PDG I
(Zentralübung)

Problem Sheet 10

Question 1

Assume that for some attenuation function $\alpha = \alpha(r)$ and delay function $\beta = \beta(r) \geq 0$, there exist for *all* profiles φ solutions of the wave equation in $\mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$ having the form

$$u(t, x) = \alpha(r)\varphi(t - \beta(r)) .$$

Here $r = |x|$ and we assume $\beta(0) = 0$. Show that this is possible only if $n = 1$ or $n = 3$, and compute the form of the functions α, β.

Question 2

Let $g = \mathrm{diag}(-1, 1, 1, 1) \in \mathbb{R}^{4\times4}$. A real 4×4 matrix $\Lambda \in \mathbb{R}^{4\times4}$ is called a *Lorentz transformation* if and only if $\Lambda^T g\Lambda = g$, where Λ^T denotes the transpose of Λ.

(a) Show that the product of two Lorentz transformations is also a Lorentz transformation.

(b) Show that every Lorentz transformation is invertible, and that its inverse is also a Lorentz transformation. (Hence the set of all Lorentz transformations is a group.)

(c) Define the quadratic form $\langle x, y\rangle_g := x^T gy$, for x, $y \in \mathbb{R}^4$. Show that for every Lorentz transformation Λ we have $\langle \Lambda x, \Lambda y\rangle_g = \langle x, y\rangle_g$.

(d) Show that the following are Lorentz transformations (where $(t, x) \in \mathbb{R}^4$, with $t \in \mathbb{R}$, $x \in \mathbb{R}^3$):

(i) $(t, x) \mapsto (t, Ox)$, where O is an othogonal transformation of $\mathbb{R}^3$,

(ii) $(t, x) \mapsto (-t, x)$,

(iii) $(t, x) \mapsto \left(\frac{t-ax_1}{\sqrt{1-a^2}}, \frac{x_1-at}{\sqrt{1-a^2}}, x_2, x_3\right)$, where $0 < a < 1$.
(This transformation is called a *Lorentz boost*.)

(e) Show that the wave equation is Lorentz-covariant. That is, if u is a solution to the wave equation $u_{tt} - \Delta u = 0$ in $\mathbb{R}^4$, then for every Lorentz transformation Λ, the function $v(t, x) := u(\Lambda(t, x))$ is also a solution.

Question 3

Let $S := \{(\varphi(x), x) : x \in \mathbb{R}^3\} \subset \mathbb{R}^4$ be a smooth hypersurface (i.e. $\varphi \in C^\infty(\mathbb{R}^3)$). The *Cauchy problem for the wave equation with initial surface* S is:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^4 \\ u = g \quad u_t = h & \text{on } S\,. \end{cases}$$

We say that S is *space-like* if $1 - |\nabla\varphi|^2 > 0$ on $\mathbb{R}^3$.

Show that the Cauchy problem for the wave equation with the space-like initial surface $S = \{(t,x) \in \mathbb{R}^4 : t = ax_1\}$, $0 < a < 1$, is equivalent to the initial-value problem (i.e. when $S = \{(t,x) \in \mathbb{R}^4 : t = 0\}$). *Hint:* Use a Lorentz transformation.

Holiday Problems

(Not for handing in: we will go through this in the first tutorials of the new year)

Question 1 (Vitali's Covering Lemma)

Let $E \subset \mathbb{R}^n$ be the union of a finite number of balls $B(x_i, r_i)$, $i = 1, 2 \ldots k$. Show that there exists a subset $I \subset \{1, \ldots k\}$ such that the balls $B(x_i, r_i)$ with $i \in I$ are pairwise disjoint (that is, $B(x_i, r_i) \cap B(x_j, r_j) =$ whenever $i, j \in I$ and $i \neq j$), and

$$E \subset \bigcup_{i \in I} B(x_i, 3r_i)\,.$$

Question 2 (A "weak-type" inequality for the Hardy-Littlewood Maximal Function)

Let $u \in L^1(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$, define the *Maximal Function* Mu as

$$(Mu)(x) := \sup_{r>0} \frac{1}{\mathscr{L}^n(B(x,r))} \int_{B(x,r)} |u(y)|\,\mathrm{d}y \quad \left(= \sup_{r>0} ⨍_{B(x,r)} |u(y)|\,\mathrm{d}y \right),$$

where $\mathscr{L}^n(\cdot)$ denotes the n-dimensional Lebesgue measure/volume (so $\mathscr{L}^n(B(x,r)) = r^n\alpha_n$). You may assume this is a measurable function. Let $t \in (0, \infty)$. Show that

$$\mathscr{L}^n\big(\{x \in \mathbb{R}^n : (Mu)(x) > t\}\big) \leq \frac{3^n}{t} \int_{\mathbb{R}^n} |u(y)|\,\mathrm{d}y\,. \tag{1}$$

To do this, use Vitali's Covering Lemma from Question 1. You may also use the fact that Lebesgue Measure is "inner regular" i.e. for any Lebesgue-measurable set E,

$$\mathscr{L}^n(E) = \sup\{\,\mathscr{L}^n(K) : K \subset E \text{ and } K \text{ compact}\}\,.$$

This means that it suffices to prove the estimate

$$\mathscr{L}^n(K) \leq \frac{3^n}{t} \int_{\mathbb{R}^n} |u(y)|\,\mathrm{d}y$$

for any compact set $K \subset \{x \in \mathbb{R}^n : (Mu)(x) > t\}$.

Question 3 (Lebesgue's Differentiation Theorem)

In this quetion we shall prove *Lebesgue's Differentiation Theorem*: i.e. if $u \in L^1(\mathbb{R}^n)$ then for almost all $x \in \mathbb{R}^n$,

$$\limsup_{r \searrow 0} \fint_{B(x,r)} |u(y) - u(x)| \, \mathrm{d}y = 0 \,. \tag{2}$$

Recall that if u is continuous, then this holds for all $x \in \mathbb{R}^n$ (in one dimension this is just the Fundamental Theorem of Calculus!) Our aim is to extend such a result to integrable functions.

(i) Let $u \in L^1(\mathbb{R}^n)$, and let $\varphi \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$. Fix $x \in \mathbb{R}^n$ and show that

$$\limsup_{r \searrow 0} \fint_{B(x,r)} |u(y) - u(x)| \, \mathrm{d}y \leq M(|u - \varphi|)(x) + |u(x) - \varphi(x)|$$

where $M(|u - \varphi|)$ is the Maximal Function of $(u - \varphi)$ at x, defined as in Question 2.

(ii) Let $\epsilon > 0$ and observe that the previous part implies that, for fixed x, if

$$\limsup_{r \searrow 0} \fint_{B(x,r)} |u(y) - u(x)| \, \mathrm{d}y > \epsilon \,,$$

then

$$M(|u - \varphi|)(x) > \tfrac{\epsilon}{2} \text{ or } |u(x) - \varphi(x)| > \tfrac{\epsilon}{2} \,.$$

Using the Maximal inequality (1) from Question 2 and the density of continuous functions in $L^1(\Omega)$ (in the $\| \cdot \|_1$ norm topology), deduce

$$\mathscr{L}^n\left(\left\{x : \limsup_{r \searrow 0} \fint_{B(x,r)} |u(y) - u(x)| \, \mathrm{d}y > \epsilon\right\}\right) = 0 \,,$$

and then use this to establish (2).

Deadline for handing in: 0800 Wednesday 7 January

Please put solutions in Box 17, 1st floor (near the library)

Homepage: `http://www.mathematik.uni-muenchen.de/~soneji/pde1.php`