \sum_{|\alpha| = 2} D^\alpha u = \sum_{j=1}^d \left(-\partial^2_{x_j} u\right) = f.$$

By Fourier analysis (see later), $|\xi|^2 \hat{u}(\xi) = \widehat{\Delta u}(\xi) = \hat{f}(\xi) = e^{-c|\xi|^2}$, $\xi \in \mathbb{R}^d$.
So $\hat{u}(\xi)$ decays rapidly as $|\xi| \to \infty$.
But $u(x) = c \int e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi$.

By derivation under the int. sign, there exist

$$D^\alpha u(x) = \int \underbrace{e^{ix\cdot\xi}\, \xi^\alpha \hat{u}(\xi)\, d\xi}_{\text{abs. integrable}}$$

for all α. Thus $u \in C^\infty$.

We want to extend this treatment to more general situations
(u more general, pde with variables coeff.)

Rk.: * Δ is elliptic (see below)

* $u_1 : \mathbb{R}^2 \to \mathbb{R}$, $(x_1,x_2) \mapsto 1/2$ if $x_1>0$, $-1/2$ if $x_1<0$, 0 if $x_1=0$, is solution of $D_{x_2} u_1 = 0$.

We have a non smooth sol. of $D_y u = f$ with smooth f. We shall see that D_y is not elliptic.

* It is convenient to allow a much larger class of sol. u of $Pu = f$. We shall work with <u>distributions</u>.

Schedule of the Lecture:

I Review on distributions

II Wave front set (microlocal analysis)

III Oscillatory Integrals.

IV Pseudodifferential Calculus.

V Elliptic regularity.

VI Extension

I- Distributions.

Prop. and Def : * a distrib. u on $\mathbb{R}^d$ is a (complex) linear form on $C_0^\infty(\mathbb{R}^d)$ s.th.

$\forall K \subset\subset \mathbb{R}^d, \exists C>0, \exists k\in\mathbb{N}; \forall \phi \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\phi\subset K$

$$|T(\phi)| \le C \sup_{|\alpha|\le k} |\partial^\alpha \phi|.$$

$\mathcal{D}'(\mathbb{R}^d)$ is the vector space of distrib.

$\iff \forall K\subset\subset\mathbb{R}^d, \forall \phi_j \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\phi_j\subset K$ s.t. $\sup_K |\partial^\alpha \phi_j| \to 0$ for all α, $u(\phi_j)\to 0$.

* On $\mathcal{D}'(\mathbb{R}^d)$, we use the weak topology defined by the semi-norms: $T \mapsto |T(\phi)|$ (for $\phi\in C_0^\infty(\mathbb{R}^d)$)

* If k is independent with K above, $T\in \mathcal{D}'^k(\mathbb{R}^d)$.

$\mathcal{D}'^k$ can be identified to the dual of $C_0^k(\mathbb{R}^d)$ (for $k=0$, $\mathcal{D}'^0$ is a measure space)

(Rk: If $T_j\in\mathcal{D}'$ and for $\phi\in C_0^\infty$, $T_j(\phi)\to T(\phi)$. Then $T\in\mathcal{D}'$ and $T_j\to T$ in $\mathcal{D}'$.)

* If $\psi\in C^\infty$ and $T\in\mathcal{D}'$, then we define

$$\psi T: \phi \mapsto T(\psi\phi)$$

$$\partial_{x_k} T: \phi\mapsto -T(\partial_{x_k}\phi)$$

ψT and $\partial_{x_k} T \in \mathcal{D}'$.

* $\operatorname{supp} T$ = complement of the biggest open set U s.th. $\forall\phi\in C_0^\infty$ with $\operatorname{supp}\phi\subset U$, $T(\phi)=0$.
→ closed set

* If $f\in L^1_{loc}$, $T_f: \phi\mapsto \int f\phi$ belongs to $\mathcal{D}'$.

f identified to T_f.

Now we can view $Pu=f$ in the distribution sense.

Rk.: if $g\in C^1$, $\forall\phi\in C_0^\infty$, $T_{\partial_{x_i} g}(\phi) = \int \partial_{x_i} g\,\phi \underset{\text{int. by parts}}{=} -\int g\,\partial_{x_i}\phi = (\partial_{x_i} T_g)(\phi)$.

Example: M submanifold of $\mathbb{R}^d$.

Let $\delta_M : \phi \longmapsto \int_M \phi \, d\sigma_M$. Then $\delta_M \in \mathcal{D}'^0$.

If $M = \{x_0\}$, $\delta_{(\cdot - x_0)} = \delta_M : \phi \longmapsto \phi(x_0)$. $\delta_{(\cdot - x_0)} \in \mathcal{D}'^0$.

$$\partial^\alpha \delta_{(\cdot - x_0)} \in \mathcal{D}'^{|\alpha|} \text{ for all } \alpha.$$

Let $T \in \mathcal{D}'$ and $\operatorname{supp} T$ is compact, T can be identified to a element of the dual $\mathcal{E}'$ of $C^\infty(\mathbb{R}^d)$ (equiped with the semi-norms $\phi \mapsto \sup_K |\partial^\alpha \phi|$).

If $\phi \in C^\infty$, $T(\phi) := T(\chi \phi)$

where $\chi \in C_0^\infty$ s.th. $\chi = 1$ on $\operatorname{supp} T$.

We can view $\mathcal{E}'$ as subset of $\mathcal{D}'$.

~~(Def: if $f \in C^1$ then for all $\phi \in C_0^\infty$,~~

~~$u_{\partial_x f}(\phi) = \int \partial_x f \, \phi = -\int f \, \partial_x \phi = (\partial_x u_f)(\phi)$ integrate by part in x_1)~~

Rk: the product of distribution is not defined in general.

Singular support of a distribution: let Ω open & $T \in \mathcal{D}'$.

We say that T is C^∞ in Ω if there exists a C^∞ function f s.t. $T = T_f$ on Ω

(i.e. $\forall \phi \in C_0^\infty$ with $\operatorname{supp} \phi \subset \Omega$, $T(\phi) = T_f(\phi)$).

$\operatorname{singsupp} u :=$ complement the biggest open set where u is C^∞.

closed set

<u>Example</u> : sing supp $\delta_M \subset M$

(since $\delta_M = 0$ away from M).

Actually sing supp $\delta_M = M$.

<u>Proof in the case</u> $M = \{x_1 = 0\}$.

Assume δ_M is smooth on Ω neighborhood of $x^0 \in M$. Let $\varphi_n(x_1, y) = \chi(n(x_1 - x_1^0))\,\theta(y)$ with $\theta \in C_0^\infty(\mathbb{R}^{d-1})$, $\int \theta = 1$

$\chi \in C_0^\infty(\mathbb{R})$ with $\chi'(0) = 1$.

s.t. $\operatorname{supp} \varphi_n \subset \Omega$, $\forall n$.

Then $\forall n$, $\delta_M(\partial_{x_1}\varphi_n) = T_\psi(\partial_{x_1}\varphi_n)$ for some smooth function ψ (indep. of n).

$$= -\int (\partial_{x_1}\psi)\,\varphi_n \quad \text{cv. as } n \to \infty \text{ by dom. cv.}$$

But $$\delta_M(\partial_{x_1}\varphi_n) = \int_M n\,\chi'(n(x_1 - x_1^0))\,\theta(y)\,d\sigma_M(x_1; y)$$

$$= n \int \theta(y)\,dy = n \xrightarrow[n \to \infty]{} \infty.$$

Contr. So $x^0 \in$ sing supp δ_M and sing supp $\delta_M = M$.

skipped

<u>Rk</u>. : * as distribution $\partial_{x_1} u_1 = \delta_M$ (in $\mathbb{R}^2$)

with $M = \{x_1 = 0\}$.

* If $u \in \mathcal{D}'$ and $x_1 u = f \in C^\infty$

then sing supp $u \subset \{x_1 = 0\} = M$

(indeed on Ω open set s.t. $\Omega \cap M = \emptyset$ $u(\phi) = \int \frac{f(x)}{x_1}\phi$ and $\frac{f(x)}{x_1}$ is smooth on Ω).