Mathematisches Institut LMU
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# Übungsblatt 9 zu MPIIA

**Aufgabe 33: (4 Punkte)**

a) Es seien $f, g : \mathbb{R}^3 \to \mathbb{R}$ und $u : \mathbb{R}^3 \to \mathbb{R}^3$ alle aus $C^2(\mathbb{R}^3)$. Zeigen Sie:

1) $\operatorname{div}(\operatorname{rot} u) = 0$.
2) $\nabla(\operatorname{div} u) - \operatorname{rot}(\operatorname{rot} u) = \Delta u$.
3) $\operatorname{rot}(fu) = \nabla f \times u + f(\operatorname{rot} u)$.

b) Zeigen Sie: Sind $f, g : M \subset \mathbb{R}^n \to \mathbb{R}$ aus $C^2(M)$ , so gilt

$$\Delta(fg) = (\Delta f)g + 2\nabla f \cdot \nabla g + f\Delta g.$$

c) Zeigen Sie: Ist $F : I \to \mathbb{R}$, $I \subset ]0, \infty[$ , eine $C^2$-Funktion und $f(x) := F(r)$, wobei $r := |x|$, $x \in \mathbb{R}^n$, so gilt

$$\Delta f(x) = F''(r) + \frac{n-1}{r} F'(r).$$

d) Zeigen Sie: Ist $f : \mathbb{R}^2 \to \mathbb{R}$ aus $C^2(\mathbb{R}^2)$ und $F(r, \varphi) := f(r\cos\varphi, r\sin\varphi)$, so gilt

$$(\Delta f)(r\cos\varphi, r\sin\varphi) = \Big(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2} + \frac{1}{r}\frac{\partial F}{\partial r}\Big)(r, \varphi).$$

*Lösungsvorschlag:* a) 1)

$$\begin{aligned}
\operatorname{div}(\operatorname{rot} u) &= \frac{\partial}{\partial x}\big((\operatorname{rot} u)_x\big) + \frac{\partial}{\partial y}\big((\operatorname{rot} u)_y\big) + \frac{\partial}{\partial z}\big((\operatorname{rot} u)_z\big) \\
&= \frac{\partial}{\partial x}\Big(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\Big) + \frac{\partial}{\partial z}\Big(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\Big) = 0.
\end{aligned}$$

$$\Big[\text{ Weil } \frac{\partial^2 u_3}{\partial x \partial y} = \frac{\partial^2 u_3}{\partial y \partial x},\ \frac{\partial^2 u_2}{\partial x \partial z} = \frac{\partial^2 u_2}{\partial z \partial x},\ \frac{\partial^2 u_1}{\partial y \partial z} = \frac{\partial^2 u_1}{\partial z \partial y}$$

weil $u \in C^2(\mathbb{R}^3)$, und alle partielle Ableitungen 2. Ordnung damit stetig sind. $\Big]$

2)

$$\begin{aligned}
&\nabla(\operatorname{div} u) - \operatorname{rot}(\operatorname{rot} u) = \nabla\Big(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}\Big) - \operatorname{rot}\big((\operatorname{rot} u)_x, (\operatorname{rot} u)_y, (\operatorname{rot} u)_z\big) \\
&= \Big(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_2}{\partial x \partial y} + \frac{\partial^2 u_3}{\partial x \partial z}, \frac{\partial^2 u_1}{\partial y \partial x} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_3}{\partial y \partial z}, \frac{\partial^2 u_1}{\partial z \partial x} + \frac{\partial^2 u_2}{\partial z \partial y} + \frac{\partial^2 u_3}{\partial z^2}\Big) \\
&\quad - \Big(\frac{\partial}{\partial y}\Big(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\Big) - \frac{\partial}{\partial z}\Big(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\Big), \frac{\partial}{\partial z}\Big(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\Big) - \frac{\partial}{\partial x}\Big(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\Big), \\
&\qquad\qquad \frac{\partial}{\partial x}\Big(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\Big)\Big) \\
&= \Big(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}, \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + \frac{\partial^2 u_2}{\partial z^2}, \frac{\partial^2 u_3}{\partial x^2} + \frac{\partial^2 u_3}{\partial y^2} + \frac{\partial^2 u_3}{\partial z^2}\Big) \\
&= \big(\Delta u_1, \Delta u_2, \Delta u_3\big) = \Delta u.
\end{aligned}$$

[ Weil

$$\frac{\partial^2 u_1}{\partial z\partial x}=\frac{\partial^2 u_1}{\partial x\partial z};\ \frac{\partial^2 u_2}{\partial y\partial x}=\frac{\partial^2 u_2}{\partial x\partial y};\ \frac{\partial^2 u_3}{\partial x\partial z}=\frac{\partial^2 u_3}{\partial z\partial x};$$
$$\frac{\partial^2 u_3}{\partial y\partial z}=\frac{\partial^2 u_3}{\partial z\partial y};\ \frac{\partial^2 u_3}{\partial y\partial z}=\frac{\partial^2 u_2}{\partial z\partial y};\ \frac{\partial^2 u_1}{\partial y\partial x}=\frac{\partial^2 u_1}{\partial x\partial y}$$

wie oben. ]

3)

$$\begin{aligned}
u&=(u_1,u_2,u_3);\ fu=(fu_1,fu_2,fu_3);\\
\operatorname{rot}(fu)&=\Big(\frac{\partial}{\partial y}(fu_3)-\frac{\partial}{\partial z}(fu_2),\frac{\partial}{\partial z}(fu_1)-\frac{\partial}{\partial x}(fu_3),\frac{\partial}{\partial x}(fu_2)-\frac{\partial}{\partial y}(fu_1)\Big)\\
&=\Big([u_3\frac{\partial f}{\partial y}+f\frac{\partial u_3}{\partial y}]-[u_2\frac{\partial f}{\partial z}+f\frac{\partial u_2}{\partial z}],[u_1\frac{\partial f}{\partial z}+f\frac{\partial u_1}{\partial z}]-[u_3\frac{\partial f}{\partial x}+f\frac{\partial u_3}{\partial x}],\\
&\qquad [u_2\frac{\partial f}{\partial x}+f\frac{\partial u_2}{\partial x}]-[u_1\frac{\partial f}{\partial y}+f\frac{\partial u_1}{\partial y}]\Big)\\
&=f\Big(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z},\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x},\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}\Big)\\
&\quad+\Big(\big(\nabla f\big)_2u_3-\big(\nabla f\big)_3u_2,\big(\nabla f\big)_3u_1-\big(\nabla f\big)_1u_3,\big(\nabla f\big)_1u_2-\big(\nabla f\big)_2u_1\Big)\\
&=f\operatorname{rot}(u)+\nabla f\times u.
\end{aligned}$$

b)

$$\begin{aligned}
\Delta(fg)&=\sum_{j=1}^n\frac{\partial^2}{\partial}x_j^2(fg)=\sum_{j=1}^n\frac{\partial}{\partial x_j}\big(f\frac{\partial g}{\partial x_j}+g\frac{\partial f}{\partial x_j}\big)\\
&=\sum_{j=1}^n\big(\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_j}+f\frac{\partial^2 g}{\partial x_j^2}+\frac{\partial g}{\partial x_j}\frac{\partial f}{\partial x_j}+g\frac{\partial^2 f}{\partial x_j^2}\big)\\
&=g\big(\sum_{j=1}^n\frac{\partial^2 f}{\partial x_j}\big)+2\sum_{j=1}^n\big(\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_j}\big)+f\big(\sum_{j=1}^n\frac{\partial^2 g}{\partial x_j}\big)\\
&=g\Delta f+2\nabla f\cdot\nabla g+f\Delta g.
\end{aligned}$$

c) Man verwendet die Kettenregel (mehrmals):

$$\begin{aligned}
&g(x)=|x|;\quad f=F\circ g;\ f:\mathbb{R}^n\to\mathbb{R};\\
&\quad\frac{\partial g}{\partial x_j}(x)=\frac{x_j}{|x|};\ \frac{\partial^2 g}{\partial x_j^2}(x)=\frac{\partial}{\partial x_j}\big(\frac{x_j}{|x|}\big)=\frac{|x|\frac{\partial}{\partial x_j}(x_j)-x_j\frac{\partial}{\partial x_j}(|x|)}{|x|^2}=\frac{|x|-x_j\frac{x_j}{|x|}}{|x|^2}=\frac{|x|^2-x_j^2}{|x|^3};\\
&\quad\frac{\partial f}{\partial x_j}(x)=\Big(\frac{\partial}{\partial x_j}\big(F\circ g\big)\Big)(x)=F'(g(x))\,\frac{\partial g}{\partial x_j}(x)=F'(r)\,\frac{x_j}{|x|};\\
&\quad\frac{\partial^2 f}{\partial x_j^2}(x)=\frac{\partial}{\partial x_j}\big(F'(g(x))\,\frac{x_j}{|x|}\big)=\frac{\partial}{\partial x_j}\big(F'(g(x))\big)\,\frac{x_j}{|x|}+F'(g(x))\frac{\partial}{\partial x_j}\big(\frac{x_j}{|x|}\big)\\
&\quad=\frac{\partial}{\partial x_j}\big((F'\circ g)(x)\big)\frac{x_j}{|x|}+F'(r)\,\frac{|x|^2-x_j^2}{|x|^3}=\big(F''(g(x))\frac{\partial g}{\partial x_j}\big)\frac{x_j}{|x|}+F'(r)\,\frac{|x|^2-x_j^2}{|x|^3};\\
&\quad\Delta f(x)=\sum_{j=1}^n\frac{\partial^2 f}{\partial x_j^2}(x)=\sum_{j=1}^n\Big\{F''(g(x))\frac{x_j}{|x|}\frac{x_j}{|x|}+F'(r)\,\frac{|x|^2-x_j^2}{|x|^3}\Big\}\\
&\quad=F''(r)\Big(\frac{\sum_{j=1}^n x_j^2}{|x|^2}\Big)+F'(r)\frac{1}{|x|^3}\sum_{j=1}^n\big(|x|^2-x_j^2\big)=F''(r)\cdot1+F'(r)\frac{1}{|x|^3}\big(n|x|^2-|x|^2\big)\\
&\quad=F''(r)+\frac{n-1}{r}F'(r).
\end{aligned}$$

d) Man verwendet wieder die Kettenregel.

$$\begin{aligned}
F &= f \circ g;\; g(r,\varphi) = (r\cos\varphi, r\sin\varphi) = (g_1(r,\varphi), g_1(r,\varphi));\\
\frac{\partial F}{\partial r}(r,\varphi) &= \frac{\partial f}{\partial x_1}(g(r,\varphi))\,\frac{\partial g_1}{\partial r}(r,\varphi) + \frac{\partial f}{\partial x_2}(g(r,\varphi))\,\frac{\partial g_2}{\partial r}(r,\varphi)\\
&= \frac{\partial f}{\partial x_1}(g(r,\varphi))\cos\varphi + \frac{\partial f}{\partial x_2}(g(r,\varphi))\sin\varphi;\\
\frac{\partial^2 F}{\partial r^2}(r,\varphi) &= \frac{\partial}{\partial r}\Big[\frac{\partial f}{\partial x_1}(g(r,\varphi))\cos\varphi + \frac{\partial f}{\partial x_2}(g(r,\varphi))\sin\varphi\Big]\\
&= \Big[\frac{\partial^2 f}{\partial x_1\partial x_1}(g(r,\varphi))\frac{\partial g_1}{\partial r}(r,\varphi) + \frac{\partial^2 f}{\partial x_2\partial x_1}(g(r,\varphi))\frac{\partial g_2}{\partial r}(r,\varphi)\Big]\cos\varphi\\
&\quad + \Big[\frac{\partial^2 f}{\partial x_1\partial x_2}(g(r,\varphi))\frac{\partial g_1}{\partial r}(r,\varphi) + \frac{\partial^2 f}{\partial x_2\partial x_2}(g(r,\varphi))\frac{\partial g_2}{\partial r}(r,\varphi)\Big]\sin\varphi\\
&= \frac{\partial^2 f}{\partial x_1^2}(g(r,\varphi))\cos^2\varphi + \frac{\partial^2 f}{\partial x_2^2}(g(r,\varphi))\sin^2\varphi + 2\frac{\partial^2 f}{\partial x_1\partial x_2}(g(r,\varphi))\sin\varphi\cos\varphi;\\
\frac{\partial F}{\partial \varphi}(r,\varphi) &= \frac{\partial f}{\partial x_1}(g(r,\varphi))\,\frac{\partial g_1}{\partial \varphi}(r,\varphi) + \frac{\partial f}{\partial x_2}(g(r,\varphi))\,\frac{\partial g_2}{\partial \varphi}(r,\varphi)\\
&= \frac{\partial f}{\partial x_1}(g(r,\varphi))(-r\sin\varphi) + \frac{\partial f}{\partial x_2}(g(r,\varphi))(r\cos\varphi);\\
\frac{\partial^2 F}{\partial \varphi^2}(r,\varphi) &= \frac{\partial}{\partial \varphi}\Big[\frac{\partial f}{\partial x_1}(g(r,\varphi))(-r\sin\varphi) + \frac{\partial f}{\partial x_2}(g(r,\varphi))(r\cos\varphi)\Big]\\
&= \frac{\partial}{\partial \varphi}\Big[\frac{\partial f}{\partial x_1}(g(r,\varphi))\Big](-r\sin\varphi) + \frac{\partial f}{\partial x_1}(g(r,\varphi))\frac{\partial}{\partial \varphi}\big[(-r\sin\varphi\big]\\
&\quad + \frac{\partial}{\partial \varphi}\Big[\frac{\partial f}{\partial x_2}(g(r,\varphi))\Big](r\cos\varphi) + \frac{\partial f}{\partial x_2}(g(r,\varphi))\frac{\partial}{\partial \varphi}\big[r\cos\varphi)\big]\\
&= \Big[\frac{\partial^2 f}{\partial x_1\partial x_1}(g(r,\varphi))\frac{\partial g_1}{\partial \varphi}(r,\varphi) + \frac{\partial^2 f}{\partial x_1\partial x_2}(g(r,\varphi))\frac{\partial g_2}{\partial \varphi}(r,\varphi)\Big](-r\sin\varphi)\\
&\quad + \Big[\frac{\partial f}{\partial x_1}(g(r,\varphi))\Big](-r\cos\varphi)\\
&\quad + \Big[\frac{\partial^2 f}{\partial x_1\partial x_2}(g(r,\varphi))\frac{\partial g_1}{\partial \varphi}(r,\varphi) + \frac{\partial^2 f}{\partial x_2\partial x_2}(g(r,\varphi))\frac{\partial g_2}{\partial \varphi}(r,\varphi)\Big](r\cos\varphi)\\
&\quad + \Big[\frac{\partial f}{\partial x_2}(g(r,\varphi))\Big](-r\sin\varphi)\\
&= \Big[\frac{\partial^2 f}{\partial x_1^2}(g(r,\varphi))\Big]r^2\sin^2\varphi + \Big[\frac{\partial^2 f}{\partial x_2^2}(g(r,\varphi))\Big]r^2\cos^2\varphi\\
&\quad - (2r\sin\varphi\cos\varphi)\Big[\frac{\partial^2 f}{\partial x_1\partial x_2}(g(r,\varphi))\Big]\\
&\quad - \Big[(r\cos\varphi)\frac{\partial f}{\partial x_1}(g(r,\varphi)) + (r\sin\varphi)\frac{\partial f}{\partial x_2}(g(r,\varphi))\Big]
\end{aligned}$$

$$
\begin{aligned}
\Big(\frac{\partial^2 F}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2}+\frac{1}{r}\frac{\partial F}{\partial r}\Big)(r,\varphi) &= \Big[\frac{\partial^2 f}{\partial x_1^2}(g(r,\varphi))\cos^2\varphi+\frac{\partial^2 f}{\partial x_2^2}(g(r,\varphi))\sin^2\varphi \\
&\quad +2\frac{\partial^2 f}{\partial x_1\partial x_2}(g(r,\varphi))\sin\varphi\cos\varphi\Big] \\
&+\frac{1}{r^2}\Big\{\Big[\frac{\partial^2 f}{\partial x_1^2}(g(r,\varphi))\Big]r^2\sin^2\varphi+\Big[\frac{\partial^2 f}{\partial x_2^2}(g(r,\varphi))\Big]r^2\cos^2\varphi \\
&\quad -(2r\sin\varphi\cos\varphi)\Big[\frac{\partial^2 f}{\partial x_1\partial x_2}(g(r,\varphi))\Big] \\
&\quad -\Big[(r\cos\varphi)\frac{\partial f}{\partial x_1}(g(r,\varphi))+(r\sin\varphi)\frac{\partial f}{\partial x_2}(g(r,\varphi))\Big]\Big\} \\
&+\frac{1}{r}\Big\{\frac{\partial f}{\partial x_1}(g(r,\varphi))\cos\varphi+\frac{\partial f}{\partial x_2}(g(r,\varphi))\sin\varphi\Big\} \\
&=\frac{\partial^2 f}{\partial x_1^2}(g(r,\varphi))\big[\cos^2\varphi+\sin^2\varphi\big] \\
&\quad +\frac{\partial^2 f}{\partial x_2^2}(g(r,\varphi))\big[\sin^2\varphi+\cos^2\varphi\big] \\
&=\Big(\frac{\partial^2 f}{\partial x_1^2}+\frac{\partial^2 f}{\partial x_2^2}\Big)(g(r,\varphi))=(\Delta f)(r\cos\varphi,r\sin\varphi).
\end{aligned}
$$

[ Verwendet wurde (zweimal), daß (weil $f$ aus $C^2$ ist)

$$\frac{\partial^2 f}{\partial x_1\partial x_2}(g(r,\varphi))=\frac{\partial^2 f}{\partial x_2\partial x_1}(g(r,\varphi)). \quad ]$$

**Aufgabe 34: (4 Punkte)** Es sei $f:\mathbb{R}^2\setminus\{0\}\to\mathbb{R}$ gegeben durch

$$f(x,y):=x^2y^2\ln(x^2+y^2).$$

Es ist $(1,1,\ln 2)\in \mathrm{Graph}(f):=\{(x,y,f(x,y))\mid (x,y)\in\mathbb{R}^2\setminus\{0\}\}$.

a) Stellen die Tangentialebene an Graph($f$) in diesem Punkt als Graph einer Funktion $\ell:\mathbb{R}^2\to\mathbb{R}$ dar.

b) Approximieren Sie Graph($f$) in der Nähe von $(1,1,\ln 2)$ durch den Graph eines Polynoms 2. Grades in $x$ und $y$. (Hinweis: Taylor).

*Lösungsvorschlag:* a) Man braucht das Taylorpolynom 1. Ordnung:

$$
\begin{aligned}
\ell(x,y) &= f(1,1)+\frac{\partial f}{\partial x}(1,1)(x-1)+\frac{\partial f}{\partial x}(1,1)(y-1); \\
\frac{\partial f}{\partial x}(x,y) &= 2xy^2\ln(x^2+y^2)+x^2y^2\frac{2x}{x^2+y^2};\ \frac{\partial f}{\partial x}(1,1)=2\ln 2+1; \\
\frac{\partial f}{\partial y}(x,y) &= 2yx^2\ln(x^2+y^2)+x^2y^2\frac{2y}{x^2+y^2};\ \frac{\partial f}{\partial y}(1,1)=2\ln 2+1; \\
\ell(x,y) &= \ln 2+(2\ln 2+1)(x-1)+(2\ln 2+1)(y-1) \\
&= -3\ln 2-2+(2\ln 2+1)(x+y).
\end{aligned}
$$

[ $\ell$ ist eine Polynom 2. Grad in $(x,y)$, also ist Graph($\ell$) eine Ebene; $\ell(1,1)=f(1,1)$; $\frac{\partial \ell}{\partial x}(1,1)=\frac{\partial f}{\partial x}(1,1)$; $\frac{\partial \ell}{\partial y}(1,1)=\frac{\partial f}{\partial y}(1,1)$, und damit ist Graph($\ell$) die Tangentialebene am Graph($f$) im Punkte $(1,1,\ln 2)$. ]

b) Man braucht das Taylorpolynom 2. Ordnung:

$$
\begin{aligned}
(T_2(f))(x,y) &= f(1,1) + \frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial x}(1,1)(y-1)\\
&\quad + \frac{1}{2}\Big[\frac{\partial^2 f}{\partial x^2}(1,1)\Big](x-1)^2 + \frac{1}{2}\Big[\frac{\partial^2 f}{\partial y^2}(1,1)\Big](y-1)^2\\
&\quad + \Big[\frac{\partial^2 f}{\partial x\partial y}(1,1)\Big](x-1)(y-1)\\
&= \ell(x,y) + \frac{1}{2}\Big[\frac{\partial^2 f}{\partial x^2}(1,1)\Big](x-1)^2 + \frac{1}{2}\Big[\frac{\partial^2 f}{\partial y^2}(1,1)\Big](y-1)^2\\
&\quad + \Big[\frac{\partial^2 f}{\partial x\partial y}(1,1)\Big](x-1)(y-1);\\
\frac{\partial^2 f}{\partial x^2}(x,y) &= \frac{\partial}{\partial x}\Big[2xy^2\ln(x^2+y^2) + x^2y^2\frac{2x}{x^2+y^2}\Big]\\
&= 2y^2\ln(x^2+y^2) + \frac{10x^2y^2}{x^2+y^2} - \frac{4x^4y^2}{(x^2+y^2)^2}; \quad \frac{\partial^2 f}{\partial x^2}(1,1) = 2\ln 2 + 4;\\
\frac{\partial^2 f}{\partial y^2}(x,y) &= \frac{\partial}{\partial y}\Big[2x^2y\ln(x^2+y^2) + x^2y^2\frac{2y}{x^2+y^2}\Big]\\
&= 2x^2\ln(x^2+y^2) + \frac{10x^2y^2}{x^2+y^2} - \frac{4x^2y^4}{(x^2+y^2)^2}; \quad \frac{\partial^2 f}{\partial y^2}(1,1) = 2\ln 2 + 4;\\
\frac{\partial^2 f}{\partial y\partial x}(x,y) &= \frac{\partial}{\partial y}\Big[2xy^2\ln(x^2+y^2) + x^2y^2\frac{2x}{x^2+y^2}\Big]\\
&= 4xy\ln(x^2+y^2) + \frac{4(xy^3+x^3y)}{x^2+y^2} - \frac{4x^3y^3}{(x^2+y^2)^2};\\
\frac{\partial^2 f}{\partial y\partial x}(1,1) &= 4\ln 2 + 3;\\
(T_2(f))(x,y) &= -3\ln 2 - 2 + (2\ln 2 + 1)(x+y) + \frac{1}{2}(2\ln 2 + 4)(x-1)^2\\
&\quad + \frac{1}{2}(2\ln 2 + 4)(y-1)^2 + (4\ln 2 + 3)(x-1)(y-1)\\
&= (2+\ln 2)(x^2+y^2) + (4\ln 2 + 3)xy - (4\ln 2 + 6)(x+y) + (3\ln 2 + 5)
\end{aligned}
$$

[ Verwendet wurde, daß (weil $f$ aus $C^2$ ist)

$$\frac{\partial^2 f}{\partial x\partial y}(x,y) = \frac{\partial^2 f}{\partial y\partial x}(x,y).$$

Man sieht dieses auch durch direktes Ausrechnen. ]

Für Grafen von $\ell$ und $T_2(f)$, siehe PS-Datei auf der Homepage.

**Aufgabe 35: (4 Punkte)** Eine Funktion $u : \mathbb{R}^2 \supset U \to \mathbb{R}$ heißt *harmonisch*, falls $u$ aus $C^2(U)$ ist und die Gleichung $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ erfüllt ist.

Sei $f = u + iv : \mathbb{C} \supset U \to \mathbb{C}$ *komplex* differenzierbar, und $u, v \in C^2(U, \mathbb{R})$ (hier interpretieren wir $u(x+iy) = u(x,y), v(x+iy) = v(x,y)$). Zeigen Sie, daß $u$ und $v$ harmonisch sind.
*Lösungsvorschlag:* Man hat, daß

$$
\begin{aligned}
&\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\\
&\quad = \lim_{(s,t)\to(0,0)}\frac{\big[u(x_0+s,y_0+t) + iv(x_0+s,y_0+t)\big] - \big[u(x_0,y_0) + iv(x_0,y_0)\big]}{s+it}
\end{aligned}
$$

existiert für alle $z_0 = x_0 + iy_0 \in U$ (Analysis I). Insbesonders muss $((s,t) = (r,0))$

$$\begin{aligned}
&\lim_{r\to 0} \frac{\big[u(x_0+r,y_0) + iv(x_0+r,y_0)\big] - \big[u(x_0,y_0) + iv(x_0,y_0)\big]}{r} \\
&\qquad = \lim_{r\to 0} \frac{u(x_0+r,y_0) - u(x_0,y_0)}{r} + i \lim_{r\to 0} \frac{v(x_0+r,y_0) - v(x_0,y_0)}{r} \\
&\qquad = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)
\end{aligned}$$

gleich $((s,t) = (0,\rho))$

$$\begin{aligned}
&\lim_{\rho\to 0} \frac{\big[u(x_0,y_0+\rho) + iv(x_0,y_0+\rho)\big] - \big[u(x_0,y_0) + iv(x_0,y_0)\big]}{i\rho} \\
&\qquad = \frac{1}{i} \lim_{\rho\to 0} \frac{u(x_0,y_0+\rho) - u(x_0,y_0)}{r} + \lim_{\rho\to 0} \frac{v(x_0,y_0+\rho) - v(x_0,y_0)}{\rho} \\
&\qquad = -i\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0)
\end{aligned}$$

sein. (Alle Grenzwerten existieren, weil $u, v$ aus $C^2$ sind). Weil $u, v$ reellwertige sind, folgt

$$\frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial y}{\partial y}(x_0,y_0), \tag{1}$$

$$\frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0). \tag{2}$$

Aus (1) (und daß $u, v$ aus $C^2$ sind) folgt, daß

$$\frac{\partial^2 u}{\partial y \partial x}(x_0,y_0) = \frac{\partial^2 v}{\partial y^2}(x_0,y_0)$$

und aus (2)

$$\frac{\partial^2 u}{\partial x \partial y}(x_0,y_0) = -\frac{\partial^2 v}{\partial x^2}(x_0,y_0).$$

Aber weil $u, v$ aus $C^2$ sind, gilt $\frac{\partial^2 u}{\partial x \partial y}(x_0,y_0) = \frac{\partial^2 u}{\partial y \partial x}(x_0,y_0)$, und damit, daß

$$\frac{\partial^2 v}{\partial y^2}(x_0,y_0) = -\frac{\partial^2 v}{\partial x^2}(x_0,y_0)$$

d.h.

$$(\Delta v)(x_0,y_0) = \frac{\partial^2 v}{\partial x^2}(x_0,y_0) + \frac{\partial^2 v}{\partial y^2}(x_0,y_0) = 0.$$

Ähnlich folgt, daß $(\Delta u)(x_0,y_0) = 0$.

**Aufgabe 36: (4 Punkte)** Sei $f : \mathbb{R}^m \to \mathbb{R}^n$ aus $C^1(\mathbb{R}^m)$.

a) Zeigen Sie, daß

$$f(x) - f(y) = \int_0^1 \big[f'(x + t(y-x))\big](y-x)\, dt$$

für alle $x, y \in \mathbb{R}^m$. (Hinweis: Kettenregel).

b) Sei $\|A\|_2 := \sqrt{\sum_{i=1}^n \sum_{j=1}^m |a_{ij}|^2}$, $A = (a_{ij}) \in \mathbb{R}^{n\times m}$. Zeigen Sie, daß

$$|Ax| \leq \|A\|_2 |x|$$

für all $x \in \mathbb{R}^m$.

c) Zeigen Sie, daß

$$\big|f(x) - f(y)\big| \leq C|x-y| \quad \text{mit} \quad C := \sup_{t\in[0,1]} \|f'(x+t(y-x))\|_2$$

für alle $x, y \in \mathbb{R}^m$.

*Lösungsvorschlag:* a) Die Funktion $g : \mathbb{R} \to \mathbb{R}^n$ definiert durch $g(t) = f(x + t(y-x))$ ist aus $C^1(\mathbb{R}$ (weil $h(t) = t(y-x)$ aus $C^1(\mathbb{R})$ ist, und $g = f \circ h$; Kettenregel). Damit ist

$$g(1) - g(0) = \int_0^1 g'(t)\,dt.$$

Nach der Kettenregel ist $g'(t) = f'(h(t)) \circ h'(t)$. Es gilt $h'(t) = y - x$ und damit

$$g'(t) = f'(h(t)) \circ h'(t) = f'(x + t(y-x)) \cdot (y-x).$$

Ausserdem ist $g(1) = f(h(1)) = f(x+1\cdot(y-x)) = f(y)$, $g(0) = f(h(0)) = f(x+0\cdot(y-x)) = f(x)$, d.h.,

$$f(y) - f(x) = g(1) - g(0) = \int_0^1 g'(t)\,dt = \int_0^1 f'(x+t(y-x))(y-x)\,dt.$$

[ Man bemerke den Druckfehler in der Aufgabeformulierung. ]

b)

$$\begin{aligned}
|Ax|^2 &= \Big|\big(\sum_{j=1}^m a_{1j}x_j, \ldots, \sum_{j=1}^m a_{nj}x_j\big)\Big|^2 = \sum_{i=1}^n \big(\sum_{j=1}^m a_{ij}x_j\big)^2 \\
&\leq \sum_{i=1}^n \big(\sum_{j=1}^m |a_{ij}|^2\big)\big(\sum_{j=1}^m {x_j}^2\big) \\
&\Big[\text{ weil (Cauchy-Schwarz) } \big(\sum_{j=1}^m a_{ij}x_j\big)^2 = \big|\langle a_{i\bullet}, x\rangle\big|^2 \leq |a_{i\bullet}|^2|x|^2 = \big(\sum_{j=1}^m |a_{ij}|^2\big)\big(\sum_{j=1}^m x_j^2\big) \Big] \\
&= \big(\sum_{j=1}^m {x_j}^2\big)\big(\sum_{i=1}^n\sum_{j=1}^m |a_{ij}|^2\big) = \|A\|_2|x|^2.
\end{aligned}$$

c)

$$\begin{aligned}
|f(x) - f(y)| = |f(y) - f(x)| &= \Big|\int_0^1 f'(x+t(y-x))(y-x)\,dt\Big| \\
&\leq \int_0^1 \big|f'(x+t(y-x))(y-x)\big|dt \\
&\leq \int_0^1 \|f'(x+t(y-x))\|_2|y-x|dt \\
&\leq |y-x| \int_0^1 \sup_{t\in[0,1]} \|f'(x+t(y-x))\|_2\,dt \\
&= |y-x| \sup_{t\in[0,1]} \|f'(x+t(y-x))\|_2 \int_0^1 1\,dt \\
&= C|x-y| \quad \text{mit} \quad C := \sup_{t\in[0,1]} \|f'(x+t(y-x))\|_2.
\end{aligned}$$

**Abgabe bis Montag 20.06.2005, 11.15 Uhr in den MPIIA Übungskasten im 1. Stock vor der Bibliothek.**
**Unter** `http://www.mathematik.uni-muenchen.de/∼sorensen` **sind die Blätter im Internet abrufbar.**

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