PDG I
(Tutorium)

Tutorial 7

(Energy methods and continuity of Green functions)

Question 1

Let $\Omega \subset \mathbb{R}^n$ be open and bounded, with C^1 boundary. Consider the *variational integral*

$$F(v) := \frac{1}{p} \int_\Omega |\nabla v(x)|^p \, \mathrm{d}x \, .$$

Let $g \in C(\partial\Omega)$ and suppose that $u \in C^2(\overline{\Omega})$ is a minimiser of F within the set

$$\{v \in C^1(\overline{\Omega}) : v = g \text{ on } \partial\Omega\} \, .$$

Prove that this minimiser u solves the *p-Laplacian equation*

$$\begin{cases} \operatorname{div}\big(|\nabla u(x)|^{p-2} \nabla u(x)\big) = 0 & x \in \Omega \\ u(x) = g(x) & x \in \partial\Omega \, . \end{cases}$$

Question 2

Let $\Omega \subset \mathbb{R}^n$ be open and bounded.

(a) Recall that for $y \in \Omega$, the function φ^y $(= \varphi^y(x))$ is defined to solve

$$\begin{cases} \Delta\varphi^y(x) = 0 & y \in \Omega \\ \varphi^y(x) = \Phi(y - x) & y \in \partial\Omega \, , \end{cases}$$

where Φ is the Fundamental solution of the Laplace equation. Recall also that for fixed y, $x \mapsto \varphi^y(x)$ is in $C(\overline{\Omega})$.

Let $\Omega_0 \subset\subset \Omega$. Show that for fixed $x \in \Omega$, $y \mapsto \varphi^y(x)$ is in *uniformly continuous* in Ω_0.

(b) Deduce that the map $(x, y) \mapsto \varphi^y(x)$ is in $C(\overline{\Omega} \times \Omega)$.