MATHEMATISCHES INSTITUT
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Übungen zur Vorlesung Mathematik für Naturwissenschaftler II

Lösungen

Aufgabe 1: a) $\frac{\partial f}{\partial x}(x,y) = y^2 \frac{\partial}{\partial x}\big[(x-1)^3\big] = y^2 \cdot 3(x-1)^2 = 3y^2(x-1)^2$.
$\frac{\partial f}{\partial y}(x,y) = (x-1)^3 \frac{\partial}{\partial y}[y^2] = 2y(x-1)^3$.
b) $\frac{\partial f}{\partial x}(x,y) = \frac{1}{y^2+1}\frac{\partial}{\partial x}\big[x^2+y\big] = \frac{2x}{y^2+1}$.
$\frac{\partial f}{\partial y}(x,y) = \frac{y^2+1\frac{\partial}{\partial y}[x^2+y]-(x^2+y)\frac{\partial}{\partial y}[y^2+1]}{(y^2+1)^2} = \frac{-y^2-2yx^2+1}{(y^2+1)^2}$.
c) $\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}[x^3]\cdot\sin(xy) + x^3\frac{\partial}{\partial x}[\sin(xy)] = 3x^2\sin(xy) + yx^3\cos(xy)$.
$\frac{\partial f}{\partial y}(x,y) = x^3\frac{\partial}{\partial y}[\sin(xy)] = x^4\cos(xy)$.

Aufgabe 2: $\frac{\partial f}{\partial x}(x,y) = 3x^2y + e^{y^2}$; $\frac{\partial f}{\partial y}(x,y) = x^3 + 2xye^{y^2}$; $\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}[3x^2y + e^{y^2}] = 6xy$; $\frac{\partial^2 f}{\partial y\partial x}(x,y) = \frac{\partial}{\partial y}[3x^2y + e^{y^2}] = 3x^2 + 2ye^{y^2}$; $\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}[x^3 + 2xye^{y^2}] = 2x(1+2y^2)e^{y^2}$; $\frac{\partial^2 f}{\partial x\partial y}(x,y) = \frac{\partial}{\partial x}[x^3 + 2xye^{y^2}] = 3x^2 + 2ye^{y^2}$ $(= \frac{\partial^2 f}{\partial y\partial x}(x,y))$. Alle die ausgerechnete partielle Ableitungen sind in ganz $\mathbb{R}^2$ *stetig*: Die sind alle von stetigen Funktionen zusammengesetzt. Die Funktionen $(x,y)\mapsto x$ und $(x,y)\mapsto y$ sind stetig, Produckten und Summen von stetigen Funtkionen sind stetig, und $t\mapsto e^t$ ist stetig. Verknüpfungen von stetigen Funktionen sind auch stetig.

Aufgabe 3: $\frac{\partial f}{\partial x}(x,y) = yx^{y-1}$; $\frac{\partial^2 f}{\partial x^2}(x,y) = y(y-1)x^{y-2}$. Verwende jetzt daß

$$\frac{d}{dt}(a^t) = \frac{d}{dt}(e^{t\ln(a)}) = \ln(a)\cdot e^{t\ln(a)} = a^t\ln(a),$$

dann: $\frac{\partial f}{\partial y}(x,y) = x^y\ln(x)$; $\frac{\partial^2 f}{\partial y^2}(x,y) = x^y(\ln(x))^2$; $\frac{\partial^2 f}{\partial x\partial y}(x,y) = \frac{\partial}{\partial x}(x^y\ln(x)) = yx^{y-1}\ln(x) + x^y\frac{1}{x} = (y\ln(x)+1)x^{y-1}$; $\frac{\partial^2 f}{\partial y\partial x}(x,y) = \frac{\partial}{\partial y}(yx^{y-1}) = x^{y-1} + y\frac{\partial}{\partial y}[x^{y-1}] = x^{y-1} + y\frac{1}{x}\frac{\partial}{\partial y}(x^y) = x^{y-1} + \frac{y}{x}x^y\ln(x) = (y\ln(x)+1)x^{y-1}$ $(= \frac{\partial^2 f}{\partial x\partial y}(x,y))$.

Aufgabe 4: Für $(x,y)\neq(0,0)$:

$$\frac{\partial f}{\partial x}(x,y) = y\frac{x^2-y^2}{x^2+y^2} + xy\frac{2x(x^2+y^2)-2x(x^2-y^2)}{(x^2+y^2)^2} = \frac{yx^4-y^5+4x^2y^3}{(x^2+y^2)^2},$$
$$\frac{\partial f}{\partial y}(x,y) = x\frac{x^2-y^2}{x^2+y^2} + xy\frac{-2y(x^2+y^2)-2y(x^2-y^2)}{(x^2+y^2)^2} = \frac{x^5-xy^4-4y^2x^3}{(x^2+y^2)^2},$$
$$\frac{\partial^2 f}{\partial y\partial x}(x,y) = \frac{\partial}{\partial y}\Big[\frac{yx^4-y^5+4x^2y^3}{(x^2+y^2)^2}\Big] = \frac{(x^4-5y^4+12x^2y^2)(x^2+y^2)^2}{(x^2+y^2)^4} - \frac{(yx^4-y^5+4x^2y^3)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4} = \frac{x^6-y^6+9x^4y^2-9x^2y^4}{(x^2+y^2)^3},$$
$$\frac{\partial^2 f}{\partial x\partial y}(x,y) = \frac{\partial}{\partial x}\Big[\frac{x^5-xy^4-4y^2x^3}{(x^2+y^2)^2}\Big] = \frac{(5x^4-y^4-12y^2x^2)(x^2+y^2)}{(x^2+y^2)^4} - \frac{(x^5-xy^4-4y^2x^3)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4} = \frac{x^6-y^6+9x^4y^2-9x^2y^4}{(x^2+y^2)^3}.$$

Im Punkt $(0,0)$ müssen wir die Definition verwenden:

$$\begin{aligned}
\frac{\partial f}{\partial x}(0,0) &:= \lim_{x\to 0, x\neq 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0, x\neq 0} \frac{x\cdot 0\cdot \frac{x^2-0^2}{x^2+0^2}-0}{x-0} = \lim_{x\to 0, x\neq 0} 0 = 0,\\
\frac{\partial f}{\partial y}(0,0) &:= \lim_{y\to 0, y\neq 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{x\to 0, x\neq 0} 0 = 0,\\
\frac{\partial^2 f}{\partial y \partial x}(0,0) &:= \lim_{y\to 0, y\neq 0} \frac{\frac{\partial f}{\partial x}(0,y)-\frac{\partial f}{\partial x}(0,0)}{y-0} = \lim_{y\to 0, y\neq 0} \frac{\frac{y\cdot 0^4-y^5+4\cdot 0^2\cdot y^2}{(0^2+y^2)^2}-0}{y-0}\\
&= \lim_{y\to 0, y\neq 0} \frac{\frac{-y^5}{y^4}}{y} = \lim_{y\to 0, y\neq 0} (-1) = -1,\\
\frac{\partial^2 f}{\partial x \partial y}(0,0) &:= \lim_{x\to 0, x\neq 0} \frac{\frac{\partial f}{\partial y}(x,0)-\frac{\partial f}{\partial y}(0,0)}{x-0} = \lim_{x\to 0, x\neq 0} \frac{\frac{x^5}{x^4}}{x} = \lim_{x\to 0, x\neq 0} 1 = 1.
\end{aligned}$$

Das heißt, $\frac{\partial^2 f}{\partial x \partial y}$ und $\frac{\partial^2 f}{\partial y \partial x}$ existieren in ganz $\mathbb{R}^2$, und

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y}(x,y) &= \begin{cases} \frac{x^6-y^6+9x^4y^2-9x^2y^4}{(x^2+y^2)^3}\ , & (x,y)\neq(0,0),\\ 1\ , & (x,y)=(0,0)\end{cases}\\
\frac{\partial^2 f}{\partial y \partial x}(x,y) &= \begin{cases} \frac{x^6-y^6+9x^4y^2-9x^2y^4}{(x^2+y^2)^3}\ , & (x,y)\neq(0,0),\\ -1\ , & (x,y)=(0,0).\end{cases}
\end{aligned}$$

Also ist $f_{xy}(x^*,y^*) = f_{yx}(x^*,y^*)$ für alle $(x^*,y^*)\in\mathbb{R}^2\setminus\{(0,0)\}$.

Sprechstunden :

Prof. Dr. W. Richert, Mo. $15^{00} - 16^{00}$ Uhr, Zi. 333.
Dr. T. Ø. Sørensen, Do. $12^{00} - 13^{00}$ Uhr, Zi. 335.
I. Hoffmann, Mo. $12^{00} - 13^{00}$ Uhr, Zi. 235.