③ $n \in \mathbb{N}$ , $x_0, x_1, \dots, x_n \in \mathbb{R}$

$$U := \begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \vdots & & & & \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

(or $V_n$)

Claim: $\det U = \prod_{\substack{i,j=0 \\ i>j}}^{n} (x_i - x_j)$

Pf: By induction on $n$.

Base case $n=1$: $\det U = \det \begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix} = x_1 - x_0$ ✓

Also note that if $x_i = x_j$ for some $i \neq j$, then $\det U = 0$ (since 2 rows are identical).

Then also $\prod_{\substack{i,j=0 \\ i>j}}^{n} (x_i - x_j) = 0$ (zero in product)

So suppose all the $x_i$ are distinct

Induction Step: Suppose $n > 1$ and that for all $1 \leq k < n$ (in particular $k = n-1$),

$$\det \underbrace{\begin{pmatrix} 1 & x_0 & \dots & x_0^k \\ \vdots & & & \\ \vdots & & & \\ 1 & x_k & \dots & x_k^k \end{pmatrix}}_{"V_k"} = \prod_{\substack{i,j=0 \\ i>j}}^{n} (x_i - x_j)$$

~~Write~~ Now: Treat $x_0$ as a variable (and $x_1 \dots x_n$ as constants) and write

$\det(V_n)$ as a polynomial $f(x_0)$ in $x_0$.

This has degree at most $n$, since

$$\det(V_n) = \det\begin{pmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & & & \\ 1 & x_n & \cdots & x_n^n \end{pmatrix} = 1\det U_0 - x_0 \det U_1 + x_0^2 \det U_2 \cdots + (-1)^{n+2} x_0^{n} \det U_n \quad (1)$$

where all matrices $U_i$ do not contain $x_0$ terms.

Now note that again that if $x_0$ (variable!) $= x_i$ for any $1 \le i \le n$

Then $f(x_i) = \det\Big( \quad \Big\}$ repeated row $= 0$.

So $(x_i - x_0)$ are all factors of $f$!

So $f(x_0) = C(x_1 - x_0)(x_2 - x_0)\ldots(x_n - x_0)$ $(*)$

$(x_1 - x_0)\cdots(x_n - x_0)$ is a polynomial of degree $n$ with leading term $(-1)^n x_0^n$

So $C$ cannot contain any more $x_0$ terms (it's "constant") : depends only on $x_1, \ldots, x_n$).

So we have, from $(*)$,

$\det V_n = f(x_0) = (-1)^n C x_0^n$ + lower order terms of $x_0$

But also, we can use Laplace expansion of determinant in (1) ~~along the first column (or row)~~ ~~of $V_n$~~ to see that

$$\det V_n = (-1)^{n+2} x_0^n \det \underbrace{\begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & & & \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix}}_{U_n} + \text{lower order terms of } x_0$$

Hence $C = \det\begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & & & \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{\substack{i,j=1 \\ i>j}}^{n} (x_i - x_j)$ induction hypothesis

Substitute this expression for $C$ into $(*)$, and we have proved the ~~claim~~. inductive step, and hence the claim. $\square$