LMU | LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN | MATHEMATISCHES INSTITUT

Spring term 2012 / *Sommersemester 2012*

Functional Analysis – Final Test, 20.10.2012

Funktionalanalysis – Endklausur, 20.10.2012

Name:/*Name*: ______________________________

Matriculation number:/*Matrikelnr.*: ______________ **Semester:**/*Fachsemester*: ______________

Degree course:/*Studiengang*: ❑ Bachelor PO 2007 ❑ Lehramt Gymnasium (modularisiert)
❑ Bachelor PO 2010 ❑ Lehramt Gymnasium (nicht modularisiert)
❑ Diplom ❑ Master ❑ TMP ❑ ______________

Major:/*Hauptfach*: ❑ Mathematik ❑ Wirtschaftsm. ❑ Informatik ❑ Physik ❑ Statistik ❑ ________

Minor:/*Nebenfach*: ❑ Mathematik ❑ Wirtschaftsm. ❑ Informatik ❑ Physik ❑ Statistik ❑ ________

Credits needed for:/*Anrechnung der Credit Points für das*: ❑ Hauptfach ❑ Nebenfach (Bachelor/Master)

Extra solution sheets submitted:/*Zusätzlich abgegebene Lösungsblätter*: ❑ Yes ❑ No

problem	**1**	**2**	**3**	**4**	**5**	$\sum$
total marks	**10**	**10**	**10**	**10**	**10**	50
scored marks						

homework bonus		**final test performance**		**total performance**		**FINAL MARK**	

INSTRUCTIONS:

- This booklet is made of fourteen pages, including the cover, numbered from 1 to 14. The test consists of **five** problems. Each problem is worth the number of marks specified in the table above. **40** marks are counted as 100% performance in this test. You are free to attempt any problem and collect partial credits.
- The only material that you are allowed to use is black or blue pens/pencils and one hand-written, two-sided, A4-paper "cheat sheet" (Spickzettel). You cannot use your own paper: should you need more paper, raise your hand and you will be given extra sheets.
- Prove all your statements or refer to the standard material discussed in class.
- Work individually. Write with legible handwriting. You may hand in your solution in English or in German. Put your name on every sheet you hand in.
- You have 120 minutes.

GOOD LUCK!

Fill in the form here below only if you need the certificate (Schein).

Dieser Leistungsnachweis entspricht auch den Anforderungen
nach § Abs. Nr. Buchstabe LPO I
nach § Abs. Nr. Buchstabe LPO I

UNIVERSITÄT MÜNCHEN

ZEUGNIS

Der / Die Studierende der __________
Herr / Frau __________ *aus* __________
geboren am ______ *in* __________ *hat im* **SoSe** ______ *–Halbjahr* **2012** ______
meine Übungen **zur Funktionalanalysis** __________

mit __________ *besucht.*
Er / Sie hat __________
schriftliche Arbeiten geliefert, die mit ihm / ihr besprochen wurden. __________

MÜNCHEN, *den* **20 Oktober 2012**

Name

PROBLEM 1. (10 marks)

Consider the vector space $C^1([0,1])$ and define the following functionals on it:

- $F_1(f) := |f(0)| + \max_{x\in[0,1]} |f'(x)|$,
- $F_2(f) := \int_0^1 |f(x)|\,\mathrm{d}x + \max_{x\in[0,1]} |f'(x)|$.

(i) Prove that both F_1 and F_2 are norms.

(ii) Prove that F_1 and F_2 are equivalent (as norms).
(Hint: $f(x) - f(0) = \int_0^x f'(t)dt$.)

SOLUTION:

(i) Both F_1 and F_2 are clearly well-defined. The positivity, triangle inequality and linear scaling under multiplication clearly hold for both F_1 and F_2 (but most be proved!). Assume $F_1(f) = 0$, then $f(0) = 0$ and $f' \equiv 0$, hence $|f(x)| = |\int_0^x f'(t)dt| = 0$ for all x, so that $f \equiv 0$. Thus, F_1 is a norm. Assume $F_2(f) = 0$, then $f' \equiv 0$, hence $|f(x) - f(0)| = |\int_0^x f'(t)dt| = 0$, i.e., $f(x) = f(0)$ for all x. Also, $\int_0^1 |f(x)|dx = 0$ (since $F_2(f) = 0$), so $0 = \int_0^1 |f(0)|dx = f(0)$. It follows that $f(x) = 0$ for all x, and so F_2 is also a norm.

(ii) To prove is: There exists constants $C, C' > 0$ such that $C\,F_1(f) \le F_2(f) \le C'\,F_1(f)$ for all $f \in C^1([0,1])$.

We have

$$\begin{aligned} F_2(f) &= \int_0^1 dx \Big| f(0) + \int_0^x f'(t)dt \Big| + \max_{x\in[0,1]} |f'(x)| \\ &\le |f(0)| + \int_0^1 \big(|x| \max_{t\in[0,1]} |f'(t)|\big) dx + \max_{x\in[0,1]} |f'(x)| \\ &\le |f(0)| + 2 \max_{x\in[0,1]} |f'(x)| \le 2\,F_1(f)\,. \end{aligned}$$

Conversely, for all $x \in [0,1]$,

$$|f(0)| \le |f(x) - f(0)| + |f(x)| \le \Big| \int_0^x f'(t)dt \Big| + |f(x)| \le \max_{x\in[0,1]} |f'(x)| + |f(x)|\,,$$

and integrating both sides in dx from $x = 0$ to $x = 1$ yields $|f(0)| \le \int_0^1 |f(x)|dx + \max_{x\in[0,1]} |f'(x)|$, hence $F_1(f) \le 2\,F_2(f)$.

SOLUTION TO PROBLEM 1 (CONTINUATION):

Name

PROBLEM 2. (10 marks)

Let (X, d) be a compact metric space. For each $n \in \mathbb{N}$ let $f_n : X \to \mathbb{R}$ be a continuous function such that $f_n(x) \geqslant 0 \; \forall x \in X$. Assume that $\forall x \in X$ the sequence $\{f_n(x)\}_{n=1}^{\infty}$ decreases monotonically and $\lim_{n\to\infty} f_n(x) = 0$. Prove that

$$\sup_{x\in X} |f_n(x)| \xrightarrow{n\to\infty} 0 \, .$$

(Hint: Using the assumptions, write X as a suitable union of open balls.)

SOLUTION:

Let $\epsilon > 0$. For all $x \in X$ there exists $n_{\epsilon,x} \in \mathbb{N}$ such that $f_{n_{\epsilon,x}}(x) < \epsilon/2$ (since $\{f_n(x)\}_{n=1}^{\infty}$ decreases monotonically and $\lim_{n\to\infty} f_n(x) = 0$). Also (by continuity of $f_{n_{\epsilon,x}}$ at x), there exists $\delta_{\epsilon,x} > 0$ such that $|f_{n_{\epsilon,x}}(y) - f_{n_{\epsilon,x}}(x)| < \epsilon/2$ for all $y \in B_{\delta_{\epsilon,x}}(x)$. By compactness of X, the open cover $\bigcup_{x\in X} B_{\delta_{\epsilon,x}}(x) = X$ contains a finite subcover, $\bigcup_{i=1}^{m} B_{\delta_{x_i,\epsilon}}(x_i) = X$. Let $N := \max\{n_{\epsilon,x_1}, \ldots, n_{\epsilon,x_m}\}$.

Let now $y \in X$, then there exists some $i \in \{1, \ldots, m\}$ such that $y \in B_{\delta_{x_i,\epsilon}}(x_i)$, and hence, by the above: For all $n \geq N$:

$$0 \leq f_n(y) \leq f_N(y) \leq f_{n_{\epsilon,x_i}}(y) \leq f_{n_{\epsilon,x_i}}(x_i) + |f_{n_{\epsilon,x_i}}(y) - f_{n_{\epsilon,x_i}}(x_i)| < \epsilon/2 + \epsilon/2 = \epsilon \, ,$$

hence, $\sup_{y\in X} |f_n(y)| \leq \epsilon$ for all $n \geq N$. That is, $\sup_{x\in X} |f_n(x)| \xrightarrow{n\to\infty} 0$.

SOLUTION TO PROBLEM 2 (CONTINUATION):

Name

PROBLEM 3. (10 marks)

Consider the set

$$c_{00} \;=\; \big\{x = (x_1, x_2, x_3, \dots) \,|\, x_n \in \mathbb{C} \text{ and } x \text{ has finitely many non-zero entries}\big\}$$

equipped with the natural structure of vector space given by componentwise sum and multiplication by a scalar. Let $\| \; \|$ be an arbitrary norm on c_{00}.

(i) Prove that c_{00} can be written as a countable union of finite dimensional subspaces.

(ii) Prove that $(c_{00}, \| \; \|)$ is not a Banach space.

SOLUTION:

(i) Let, for all $n \in \mathbb{N}$, $V_n = \{x \in c_{00} \,|\, x = (x_1, \dots, x_n, 0, 0, \dots)\}$. Then $V_n \subset c_{00}$ is clearly a linear subspace, and $\dim V_n = n < \infty$, and, if $x = (x_1, x_2, x_3, \dots) \in c_{00}$, then there is an $N \in \mathbb{N}$ such that $x_n = 0$ for $n \geq N$. Hence $x \in V_N$. So $c_{00} = \bigcup_{n=1}^{\infty} V_n$.

(ii) Assume for contradiction that $(c_{00}, \|\cdot\|)$ is a Banach space, in particular, a complete metric space.

From (i), $c_{00} = \bigcup_{n=1}^{\infty} V_n$ with $\dim V_n = n$. Hence each V_n is closed (all normed finite dimensional vector spaces are complete in themselves). Claim: Any proper subspace V of a normed space X has empty interior (hence, $(\overline{V_n})^\circ = (V_n)^\circ = \emptyset$). Proof: Let $v \in V$ and $x \in X \setminus V$ (in particular, $x \neq 0$). Then, for all $\epsilon > 0$, $v + \frac{\epsilon}{2\|x\|} x \notin V$, otherwise, by linearity,

$$x = \frac{2\|x\|}{\epsilon}\Big[\big(v + \frac{\epsilon}{2\|x\|}x\big) - v\Big] \in V$$

too. But $v + \frac{\epsilon}{2\|x\|} x \in B_\epsilon(v)$. Hence, $B_\epsilon(v) \cap (X \setminus V) \neq \emptyset$ for all $\epsilon > 0$, so v cannot be inner point of V.

Hence, $c_{00} = \bigcup_{n=1}^{\infty} V_n$, with $(\overline{V_n})^\circ = \emptyset$ for all $n \in \mathbb{N}$. This contradicts Baire's Category Theorem (or, one of its corollaries), since $(c_{00}, \|\cdot\|)$ is assumed complete. So $(c_{00}, \|\cdot\|)$ is not Banach.

SOLUTION TO PROBLEM 3 (CONTINUATION):

Name

PROBLEM 4. (10 marks)

In $L^2([\frac{1}{2},2])$ consider the subspace

$$\mathcal{M} := \Big\{ f \in L^2([\tfrac{1}{2},2]) \mid f(x) = f(\tfrac{1}{x}) \text{ a.e. for } x \in [\tfrac{1}{2},2] \Big\}.$$

(i) Prove that $\mathcal{M}^\perp = \{ g \in L^2([\frac{1}{2},2]) \mid g(\frac{1}{x}) = -x^2 g(x) \text{ a.e. for } x \in [\frac{1}{2},2] \}$.
(Hint: $[\frac{1}{2},2] = [\frac{1}{2},1] \cup [1,2]$.)

(ii) Find the orthogonal projection of the function $f_0(x) = x$ onto the subspace $\mathcal{M}$.
(Hint: $L^2([\frac{1}{2},2]) = \mathcal{M} \oplus \mathcal{M}^\perp$.)

SOLUTION:

(i) Any $g \in \mathcal{M}^\perp$ is characterised by $\langle g, f \rangle = 0$ for all $f \in \mathcal{M}$. From

$$\begin{aligned} 0 = \langle g, f \rangle &= \int_{1/2}^2 \overline{g(x)} f(x) dx = \int_{1/2}^1 \overline{g(x)} f(x) dx + \int_1^2 \overline{g(x)} f(x) dx \\ &= \int_2^1 \overline{g(\tfrac{1}{y})} f(\tfrac{1}{y})(-\tfrac{dy}{y^2}) + \int_1^2 \overline{g(x)} f(x) dx = \int_1^2 \overline{(g(\tfrac{1}{x}) + \tfrac{1}{x^2} g(x))} f(x) dx \quad \text{for all } \ f \in \mathcal{U} \end{aligned}$$

(where we used $f(x) = f(\frac{1}{x})$) and from the fact that f on the right hand side above is an arbitrary L^2-function on $[1,2]$, that is,

$$\Big\{ \widetilde{f} \ \text{ such that } \widetilde{f} = f\big|_{[1,2]} \ \text{ with } f \in \mathcal{M} \Big\} = L^2([1,2]),$$

one has $g(\frac{1}{x}) = -x^2 g(x)$ a.e. for $x \in [1,2]$. The same relation for g clearly holds a.e. for $x \in [\frac{1}{2},1]$ too.

(ii) To prove that $\mathcal{M}$ is closed, take $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{M}$, with $f_n \to f \in L^2([\frac{1}{2},2])$ (convergence in $L^2([\frac{1}{2},2])$). Then there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ which converges pointwise to f a.e. $x \in [\frac{1}{2},2]$. Since $f_{n_k}(x) = f_{n_k}(\frac{1}{x})$ for a.e. x, it follows that $f(x) = f(\frac{1}{x})$ for a.e. x, and so $f \in \mathcal{M}$.

Hence, by the Projection Theorem, $L^2([\frac{1}{2},2]) = \mathcal{M} \oplus \mathcal{M}^\perp$.

Decompose $f_0 = f + g$ with $f \in \mathcal{M}$ and $g \in \mathcal{M}^\perp$ (clearly, $f_0 \in L^2([\frac{1}{2},2])$). The orthogonal projection of f_0 on $\mathcal{M}$ is exactly f. For almost all $x \in [\frac{1}{2},2]$ one has

$$\begin{aligned} f_0(x) &= x = f(x) + g(x) \\ f_0(\tfrac{1}{x}) &= \tfrac{1}{x} = f(\tfrac{1}{x}) + g(\tfrac{1}{x}) = f(x) - x^2 g(x). \end{aligned}$$

Solving for $f(x), g(x)$, this gives

$$f(x) = \frac{x^2 + x^{-2}}{x + x^{-1}}, \qquad g(x) = \frac{x - x^{-1}}{x^2 + 1}.$$

SOLUTION TO PROBLEM 4 (CONTINUATION):

Name

PROBLEM 5. (10 marks)

Let $(X, \|\ \|_X)$ be a Banach space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Let $Y = \ell_1$, the Banach space of summable sequences in $\mathbb{K}$ equipped with the usual $\|\ \|_1$ norm. Let $T : X \to Y$ be a linear, bounded, and surjective operator.

(i) Prove that there exists $c > 0$ such that $cB_Y \subset T(B_X)$. (Here, B_X, B_Y are the open unit balls in X, Y and $cB_Y = B_c(0_Y)$.)

(ii) Prove that there exists a bounded linear operator $S : Y \to X$ such that TS is the identity operator on Y.
(Hint: Use (i), and try to construct S by hand.)

SOLUTION:

(i) By the Open Mapping Theorem, T is an open map. Let B_X be the open unit ball in X, then $T(B_X)$ is open in Y and contains $0_Y (= T0_X, 0_X \in B_X)$. Hence, there exists $c > 0$ such that $cB_Y = B_c(0_Y) \subset T(B_X)$.

(ii) Let $\{e_n\}_{n\in\mathbb{N}}$ denote the canonical "basis" of ℓ_1, i.e.,

$$e_n = \{0, 0, \ldots, 0, \underset{(n)}{1}, 0, \ldots\} \in \ell_1\,, \quad \|e_n\|_1 = 1\,.$$

There exists some $u_n \in X$ such that $\|u_n\|_X < 2/c$ and $T(u_n) = e_n$. (Proof: $\frac{c}{2}e_n \in cB_Y$, so there exists $\tilde{u}_n \in B_X$ such that $T(\tilde{u}_n) = \frac{c}{2}e_n$. Then let $u_n = \frac{2}{c}\tilde{u}_n$). Given $y = \{y_1, y_2, \ldots, y_n, \ldots\} \in \ell_1$, set $Sy := \sum_{i=1}^{\infty} y_i u_i$. Clearly the series converges in X (since it converges absolutely, $\sum_{i=1}^{\infty} \|y_i u_i\|_X \le \frac{2}{c}\sum_{i=1}^{\infty} |y_i| < \infty$, and X is Banach), and S is linear, and bounded ($\|Sy\|_X \le \frac{2}{c}\|y\|_1$ by the above). Also, by continuity and linearity of T,

$$TS(y) = T(\sum_{i=1}^{\infty} y_i u_i) = T(\lim_{N\to\infty}\sum_{i=1}^{N} y_i u_i) = \lim_{N\to\infty}\big[T(\sum_{i=1}^{N} y_i u_i)\big] = \lim_{N\to\infty}\big[\sum_{i=1}^{N} y_i T u_i\big]$$
$$= \lim_{N\to\infty}\big[\sum_{i=1}^{N} y_i e_i\big] = \{y_1, y_2, \ldots, y_n, \ldots\} = y \in \ell_1\,.$$

Hence, TS is the identity operator on Y.

SOLUTION TO PROBLEM 5 (CONTINUATION):