

## Übungsblatt 9 zu MPIIA

### Aufgabe 33: (4 Punkte)

a) Es seien  $f, g : \mathbb{R}^3 \rightarrow \mathbb{R}$  und  $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  alle aus  $C^2(\mathbb{R}^3)$ . Zeigen Sie:

- 1)  $\operatorname{div}(\operatorname{rot} u) = 0$ .
- 2)  $\nabla(\operatorname{div} u) - \operatorname{rot}(\operatorname{rot} u) = \Delta u$ .
- 3)  $\operatorname{rot}(fu) = \nabla f \times u + f(\operatorname{rot} u)$ .

b) Zeigen Sie: Sind  $f, g : M \subset \mathbb{R}^n \rightarrow \mathbb{R}$  aus  $C^2(M)$ , so gilt

$$\Delta(fg) = (\Delta f)g + 2\nabla f \cdot \nabla g + f\Delta g.$$

c) Zeigen Sie: Ist  $F : I \rightarrow \mathbb{R}$ ,  $I \subset ]0, \infty[$ , eine  $C^2$ -Funktion und  $f(x) := F(r)$ , wobei  $r := |x|$ ,  $x \in \mathbb{R}^n$ , so gilt

$$\Delta f(x) = F''(r) + \frac{n-1}{r}F'(r).$$

d) Zeigen Sie: Ist  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  aus  $C^2(\mathbb{R}^2)$  und  $F(r, \varphi) := f(r \cos \varphi, r \sin \varphi)$ , so gilt

$$(\Delta f)(r \cos \varphi, r \sin \varphi) = \left( \frac{\partial^2 F}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \varphi^2} + \frac{1}{r} \frac{\partial F}{\partial r} \right)(r, \varphi).$$

Lösungsvorschlag: a) 1)

$$\begin{aligned} \operatorname{div}(\operatorname{rot} u) &= \frac{\partial}{\partial x}((\operatorname{rot} u)_x) + \frac{\partial}{\partial y}((\operatorname{rot} u)_y) + \frac{\partial}{\partial z}((\operatorname{rot} u)_z) \\ &= \frac{\partial}{\partial x} \left( \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) = 0. \\ &\quad \left[ \text{Weil } \frac{\partial^2 u_3}{\partial x \partial y} = \frac{\partial^2 u_3}{\partial y \partial x}, \frac{\partial^2 u_2}{\partial x \partial z} = \frac{\partial^2 u_2}{\partial z \partial x}, \frac{\partial^2 u_1}{\partial y \partial z} = \frac{\partial^2 u_1}{\partial z \partial y} \right] \end{aligned}$$

weil  $u \in C^2(\mathbb{R}^3)$ , und alle partielle Ableitungen 2. Ordnung damit stetig sind. ]

2)

$$\begin{aligned} \nabla(\operatorname{div} u) - \operatorname{rot}(\operatorname{rot} u) &= \nabla \left( \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} \right) - \operatorname{rot}((\operatorname{rot} u)_x, (\operatorname{rot} u)_y, (\operatorname{rot} u)_z) \\ &= \left( \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial x \partial y} + \frac{\partial^2 u_3}{\partial x \partial z}, \frac{\partial^2 u_1}{\partial y \partial x} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_3}{\partial y \partial z}, \frac{\partial^2 u_1}{\partial z \partial x} + \frac{\partial^2 u_2}{\partial z \partial y} + \frac{\partial^2 u_3}{\partial z^2} \right) \\ &\quad - \left( \frac{\partial}{\partial y} \left( \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) - \frac{\partial}{\partial z} \left( \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} \right), \frac{\partial}{\partial z} \left( \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \right) - \frac{\partial}{\partial x} \left( \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right), \right. \\ &\quad \left. \frac{\partial}{\partial x} \left( \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \right) \right) \\ &= \left( \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}, \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2}, \frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} + \frac{\partial^2 u_3}{\partial z^2} \right) \\ &= (\Delta u_1, \Delta u_2, \Delta u_3) = \Delta u. \end{aligned}$$

[ Weil

$$\begin{aligned}\frac{\partial^2 u_1}{\partial z \partial x} &= \frac{\partial^2 u_1}{\partial x \partial z}; \quad \frac{\partial^2 u_2}{\partial y \partial x} = \frac{\partial^2 u_2}{\partial x \partial y}; \quad \frac{\partial^2 u_3}{\partial x \partial z} = \frac{\partial^2 u_3}{\partial z \partial x}; \\ \frac{\partial^2 u_3}{\partial y \partial z} &= \frac{\partial^2 u_3}{\partial z \partial y}; \quad \frac{\partial^2 u_3}{\partial y \partial z} = \frac{\partial^2 u_2}{\partial z \partial y}; \quad \frac{\partial^2 u_1}{\partial y \partial x} = \frac{\partial^2 u_1}{\partial x \partial y}\end{aligned}$$

wie oben. ]

3)

$$\begin{aligned}u &= (u_1, u_2, u_3); \quad fu = (fu_1, fu_2, fu_3); \\ \text{rot}(fu) &= \left( \frac{\partial}{\partial y}(fu_3) - \frac{\partial}{\partial z}(fu_2), \frac{\partial}{\partial z}(fu_1) - \frac{\partial}{\partial x}(fu_3), \frac{\partial}{\partial x}(fu_2) - \frac{\partial}{\partial y}(fu_1) \right) \\ &= \left( [u_3 \frac{\partial f}{\partial y} + f \frac{\partial u_3}{\partial y}] - [u_2 \frac{\partial f}{\partial z} + f \frac{\partial u_2}{\partial z}], [u_1 \frac{\partial f}{\partial z} + f \frac{\partial u_1}{\partial z}] - [u_3 \frac{\partial f}{\partial x} + f \frac{\partial u_3}{\partial x}], \right. \\ &\quad \left. [u_2 \frac{\partial f}{\partial x} + f \frac{\partial u_2}{\partial x}] - [u_1 \frac{\partial f}{\partial y} + f \frac{\partial u_1}{\partial y}] \right) \\ &= f \left( \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) \\ &\quad + \left( (\nabla f)_2 u_3 - (\nabla f)_3 u_2, (\nabla f)_3 u_1 - (\nabla f)_1 u_3, (\nabla f)_1 u_2 - (\nabla f)_2 u_1 \right) \\ &= f \text{rot}(u) + \nabla f \times u.\end{aligned}$$

b)

$$\begin{aligned}\Delta(fg) &= \sum_{j=1}^n \frac{\partial^2}{\partial^2} x_j^2(fg) = \sum_{j=1}^n \frac{\partial}{\partial x_j} \left( f \frac{\partial g}{\partial x_j} + g \frac{\partial f}{\partial x_j} \right) \\ &= \sum_{j=1}^n \left( \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_j} + f \frac{\partial^2 g}{\partial x_j^2} + \frac{\partial g}{\partial x_j} \frac{\partial f}{\partial x_j} + g \frac{\partial^2 f}{\partial x_j^2} \right) \\ &= g \left( \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2} \right) + 2 \sum_{j=1}^n \left( \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_j} \right) + f \left( \sum_{j=1}^n \frac{\partial^2 g}{\partial x_j^2} \right) \\ &= g \Delta f + 2 \nabla f \cdot \nabla g + f \Delta g.\end{aligned}$$

c) Man verwendet die Kettenregel (mehrmals):

$$\begin{aligned}g(x) &= |x|; \quad f = F \circ g; \quad f : \mathbb{R}^n \rightarrow \mathbb{R}; \\ \frac{\partial g}{\partial x_j}(x) &= \frac{x_j}{|x|}; \quad \frac{\partial^2 g}{\partial x_j^2}(x) = \frac{\partial}{\partial x_j} \left( \frac{x_j}{|x|} \right) = \frac{|x| \frac{\partial}{\partial x_j}(x_j) - x_j \frac{\partial}{\partial x_j}(|x|)}{|x|^2} = \frac{|x| - x_j \frac{x_j}{|x|}}{|x|^2} = \frac{|x|^2 - x_j^2}{|x|^3}; \\ \frac{\partial f}{\partial x_j}(x) &= \left( \frac{\partial}{\partial x_j} (F \circ g) \right)(x) = F'(g(x)) \frac{\partial g}{\partial x_j}(x) = F'(r) \frac{x_j}{|x|}; \\ \frac{\partial^2 f}{\partial x_j^2}(x) &= \frac{\partial}{\partial x_j} \left( F'(g(x)) \frac{x_j}{|x|} \right) = \frac{\partial}{\partial x_j} (F'(g(x))) \frac{x_j}{|x|} + F'(g(x)) \frac{\partial}{\partial x_j} \left( \frac{x_j}{|x|} \right) \\ &= \frac{\partial}{\partial x_j} ((F' \circ g)(x)) \frac{x_j}{|x|} + F'(r) \frac{|x|^2 - x_j^2}{|x|^3} = (F''(g(x)) \frac{\partial g}{\partial x_j}) \frac{x_j}{|x|} + F'(r) \frac{|x|^2 - x_j^2}{|x|^3}; \\ \Delta f(x) &= \sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}(x) = \sum_{j=1}^n \left\{ F''(g(x)) \frac{x_j}{|x|} \frac{x_j}{|x|} + F'(r) \frac{|x|^2 - x_j^2}{|x|^3} \right\} \\ &= F''(r) \left( \sum_{j=1}^n \frac{x_j^2}{|x|^2} \right) + F'(r) \frac{1}{|x|^3} \sum_{j=1}^n (|x|^2 - x_j^2) = F''(r) \cdot 1 + F'(r) \frac{1}{|x|^3} (n|x|^2 - |x|^2) \\ &= F''(r) + \frac{n-1}{r} F'(r).\end{aligned}$$

d) Man verwendet wieder die Kettenregel.

$$\begin{aligned}
F &= f \circ g; \quad g(r, \varphi) = (r \cos \varphi, r \sin \varphi) = (g_1(r, \varphi), g_2(r, \varphi)); \\
\frac{\partial F}{\partial r}(r, \varphi) &= \frac{\partial f}{\partial x_1}(g(r, \varphi)) \frac{\partial g_1}{\partial r}(r, \varphi) + \frac{\partial f}{\partial x_2}(g(r, \varphi)) \frac{\partial g_2}{\partial r}(r, \varphi) \\
&= \frac{\partial f}{\partial x_1}(g(r, \varphi)) \cos \varphi + \frac{\partial f}{\partial x_2}(g(r, \varphi)) \sin \varphi; \\
\frac{\partial^2 F}{\partial r^2}(r, \varphi) &= \frac{\partial}{\partial r} \left[ \frac{\partial f}{\partial x_1}(g(r, \varphi)) \cos \varphi + \frac{\partial f}{\partial x_2}(g(r, \varphi)) \sin \varphi \right] \\
&= \left[ \frac{\partial^2 f}{\partial x_1 \partial x_1}(g(r, \varphi)) \frac{\partial g_1}{\partial r}(r, \varphi) + \frac{\partial^2 f}{\partial x_2 \partial x_1}(g(r, \varphi)) \frac{\partial g_2}{\partial r}(r, \varphi) \right] \cos \varphi \\
&\quad + \left[ \frac{\partial^2 f}{\partial x_1 \partial x_2}(g(r, \varphi)) \frac{\partial g_1}{\partial r}(r, \varphi) + \frac{\partial^2 f}{\partial x_2 \partial x_2}(g(r, \varphi)) \frac{\partial g_2}{\partial r}(r, \varphi) \right] \sin \varphi \\
&= \frac{\partial^2 f}{\partial x_1^2}(g(r, \varphi)) \cos^2 \varphi + \frac{\partial^2 f}{\partial x_2^2}(g(r, \varphi)) \sin^2 \varphi + 2 \frac{\partial^2 f}{\partial x_1 \partial x_2}(g(r, \varphi)) \sin \varphi \cos \varphi; \\
\frac{\partial F}{\partial \varphi}(r, \varphi) &= \frac{\partial f}{\partial x_1}(g(r, \varphi)) \frac{\partial g_1}{\partial \varphi}(r, \varphi) + \frac{\partial f}{\partial x_2}(g(r, \varphi)) \frac{\partial g_2}{\partial \varphi}(r, \varphi) \\
&= \frac{\partial f}{\partial x_1}(g(r, \varphi))(-r \sin \varphi) + \frac{\partial f}{\partial x_2}(g(r, \varphi))(r \cos \varphi); \\
\frac{\partial^2 F}{\partial \varphi^2}(r, \varphi) &= \frac{\partial}{\partial \varphi} \left[ \frac{\partial f}{\partial x_1}(g(r, \varphi))(-r \sin \varphi) + \frac{\partial f}{\partial x_2}(g(r, \varphi))(r \cos \varphi) \right] \\
&= \frac{\partial}{\partial \varphi} \left[ \frac{\partial f}{\partial x_1}(g(r, \varphi)) \right](-r \sin \varphi) + \frac{\partial f}{\partial x_1}(g(r, \varphi)) \frac{\partial}{\partial \varphi}(-r \sin \varphi) \\
&\quad + \frac{\partial}{\partial \varphi} \left[ \frac{\partial f}{\partial x_2}(g(r, \varphi)) \right](r \cos \varphi) + \frac{\partial f}{\partial x_2}(g(r, \varphi)) \frac{\partial}{\partial \varphi}(r \cos \varphi) \\
&= \left[ \frac{\partial^2 f}{\partial x_1 \partial x_1}(g(r, \varphi)) \frac{\partial g_1}{\partial \varphi}(r, \varphi) + \frac{\partial^2 f}{\partial x_1 \partial x_2}(g(r, \varphi)) \frac{\partial g_2}{\partial \varphi}(r, \varphi) \right](-r \sin \varphi) \\
&\quad + \left[ \frac{\partial f}{\partial x_1}(g(r, \varphi)) \right](-r \cos \varphi) \\
&\quad + \left[ \frac{\partial^2 f}{\partial x_1 \partial x_2}(g(r, \varphi)) \frac{\partial g_1}{\partial \varphi}(r, \varphi) + \frac{\partial^2 f}{\partial x_2 \partial x_2}(g(r, \varphi)) \frac{\partial g_2}{\partial \varphi}(r, \varphi) \right](r \cos \varphi) \\
&\quad + \left[ \frac{\partial f}{\partial x_2}(g(r, \varphi)) \right](-r \sin \varphi) \\
&= \left[ \frac{\partial^2 f}{\partial x_1^2}(g(r, \varphi)) \right] r^2 \sin^2 \varphi + \left[ \frac{\partial^2 f}{\partial x_2^2}(g(r, \varphi)) \right] r^2 \cos^2 \varphi \\
&\quad - (2r \sin \varphi \cos \varphi) \left[ \frac{\partial^2 f}{\partial x_1 \partial x_2}(g(r, \varphi)) \right] \\
&\quad - \left[ (r \cos \varphi) \frac{\partial f}{\partial x_1}(g(r, \varphi)) + (r \sin \varphi) \frac{\partial f}{\partial x_2}(g(r, \varphi)) \right]
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \varphi^2} + \frac{1}{r} \frac{\partial F}{\partial r}\right)(r, \varphi) &= \left[\frac{\partial^2 f}{\partial x_1^2}(g(r, \varphi)) \cos^2 \varphi + \frac{\partial^2 f}{\partial x_2^2}(g(r, \varphi)) \sin^2 \varphi\right. \\
&\quad \left.+ 2 \frac{\partial^2 f}{\partial x_1 \partial x_2}(g(r, \varphi)) \sin \varphi \cos \varphi\right] \\
&\quad + \frac{1}{r^2} \left\{ \left[\frac{\partial^2 f}{\partial x_1^2}(g(r, \varphi))\right] r^2 \sin^2 \varphi + \left[\frac{\partial^2 f}{\partial x_2^2}(g(r, \varphi))\right] r^2 \cos^2 \varphi \right. \\
&\quad \left.- (2r \sin \varphi \cos \varphi) \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}(g(r, \varphi))\right] \right. \\
&\quad \left.- \left[(r \cos \varphi) \frac{\partial f}{\partial x_1}(g(r, \varphi)) + (r \sin \varphi) \frac{\partial f}{\partial x_2}(g(r, \varphi))\right] \right\} \\
&\quad + \frac{1}{r} \left\{ \frac{\partial f}{\partial x_1}(g(r, \varphi)) \cos \varphi + \frac{\partial f}{\partial x_2}(g(r, \varphi)) \sin \varphi \right\} \\
&= \frac{\partial^2 f}{\partial x_1^2}(g(r, \varphi)) [\cos^2 \varphi + \sin^2 \varphi] \\
&\quad + \frac{\partial^2 f}{\partial x_2^2}(g(r, \varphi)) [\sin^2 \varphi + \cos^2 \varphi] \\
&= \left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}\right)(g(r, \varphi)) = (\Delta f)(r \cos \varphi, r \sin \varphi).
\end{aligned}$$

[ Verwendet wurde (zweimal), daß (weil  $f$  aus  $C^2$  ist)

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(g(r, \varphi)) = \frac{\partial^2 f}{\partial x_2 \partial x_1}(g(r, \varphi)). \quad ]$$

**Aufgabe 34: (4 Punkte)** Es sei  $f : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$  gegeben durch

$$f(x, y) := x^2 y^2 \ln(x^2 + y^2).$$

Es ist  $(1, 1, \ln 2) \in \text{Graph}(f) := \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2 \setminus \{0\}\}$ .

- Stellen die Tangentialebene an  $\text{Graph}(f)$  in diesem Punkt als Graph einer Funktion  $\ell : \mathbb{R}^2 \rightarrow \mathbb{R}$  dar.
- Approximieren Sie  $\text{Graph}(f)$  in der Nähe von  $(1, 1, \ln 2)$  durch den Graph eines Polynoms 2. Grades in  $x$  und  $y$ . (Hinweis: Taylor).

*Lösungsvorschlag:* a) Man braucht das Taylorpolynom 1. Ordnung:

$$\begin{aligned}
\ell(x, y) &= f(1, 1) + \frac{\partial f}{\partial x}(1, 1)(x - 1) + \frac{\partial f}{\partial y}(1, 1)(y - 1); \\
\frac{\partial f}{\partial x}(x, y) &= 2xy^2 \ln(x^2 + y^2) + x^2 y^2 \frac{2x}{x^2 + y^2}; \quad \frac{\partial f}{\partial x}(1, 1) = 2 \ln 2 + 1; \\
\frac{\partial f}{\partial y}(x, y) &= 2yx^2 \ln(x^2 + y^2) + x^2 y^2 \frac{2y}{x^2 + y^2}; \quad \frac{\partial f}{\partial y}(1, 1) = 2 \ln 2 + 1; \\
\ell(x, y) &= \ln 2 + (2 \ln 2 + 1)(x - 1) + (2 \ln 2 + 1)(y - 1) \\
&= -3 \ln 2 - 2 + (2 \ln 2 + 1)(x + y).
\end{aligned}$$

[  $\ell$  ist eine Polynom 2. Grad in  $(x, y)$ , also ist  $\text{Graph}(\ell)$  eine Ebene;  $\ell(1, 1) = f(1, 1)$ ;  $\frac{\partial \ell}{\partial x}(1, 1) = \frac{\partial f}{\partial x}(1, 1)$ ;  $\frac{\partial \ell}{\partial y}(1, 1) = \frac{\partial f}{\partial y}(1, 1)$ , und damit ist  $\text{Graph}(\ell)$  die Tangentialebene am  $\text{Graph}(f)$  im Punkte  $(1, 1, \ln 2)$ . ]

b) Man braucht das Taylorpolynom 2. Ordnung:

$$\begin{aligned}
(T_2(f))(x, y) &= f(1, 1) + \frac{\partial f}{\partial x}(1, 1)(x - 1) + \frac{\partial f}{\partial y}(1, 1)(y - 1) \\
&\quad + \frac{1}{2} \left[ \frac{\partial^2 f}{\partial x^2}(1, 1) \right] (x - 1)^2 + \frac{1}{2} \left[ \frac{\partial^2 f}{\partial y^2}(1, 1) \right] (y - 1)^2 \\
&\quad + \left[ \frac{\partial^2 f}{\partial x \partial y}(1, 1) \right] (x - 1)(y - 1) \\
&= \ell(x, y) + \frac{1}{2} \left[ \frac{\partial^2 f}{\partial x^2}(1, 1) \right] (x - 1)^2 + \frac{1}{2} \left[ \frac{\partial^2 f}{\partial y^2}(1, 1) \right] (y - 1)^2 \\
&\quad + \left[ \frac{\partial^2 f}{\partial x \partial y}(1, 1) \right] (x - 1)(y - 1); \\
\frac{\partial^2 f}{\partial x^2}(x, y) &= \frac{\partial}{\partial x} \left[ 2xy^2 \ln(x^2 + y^2) + x^2 y^2 \frac{2x}{x^2 + y^2} \right] \\
&= 2y^2 \ln(x^2 + y^2) + \frac{10x^2 y^2}{x^2 + y^2} - \frac{4x^4 y^2}{(x^2 + y^2)^2}; \quad \frac{\partial^2 f}{\partial x^2}(1, 1) = 2 \ln 2 + 4; \\
\frac{\partial^2 f}{\partial y^2}(x, y) &= \frac{\partial}{\partial y} \left[ 2x^2 y \ln(x^2 + y^2) + x^2 y^2 \frac{2y}{x^2 + y^2} \right] \\
&= 2x^2 \ln(x^2 + y^2) + \frac{10x^2 y^2}{x^2 + y^2} - \frac{4x^2 y^4}{(x^2 + y^2)^2}; \quad \frac{\partial^2 f}{\partial y^2}(1, 1) = 2 \ln 2 + 4; \\
\frac{\partial^2 f}{\partial y \partial x}(x, y) &= \frac{\partial}{\partial y} \left[ 2xy^2 \ln(x^2 + y^2) + x^2 y^2 \frac{2x}{x^2 + y^2} \right] \\
&= 4xy \ln(x^2 + y^2) + \frac{4(xy^3 + x^3 y)}{x^2 + y^2} - \frac{4x^3 y^3}{(x^2 + y^2)^2}; \\
\frac{\partial^2 f}{\partial y \partial x}(1, 1) &= 4 \ln 2 + 3; \\
(T_2(f))(x, y) &= -3 \ln 2 - 2 + (2 \ln 2 + 1)(x + y) + \frac{1}{2}(2 \ln 2 + 4)(x - 1)^2 \\
&\quad + \frac{1}{2}(2 \ln 2 + 4)(y - 1)^2 + (4 \ln 2 + 3)(x - 1)(y - 1) \\
&= (2 + \ln 2)(x^2 + y^2) + (4 \ln 2 + 3)xy - (4 \ln 2 + 6)(x + y) + (3 \ln 2 + 5)
\end{aligned}$$

[ Verwendet wurde, daß (weil  $f$  aus  $C^2$  ist)

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y).$$

Man sieht dieses auch durch direktes Ausrechnen. ]

**Aufgabe 35: (4 Punkte)** Eine Funktion  $u : \mathbb{R}^2 \supset U \rightarrow \mathbb{R}$  heißt *harmonisch*, falls  $u$  aus  $C^2(U)$  ist und die Gleichung  $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$  erfüllt ist.

Sei  $f = u + iv : \mathbb{C} \supset U \rightarrow \mathbb{C}$  komplex differenzierbar, und  $u, v \in C^2(U, \mathbb{R})$  (hier interpretieren wir  $u(x + iy) = u(x, y), v(x + iy) = v(x, y)$ ). Zeigen Sie, daß  $u$  und  $v$  harmonisch sind.  
*Lösungsvorschlag:* Man hat, daß

$$\begin{aligned}
&\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \\
&= \lim_{(s, t) \rightarrow (0, 0)} \frac{[u(x_0 + s, y_0 + t) + iv(x_0 + s, y_0 + t)] - [u(x_0, y_0) + iv(x_0, y_0)]}{s + it}
\end{aligned}$$

existiert für alle  $z_0 = x_0 + iy_0 \in U$  (Analysis I). Insbesondere muss  $((s, t) = (r, 0))$

$$\begin{aligned} & \lim_{r \rightarrow 0} \frac{[u(x_0 + r, y_0) + iv(x_0 + r, y_0)] - [u(x_0, y_0) + iv(x_0, y_0)]}{r} \\ &= \lim_{r \rightarrow 0} \frac{u(x_0 + r, y_0) - u(x_0, y_0)}{r} + i \lim_{r \rightarrow 0} \frac{v(x_0 + r, y_0) - v(x_0, y_0)}{r} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) \end{aligned}$$

gleich  $((s, t) = (0, \rho))$

$$\begin{aligned} & \lim_{\rho \rightarrow 0} \frac{[u(x_0, y_0 + \rho) + iv(x_0, y_0 + \rho)] - [u(x_0, y_0) + iv(x_0, y_0)]}{i\rho} \\ &= \frac{1}{i} \lim_{\rho \rightarrow 0} \frac{u(x_0, y_0 + \rho) - u(x_0, y_0)}{\rho} + \lim_{\rho \rightarrow 0} \frac{v(x_0, y_0 + \rho) - v(x_0, y_0)}{\rho} \\ &= -i \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) \end{aligned}$$

sein. (Alle Grenzwerte existieren, weil  $u, v$  aus  $C^2$  sind). Weil  $u, v$  reellwertig sind, folgt

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad (1)$$

$$\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \quad (2)$$

Aus (1) (und daß  $u, v$  aus  $C^2$  sind) folgt, daß

$$\frac{\partial^2 u}{\partial y \partial x}(x_0, y_0) = \frac{\partial^2 v}{\partial y^2}(x_0, y_0)$$

und aus (2)

$$\frac{\partial^2 u}{\partial x \partial y}(x_0, y_0) = -\frac{\partial^2 v}{\partial x^2}(x_0, y_0).$$

Aber weil  $u, v$  aus  $C^2$  sind, gilt  $\frac{\partial^2 u}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 u}{\partial y \partial x}(x_0, y_0)$ , und damit, daß

$$\frac{\partial^2 v}{\partial y^2}(x_0, y_0) = -\frac{\partial^2 v}{\partial x^2}(x_0, y_0)$$

d.h.

$$(\Delta v)(x_0, y_0) = \frac{\partial^2 v}{\partial x^2}(x_0, y_0) + \frac{\partial^2 v}{\partial y^2}(x_0, y_0) = 0.$$

Ähnlich folgt, daß  $(\Delta u)(x_0, y_0) = 0$ .

**Aufgabe 36: (4 Punkte)** Sei  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  aus  $C^1(\mathbb{R}^m)$ .

a) Zeigen Sie, daß

$$f(x) - f(y) = \int_0^1 [f'(x + t(y - x))](y - x) dt$$

für alle  $x, y \in \mathbb{R}^m$ . (Hinweis: Kettenregel).

b) Sei  $\|A\|_2 := \sqrt{\sum_{i=1}^n \sum_{j=1}^m |a_{ij}|^2}$ ,  $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ . Zeigen Sie, daß

$$|Ax| \leq \|A\|_2 |x|$$

für all  $x \in \mathbb{R}^m$ .

c) Zeigen Sie, daß

$$|f(x) - f(y)| \leq C|x - y| \quad \text{mit} \quad C := \sup_{t \in [0,1]} \|f'(x + t(y - x))\|_2$$

für alle  $x, y \in \mathbb{R}^m$ .

*Lösungsvorschlag:* a) Die Funktion  $g : \mathbb{R} \rightarrow \mathbb{R}^n$  definiert durch  $g(t) = f(x + t(y - x))$  ist aus  $C^1(\mathbb{R})$  (weil  $h(t) = t(y - x)$  aus  $C^1(\mathbb{R})$  ist, und  $g = f \circ h$ ; Kettenregel). Damit ist

$$g(1) - g(0) = \int_0^1 g'(t) dt.$$

Nach der Kettenregel ist  $g'(t) = f'(h(t)) \circ h'(t)$ . Es gilt  $h'(t) = y - x$  und damit

$$g'(t) = f'(h(t)) \circ h'(t) = f'(x + t(y - x)) \cdot (y - x).$$

Außerdem ist  $g(1) = f(h(1)) = f(x + 1 \cdot (y - x)) = f(y)$ ,  $g(0) = f(h(0)) = f(x + 0 \cdot (y - x)) = f(x)$ , d.h.,

$$f(y) - f(x) = g(1) - g(0) = \int_0^1 g'(t) dt = \int_0^1 f'(x + t(y - x))(y - x) dt.$$

[ Man bemerke den Druckfehler in der Aufgabeformulierung. ]

b)

$$\begin{aligned} |Ax|^2 &= \left| \left( \sum_{j=1}^m a_{1j}x_j, \dots, \sum_{j=1}^m a_{nj}x_j \right) \right|^2 = \sum_{i=1}^n \left( \sum_{j=1}^m a_{ij}x_j \right)^2 \\ &\leq \sum_{i=1}^n \left( \sum_{j=1}^m |a_{ij}|^2 \right) \left( \sum_{j=1}^m x_j^2 \right) \\ &\left[ \text{weil (Cauchy-Schwarz)} \left( \sum_{j=1}^m a_{ij}x_j \right)^2 = |\langle a_{i\bullet}, x \rangle|^2 \leq |a_{i\bullet}|^2 |x|^2 = \left( \sum_{j=1}^m |a_{ij}|^2 \right) \left( \sum_{j=1}^m x_j^2 \right) \right] \\ &= \left( \sum_{j=1}^m x_j^2 \right) \left( \sum_{i=1}^n \sum_{j=1}^m |a_{ij}|^2 \right) = \|A\|_2^2 |x|^2. \end{aligned}$$

c)

$$\begin{aligned} |f(x) - f(y)| &= |f(y) - f(x)| = \left| \int_0^1 f'(x + t(y - x))(y - x) dt \right| \\ &\leq \int_0^1 |f'(x + t(y - x))(y - x)| dt \\ &\leq \int_0^1 \|f'(x + t(y - x))\|_2 |y - x| dt \\ &\leq |y - x| \int_0^1 \sup_{t \in [0,1]} \|f'(x + t(y - x))\|_2 dt \\ &= |y - x| \sup_{t \in [0,1]} \|f'(x + t(y - x))\|_2 \int_0^1 1 dt \\ &= C|x - y| \quad \text{mit} \quad C := \sup_{t \in [0,1]} \|f'(x + t(y - x))\|_2. \end{aligned}$$

Abgabe bis Montag 20.06.2005, 11.15 Uhr in den MPIIA Übungskasten im 1. Stock vor der Bibliothek.

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