Lower semicontinuity and relaxation in BV of integrals with superlinear growth

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Abstract

Throughout this thesis, we shall consider the variational integral

$$F(u;\Omega) := \int_\Omega f(\nabla u(x))\,\mathrm{d}x\,,$$

for continuous functions $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfying the growth condition $0 \le f(\xi) \le L(1+|\xi|^r)$ for some exponent r.

The first main new result of this thesis is a lower semicontinuity result in BV in the case where f is quasiconvex and $r \in (1,2)$. The key steps in this proof involve obtaining boundedness properties for an extension operator, and a precise blow-up technique that makes use of a specific approximate differentiability property of Sobolev maps.

When u is a BV function, we extend the definition of $F(u;\Omega)$ by introducing the functional

$$\mathscr{F}_{\mathrm{loc}}(u,\Omega) := \inf_{(u_j)} \left\{ \liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \;\middle|\; \begin{array}{l} (u_j) \subset W^{1,r}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \\ u_j \overset{*}{\rightharpoonup} u \text{ in } \mathrm{BV}(\Omega,\mathbb{R}^N) \end{array} \right\}.$$

The second principal contribution is to adapt a result of Fonseca and Malý [50] to show that when $r \in [1, \frac{n}{n-1})$, $\mathscr{F}_{\mathrm{loc}}$ has a measure representation. We also show that the functional $\mathscr{F}$, defined similarly but requiring maps (u_j) to be in $W^{1,r}(\Omega;\mathbb{R}^N)$, has a weak measure representation.

Lastly, also for $r \in [1, \frac{n}{n-1})$, we prove that $\mathscr{F}_{\mathrm{loc}}$ satisfies the lower bound

$$\mathscr{F}_{\mathrm{loc}}(u,\Omega) \ge \int_\Omega f(\nabla u(x))\,\mathrm{d}x + \int_\Omega f_\infty\left(\frac{D^s u}{|D^s u|}\right)|D^s u|\,,$$

provided f is quasiconvex, and the recession function f_∞ (defined as $f_\infty(\xi) := \overline{\lim}_{t\to\infty} f(t\xi)/t$) is assumed to be finite in certain rank-one directions. This result is a natural extension of work by Ambrosio and Dal Maso [12], which deals with the case $r = 1$; it involves combining a lower semicontinuity result of Kristensen [64] and a technique of Braides and Coscia [22] with another new blow-up technique that exploits fine properties of BV functions.

Contents

Chapter 1

Introduction

We consider the variational integral

$$F(u;\Omega) := \int_\Omega f(\nabla u(x))\,\mathrm{d}x\,, \tag{1.1}$$

where Ω is a bounded, open subset of $\mathbb{R}^n$, $u\colon \Omega \to \mathbb{R}^N$ is a vector-valued function, ∇u denotes the Jacobian matrix of u and f is a non-negative continuous function defined in the space $\mathbb{R}^{N\times n}$ of all real $N \times n$ matrices.

We are specifically interested in the minimisation problem

$$m := \inf\{F(v;\Omega) : v \in \mathscr{A}\}\,,$$

where $\mathscr{A}$ is a space of admissible maps, defined here as the set of functions v in some function space X satisfying the *Dirichlet Boundary Condition*

$$v = g \text{ on } \partial\Omega\,,$$

which is made precise in terms of traces when the underlying space is a Sobolev Space or, more generally, the space of functions of Bounded Variation. Hence we wish to find $u \in \mathscr{A}$ satisfying

$$m = F(u;\Omega) \leq F(v;\Omega) \text{ for all } v \in \mathscr{A}\,.$$

The essence of the *Direct Method in the Calculus of Variations* is, under carefully chosen conditions, to take sequences of functions that approximate the infimum m of the functional F, and then exploit compactness properties of X and lower semicontinuity properties of the functional to show that there exists such a minimiser. That is, we first take a sequence (u_j) in X satisfying

$$F(u_j;\Omega) \to m\,.$$

We then aim to show that conditions on f and properties of the space X allows us to deduce that (taking a subsequence if necessary) the sequence (u_j) converges in some sense to a limit map u. If we can then show that F is *lower semicontinuous* with respect to this convergence, namely that

$$\liminf_{j\to\infty} F(u_j;\Omega) \geq F(u;\Omega)\,,$$

then this allows us to conclude that u is a minimiser of F on $\mathscr{A}$. This procedure is known as the Direct Method.

Systematic study of this existence problem dates back to Hilbert's 20th problem, and has a long and rich history, with contributions from many prominent researchers. In fact, for the scalar case $n = N = 1$ the study of the Calculus of Variations is even older, dating back to the beginnings of infinitesimal calculus. Key results in this setting are attributable to Euler, Weierstrass and Tonelli: here, they were not so concerned with existence, but just more-or-less assumed it. In this case, the following method (which we merely outline in general terms) plays a fundamental role: we assume that the integrand f is C^1 and that a minimiser u of F exists. We then set

$$g(t) := F(u + t\phi, \Omega)\,,$$

where $t \in \mathbb{R}$ and ϕ is a function equal to zero on $\partial\Omega$. Since u is a minimiser, the function g has a minimum at $t = 0$ and hence satisfies $g'(0) = 0$. We then compute g' by differentiating under the integral sign (provided this is permissible), obtaining a differential equation, which is known as the *Euler-Lagrange Equation*. A necessary condition for u to minimise F is therefore that it is a solution to such an equation; moreover, such a technique is useful when one wants to compute an explicit solution to the minimisation problem. For a comprehensive introduction to the Calculus of Variations in the one-dimensional case, we refer to [26].

In the vectorial case, matters become more complicated: the existence of minimisers depends even more strongly on our choice of X and conditions on the integrand f. In particular, it is too limiting to allow X to be a subspace of $C^1(\Omega;\mathbb{R}^N)$, and the strategy of considering the corresponding Euler-Lagrange equations with partial derivatives is difficult to implement. Hence the Direct Method becomes more important: however, in order to apply this method effectively, we need to enlarge the space of admissible functions to include those that are only weakly differentiable. This is a notion that lends itself much more readily to the techniques of functional analysis, for which compactness properties may be properly exploited. Indeed, the modern theory of the Calculus of Variations arose following the groundbreaking work of Morrey in the 50s and 60s, originally set in the context of the Sobolev Space $W^{1,\infty}(\Omega;\mathbb{R}^N)$. In subsequent decades, Morrey's results have been improved in numerous directions, for instance to cover the Sobolev Spaces $W^{1,r}(\Omega;\mathbb{R}^N)$ for $1 \leq r \leq +\infty$ and, more recently, the space of functions of Bounded Variation $\mathrm{BV}(\Omega;\mathbb{R}^N)$.

Another important issue in the Calculus of Variations is the regularity of minimisers. This is the subject of Hilbert's 19th problem and involves showing that the minimisers of such functionals have better properties than generic admissible maps. The extent of this topic is considerable, and beyond the scope of this present work: for details and references we may, for example, refer to the book of Giusti [59].

Remaining in the Sobolev Space setting, let us briefly indicate how such lower semicontinuity properties as described above relate to the minimisation problem

$$m := \inf\{F(v;\Omega) : v \in W^{1,r}(\Omega;\mathbb{R}^N)\,,\, v = g \text{ on } \partial\Omega\}\,. \tag{1.2}$$

If $m < +\infty$ and the integrand f also satisfies the coercivity condition

$$f(\xi) \geq c_0|\xi|^q - c_1 \tag{1.3}$$

for all $\xi \in \mathbb{R}^{N\times n}$, for some exponent $1 < q < \infty$, then it is easy to see that any minimising sequence $(u_j) \subset W^{1,r}(\Omega;\mathbb{R}^N)$ converges (taking a subsequence if necessary) weakly in $W^{1,q}(\Omega;\mathbb{R}^N)$ to a limit $u \in W^{1,q}(\Omega;\mathbb{R}^N)$. If we have established that the functional F is lower semicontinuous with respect to this convergence, then it follows that

$$F(u;\Omega) \leq F(v;\Omega) \text{ for all } v \in W^{1,r}(\Omega;\mathbb{R}^N)\,.$$

If $q \geq r$, then we may conclude that u is a minimiser of problem (1.2), subject to Dirichlet Boundary Conditions. We remark that the above "q-coercivity condition" may be weakened and generalised for our purposes. However, when $q < r$, u may not be in $W^{1,r}(\Omega;\mathbb{R}^N)$, and hence need not be a solution to (1.2). Indeed, it turns out that the choice of function space is of crucial importance in the context of existence of minimisers. This is due to the so-called *Lavrentiev Phenomenon*: it can happen that we obtain different infima for a variational integral depending on the underlying function space in the admissible set of functions $\mathscr{A}$. In this case, we need to *relax* the above problem, which involves considering appropriate *Lebesgue-Serrin Extensions* and their related minimisation problems. Further details of this are contained in subsequent sections of this introduction.

Another key notion when studying lower semicontinuity properties of a functional F is that of *quasiconvexity*, also introduced by Morrey. In the first section of this introduction, we provide further discussion of this and other notions of convexity, before then proceeding to give an account of various lower semicontinuity results in the Sobolev Space setting, as well as their related minimisation problems and relaxation. We then move on to a discussion of results in the setting of functions of Bounded Variation, and lastly give an outline of the material contained in this thesis. For a comprehensive and thorough reference work for much of what is discussed in the rest of this chapter, as well as many other related issues concerning the Direct Method in the Calculus of Variations, we refer the book of Dacorogna [33].

1.1 Notions of convexity

Convex analysis is a classical and well-studied branch of mathematics, for which many reference books exist - two key works being those of Rockafellar [83], and of Ekeland and Témam [44]. Convexity of the integrand f plays a key role in existence theorems for the scalar case ($n = 1$ or $N = 1$). Moreover, it is also a sufficient condition in the vectorial case (both n and $N > 1$) to ensure lower semicontinuity of the functional F in the sequential weak topology of $W^{1,r}(\Omega; \mathbb{R}^N)$ (weak* if $r = \infty$). However, it is far from being a necessary condition. Indeed, most interesting cases, such as the determinant, do not have an integrand that is convex. In [77], Morrey introduced the following, weaker definition of quasiconvexity that is of central importance in the modern theory of the Calculus of Variations.

Definition 1.1. A Borel measurable locally bounded function $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is said to be *quasiconvex* if

$$\int_{\mathbb{R}^n} [f(\xi + \nabla\phi(x)) - f(\xi)]\,\mathrm{d}x \geq 0$$

for all $\xi \in \mathbb{R}^{N\times n}$ and all test functions $\phi \in W_0^{1,\infty}(\mathbb{R}^n; \mathbb{R}^N)$.

It is easily shown that one may replace the set of test functions $W_0^{1,\infty}(\mathbb{R}^n; \mathbb{R}^N)$ by $C_0^\infty(\mathbb{R}^n; \mathbb{R}^N)$. Morrey showed that this is a necessary and sufficient condition for F to be sequentially weakly* lower semicontinuous in $W^{1,\infty}(\Omega; \mathbb{R}^N)$ (and hence also a necessary condition for sequential weak lower semicontinuity in $W^{1,r}(\Omega; \mathbb{R}^N)$ for

$1 \leq r < \infty$). Since the notion of quasiconvexity is not a pointwise condition, it is hard to verify that a given integrand f is quasiconvex. Hence it is useful to define other related notions of convexity, namely the weaker property of rank-one convexity, and the stronger property of polyconvexity, which we do now.

Definition 1.2. A function $f\colon \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ is said to be *rank-one convex* if

$$f(\lambda\xi + (1-\lambda)\eta) \leq \lambda f(\xi) + (1-\lambda) f(\eta)$$

for all $\lambda \in [0,1]$ and $\xi, \eta \in \mathbb{R}^{N\times n}$ satisfying $\operatorname{rank}(\xi - \eta) \leq 1$.

Definition 1.3. For a matrix $\xi \in \mathbb{R}^{N\times n}$, let $M(\xi)$ denote the vector whose components are all $s \times s$ minors of ξ, for $1 \leq s \leq \min\{n, N\}$. A function $f\colon \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ is said to be *polyconvex* if there exists a convex function g with domain all vectors of the form $M(\xi)$, $\xi \in \mathbb{R}^{N\times n}$, codomain $\mathbb{R} \cup \{+\infty\}$, such that

$$f(\xi) = g(M(\xi))$$

for all $\xi \in \mathbb{R}^{N\times n}$.

For real-valued integrands $f\colon \mathbb{R} \to \mathbb{R}^{N\times n}$, these notions of convexity are related as follows

$$f \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ quasiconvex } \Rightarrow f \text{ rank-one convex.}$$

If $n = 1$ or $N = 1$, then these notions are all equivalent. Otherwise, the converse implications are not true in general. Clearly if n, $N \geq 2$, then we may just let f be defined as the modulus of a 2×2 minor of ξ to produce an example of a polyconvex function that is not convex.

Moreover, several examples exist of quasiconvex functions that are not polyconvex, although none of these are elementary. For instance, for n, $N \geq 3$, there exist quadratic forms

$$f(\xi) := \langle M\xi, \xi \rangle \,,$$

where M is a symmetric matrix in $\mathbb{R}^{(N\times n)\times(N\times n)}$, $\xi \in \mathbb{R}^{N\times n}$, and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^{N\times n}$, such that f is quasiconvex but not polyconvex. This has been shown by Terpstra [95], and later by Serre [87] and Ball [19]. If n or $N = 2$, then quasiconvexity implies polyconvexity for all such quadratic functions - see [87]. However in [93], Šverák proves that indeed there exist quasiconvex functions on

$\mathbb{R}^{2\times 2}$ with subquadratic growth that are not polyconvex. In addition, expanding on the method of Šverák, Zhang has shown how to construct nontrivial quasiconvex functions with linear growth at infinity [97], and Müller [79] has constructed functions that are positively homogeneous of degree one and quasiconvex but not convex. Further details on the convexity properties of homogeneous functions of degree one may be found in work by Dacorogna and Haeberly [34, 36].

The non-equivalence of quasiconvexity and rank-one convexity when n, $N \geq 2$ is sometimes known as Morrey's conjecture. There is a famous example of Šverák [94] in the case $N \geq 3$, $n \geq 2$ of a function that is rank-one convex but not quasiconvex. However, it is still an open question whether or not the two notions are equivalent when $n \geq N = 2$.

Another particular interesting example is by Alibert, Dacorogna and Marcellini, and concerns the case $n = N = 2$ and a homogeneous polynomial of degree 4 that allows us to illustrate the different notions of convexity by using a single real parameter. Let $\gamma \in \mathbb{R}$ and define $f_\gamma \colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ as

$$f_\gamma(\xi) = |\xi|^2(|\xi|^2 - 2\gamma \det \xi)\,.$$

Then

$$\begin{aligned} f_\gamma \text{ is convex } &\Leftrightarrow |\gamma| \leq \gamma_c = \frac{2}{3}\sqrt{2}\,,\\ f_\gamma \text{ is polyconvex } &\Leftrightarrow |\gamma| \leq \gamma_p = 1\,,\\ f_\gamma \text{ is quasiconvex } &\Leftrightarrow |\gamma| \leq \gamma_q \text{ and } \gamma_q > 1\,,\\ f_\gamma \text{ is rank-one convex } &\Leftrightarrow |\gamma| \leq \gamma_r = \frac{2}{\sqrt{3}}\,. \end{aligned}$$

The conditions for f_γ to be rank-one convex and polyconvex were proved by Dacorogna and Marcellini in [37]. The other results were established by Alibert and Dacorogna in [7]. In [62], Iwaniec and Kristensen outline a method for constructing quasiconvex functions, which can also be applied to establish the third fact above. Note that this example also provides a quasiconvex function that is not polyconvex. The issue of whether $\gamma_q = \frac{2}{\sqrt{3}}$ is still open: if it were not the case, then this would give a complete answer to Morrey's conjecture.

Lastly, we consider the notion of $W^{1,r}$-quasiconvexity, introduced and studied in a well-known paper by Ball and Murat [20], which generalises in a natural way the quasiconvexity condition of Morrey.

Definition 1.4. Let $1 \leq r \leq \infty$. Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ be Borel measurable and bounded below. Then f said to be $W^{1,r}$*-quasiconvex* if

$$\int_E f(\xi + \nabla\phi(x))\,\mathrm{d}x \geq \mathscr{L}^n(E)f(\xi)$$

for every bounded open set $E \subset \mathbb{R}^n$ with $\mathscr{L}^n(\partial E) = 0$, for all $\xi \in \mathbb{R}^{N\times n}$, and all test functions $\phi \in W_0^{1,r}(E;\mathbb{R}^N)$.

Note that if a function f is $W^{1,r}$-quasiconvex, it is clearly also $W^{1,q}$ -quasiconvex for all $r \leq q \leq \infty$. Thus $W^{1,1}$-quasiconvexity is the strongest condition and $W^{1,\infty}$-quasiconvexity is the weakest. For $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ bounded below and locally bounded, $W^{1,\infty}$-quasiconvexity is just the usual definition of quasiconvexity. In their paper, Ball and Murat demonstrate, similarly to Morrey, that $W^{1,r}$-quasiconvexity is a necessary condition for sequential weak lower semicontinuity in $W^{1,r}(\Omega;\mathbb{R}^N)$ (weak* if $r = \infty$). They also show that if f is upper semicontinuous, bounded below, and satisfies the growth condition, for some exponent $1 \leq r < \infty$,

$$f(\xi) \leq L(|\xi|^r + 1)$$

for all $\xi \in \mathbb{R}^{N\times n}$, for some constant $L > 0$, then f is $W^{1,r}$-quasiconvex if and only if it is $W^{1,\infty}$-quasiconvex. One of the reasons why this is a valuable definition is because it allows us to easily reformulate lower semicontinuity theorems with different conditions (see, for example, Theorem 4.6 in this thesis).

1.2 Lower semicontinuity in Sobolev Spaces

The classical lower semicontinuity result for quasiconvex integrands is as follows. For the rest of this chapter (and indeed for this thesis) we shall assume that $n \geq 2$. We shall also require on numerous occasions throughout that a function $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfies the growth condition

$$0 \leq f(\xi) \leq L(1 + |\xi|^r) \tag{1.4}$$

for all $\xi \in \mathbb{R}^{N\times n}$, for some constant $L > 0$, and some exponent $1 \leq r < \infty$.

Theorem 1.5. *Let $1 \leq r \leq \infty$. Let Ω be an open bounded subset of $\mathbb{R}^n$ and $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous quasiconvex function. If $1 \leq r < \infty$, suppose f satisfies the growth condition* (1.4)*. Now define*

$$F(u;\Omega) := \int_\Omega f(\nabla u)\,\mathrm{d}x\,.$$

Then F is sequentially weakly lower semicontinuous in $W^{1,r}(\Omega;\mathbb{R}^N)$ (weak lower semicontinuous if $r=\infty$):*

i.e. if (u_j) is a sequence in $W^{1,r}(\Omega;\mathbb{R}^N)$ converging weakly (weakly if $r=\infty$) to a map $u \in W^{1,r}(\Omega;\mathbb{R}^N)$, then*

$$\liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \geq \int_\Omega f(\nabla u)\,\mathrm{d}x\,.$$

This theorem is essentially due to Morrey [77, 78], who proved it in the case $r = \infty$. Refinements were made most notably by Meyers [75], Acerbi and Fusco [3], and Marcellini [71]. It obviously follows that when f is quasiconvex and satisfies (1.4), then F is also sequentailly weakly lower semicontinuous in $W^{1,q}(\Omega;\mathbb{R}^N)$ for any q satisfying $1 \leq r \leq q \leq \infty$ (weak* if $q = \infty$). Henceforth, we shall just write "weak (or weak*) lower semicontinuity" to mean sequential weak (or weak*) lower semicontinuity.

If we wish to further refine this result there are, broadly speaking, three principal components we can adjust to achieve this aim.

1. Conditions on f.
2. The notion of convergence.
3. Regularity of the maps (u_j), u.

In practice, Theorem 1.5 is close to optimal. It is difficult to simply weaken one of the above parameters without having to "pay for it" by strengthening another, as the counterexamples below will demonstrate. With regards to the first point, we can for example adjust the notion of convexity or the exponent r in growth condition (1.4). We can also consider more general integrands $f = f(x, \nabla u)$ or $f = f(x, u(x), \nabla u)$. In such cases, many more problems present themselves: important recent results in this context may be found in work of Acerbi, Bouchitté and Fonseca [1], and of Mingione and Mucci [76].

For the second point, we might for example seek to generalise the above theorem by fixing an exponent $1 \leq r \leq \infty$, keeping the same corresponding growth and quasiconvexity conditions on f, as well as the requirement that (u_j), u are maps in $W^{1,r}(\Omega;\mathbb{R}^N)$; but we may aim to weaken the notion of convergence of the maps (u_j) to u, for instance by having the $u_j \rightharpoonup u$ weakly in some Sobolev Space $W^{1,q}(\Omega;\mathbb{R}^N)$ for $1 \leq q < r$.

In fact, for quasiconvex f satisfying (1.4), lower semicontinuity of F in $W^{1,q}(\Omega;\mathbb{R}^N)$ for $q > r\frac{n-1}{n}$ $(q > 1)$ was proved by Fonseca and Malý in [69, 50]. Previously, work by Marcellini in [72], and by Carbone and De Arcangelis in [28] established lower semicontinuity for $q > r\frac{n}{n+1}$ by imposing additional structural conditions on f. Fonseca and Marcellini obtained a proof in the case $q > r-1$ [52], and Malý for $q \geq r-1$: both these results require further assumptions on f in addition to quasiconvexity.

If we restrict f to be polyconvex then, again with certain structural conditions, we obtain the following lower semicontinuity result: take $r = N = n$ and let (u_j) be a sequence of functions in $W^{1,n}(\Omega;\mathbb{R}^n)$ converging weakly in $W^{1,q}(\Omega;\mathbb{R}^n)$ to $u \in W^{1,q}(\Omega;\mathbb{R}^n)$. Then

$$\liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \geq \int_\Omega f(\nabla u)\,\mathrm{d}x$$

provided $q \geq n-1$. Such a result was first established by Marcellini for $q > \frac{n^2}{n+1}$ [72], and then by Dacorogna and Marcellini [38] for $q > n-1$. The borderline case $q = n-1$ was first considered by Malý [70] with partial results, and completely established by Acerbi, Dal Maso and Sbordone [2, 40]. Improvements have since been discovered by Gangbo [57], and Celada and Dal Maso [30]; an elementary approach that avoids the use of Cartesian currents has been found by Fusco and Hutchinson [56]. Note that these results only deal with special kinds of polyconvex integrands; a more general result is obtained in [29]. For weak continuity properties of specifically the determinant we may, for example, refer to work by Ball [18] and, more recently, Brezis and Nguyen [24].

Now we shall consider some counterexamples that demonstrate clear limits to what can be achieved in obtaining lower semicontinuity results. The following counterexample, given by Ball and Murat in [20], addresses the third point above. Namely, it shows that when r is an integer strictly larger than 1 and $N, n \geq r$, the maps (u_j) in the statement of any potential lower semicontinuity result need to be at least $W^{1,r}(\Omega;\mathbb{R}^N)$ when the integrand is polyconvex.

Counterexample 1.6. *Let $N = n \geq 2$ and Ω be an open bounded subset of $\mathbb{R}^n$. Let $1 \leq q < n$. Then there exist $(u_j) \subset W^{1,q}(\Omega;\mathbb{R}^n)$ such that u_j converge weakly in $W^{1,q}(\Omega;\mathbb{R}^n)$ to the identity map, as well as strongly in $L^\infty(\Omega;\mathbb{R}^n)$, but for all j, $\det \nabla u_j(x) = 0$ for almost all $x \in \Omega$.*

Hence if $N = n \geq 2$, we can take $f(\xi) = |\det(\xi)|$. f is polyconvex (hence also quasiconvex), and satisfies the growth condition

$$0 \leq f(\xi) \leq L(1 + |\xi|^n)$$

for all $\xi \in \mathbb{R}^{N\times n}$ (so $r = n$ in (1.4)). But, taking (u_j) as in the statement of the result and u to be the identity, we have $u_j \rightharpoonup u$ in $W^{1,q}(\Omega;\mathbb{R}^N)$ for any $1 \leq q < n$ but

$$\liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x = 0 \not\geq |\Omega| = \int_\Omega f(\nabla u)\,\mathrm{d}x\,.$$

To further generalise for integer r and N, $n \geq r$, we may just consider a suitable $r \times r$ minor. It is essential in the above result that the (u_j) are no more regular than $W^{1,q}(\Omega;\mathbb{R}^N)$ for $1 \leq q < n$. In this next counterexample, established by Malý [70], it is shown that lower semicontinuity fails even when the (u_j) are C^1-diffeomorphisms: in this case, we "lose a dimension" and only have weak convergence in $W^{1,q}(\Omega;\mathbb{R}^N)$ for $1 \leq q < n-1$. For $q \geq n-1$, as described above, there are positive results.

Counterexample 1.7. *Let Q be the cube $(0,1)^n$, and $1 < r < n-1$. There is a sequence of orientation-preserving C^1-diffeomorphisms (u_j) on Q such that u_j converge weakly to the identity in $W^{1,r}(Q,\mathbb{R}^n)$ and*

$$\lim_{j\to\infty} \int_Q \det \nabla u_j\,\mathrm{d}x = 0\,.$$

In the context of the minimisation problem (1.2), for $1 < q \leq \infty$, sequential weak compactness properties of the Sobolev Spaces $W^{1,q}$ (weak* if $q = \infty$) play an important role in establishing the existence of minimisers when f satisfies a coercivity property such as (1.3). When $q < r$, where r is the exponent in growth condition (1.4), we can relax the problem as outlined in the next section. However, when $q = 1$, it is very hard to prove that a minimising sequence of F is relatively compact in the weak topology of $W^{1,1}(\Omega;\mathbb{R}^N)$. Therefore in this case it is useful to prove lower semicontinuity without assuming that maps ∇u_j converge weakly in $L^1(\Omega;\mathbb{R}^N)$ to ∇u. This was done by Dal Maso in the scalar ($N = 1$) case [39]; in the vector-valued case for f

convex, results have been obtained, for example, by Reshetnyak [81], Ball and Murat [20], and Aviles and Giga [17]. For the quasiconvex case, a first result in this direction was obtained by Fonseca in [48], who proved that if f is quasiconvex and satisfies the linear growth condition

$$0 \leq f(\xi) \leq L(1+|\xi|)\,,$$

then lower semicontinuity obtains for a sequence $(u_j) \subset W^{1,1}(\Omega;\mathbb{R}^N)$ converging strongly in $L^1(\Omega;\mathbb{R}^N)$ to $u \in W^{1,1}(\Omega;\mathbb{R}^N)$, provided the (u_j) are also bounded in $W^{1,1}(\Omega;\mathbb{R}^N)$. Subsequently, the hypothesis of boundedness in $W^{1,1}(\Omega;\mathbb{R}^N)$ was removed by Fonseca and Muller in [53].

1.3 Relaxation of minimisation problems

Let us again consider the minimisation problem

$$m := \inf\left\{ \int_\Omega f(\nabla v(x))\,\mathrm{d}x : v \in W^{1,r}(\Omega;\mathbb{R}^N)\,,\, v = g \text{ on } \partial\Omega \right\}. \tag{1.5}$$

As indicated earlier, such lower semicontinuity results relate straightforwardly to the problem of existence of minimisers when we assume that the integrand f satisfies a "q-coercivity" property such as (1.3) for $1 < q < \infty$ and $q \geq r$. However, when $q < r$ it is possible that the limit map $u \in W^{1,q}(\Omega;\mathbb{R}^N)$ of a minimising sequence $(u_j) \subset W^{1,r}(\Omega;\mathbb{R}^N)$, although it satisfies

$$\int_\Omega f(\nabla u)\,\mathrm{d}x \leq \int_\Omega f(\nabla v)\,\mathrm{d}x \text{ for all } v \in W^{1,r}(\Omega;\mathbb{R}^N)\,,$$

is not in $W^{1,r}(\Omega;\mathbb{R}^N)$ and hence not a solution of (1.5). Indeed, due to the Lavrentiev Phenomenon, it need not even satisfy the related minimisation problem to (1.5) for the case where admissible maps can be in the larger space $W^{1,q}(\Omega;\mathbb{R}^N)$. That is, it is possible that

$$\int_\Omega f(\nabla u)\,\mathrm{d}x > \inf\left\{ \int_\Omega f(\nabla v(x))\,\mathrm{d}x : v \in W^{1,q}(\Omega;\mathbb{R}^N)\,,\, v = g \text{ on } \partial\Omega \right\}.$$

In this case, we may *relax* the problem (1.5) in such a way: following a method that was first used by Lebesgue for the area integral [66], and then adopted by Serrin [88, 89] and, in the modern context, by Marcellini [72], we may consider the functional

$$\mathscr{F}(u,\Omega) := \inf_{(u_j)}\left\{ \liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \;\middle|\; \begin{array}{l} (u_j) \subset W^{1,r}(\Omega,\mathbb{R}^N) \\ u_j \rightharpoonup u \text{ in } W^{1,q}(\Omega,\mathbb{R}^N) \end{array} \right\}.$$

$\mathscr{F}$ is known as the *Lebesgue-Serrin Extension of* F and is an important quantity not only when we want to define $F(u;\Omega)$ for a wider class of functions u but also, for example, when there is a lack of convexity. Key results concerning properties of such functionals may be found in work by Bouchitté, Fonseca and Malý (see [21, 50] - in fact they consider more general integrands of the form $f = f(x, u, \nabla u)$). Now we may consider the *relaxed problem*

$$\bar{m} := \inf \left\{ \mathscr{F}(v, \Omega) : v \in W^{1,q}(\Omega; \mathbb{R}^N)\,,\, v = g \text{ on } \partial\Omega \right\}. \tag{1.6}$$

Now suppose that f is a quasiconvex integrand satisfying the "anisotropic" growth condition

$$c_0|\xi|^q - c_1 \leq f(\xi) \leq c_2(1 + |\xi|)^r$$

for $1 < q < r < \infty$. If, recalling the discussion in the previous section, we have established that the variational integral $F(\cdot;\Omega)$ is sequentially weakly lower semicontinuous when the sequence $(u_j) \subset W^{1,r}(\Omega;\mathbb{R}^N)$ converges to $u \in W^{1,r}(\Omega;\mathbb{R}^N)$ weakly in $W^{1,q}(\Omega;\mathbb{R}^N)$, it follows that $\mathscr{F}(\cdot,\Omega)$ agrees with $F(\cdot;\Omega)$ on $W^{1,r}(\Omega;\mathbb{R}^N)$, and we may say that it is indeed an extension of the original variational integral.

In addition, since f is q-coercive, it can straightforwardly be shown that $\mathscr{F}(\cdot,\Omega)$ is lower semicontinuous in the sequential weak topology of $W^{1,q}(\Omega;\mathbb{R}^N)$. Hence a minimising sequence $(u_j) \subset W^{1,q}(\Omega;\mathbb{R}^N)$ approximating $\bar{m}$ in (1.6) (that is, satisfying $\mathscr{F}(u_j,\Omega) \to \bar{m}$) converges weakly (taking a subsequence if necessary) to a limit map $u \in W^{1,q}(\Omega;\mathbb{R}^N)$, which thus satisfies

$$\mathscr{F}(u,\Omega) = \bar{m}\,,$$

meaning that u is a solution to the relaxed problem (1.6). Regularity results for quasiconvex integrands satisfying such anisotropic growth conditions may be found in recent work by Schmidt [85, 86]. When $q = 1$, so f is linearly coercive, it is more natural to set such problems in the context of the space of functions of Bounded Variation, which the next section deals with.

This is not the only way in which minimisation problems can be relaxed: we have seen in preceding discussions in this chapter how, in the vectorial case, quasiconvexity of the function f plays a key role in obtaining theorems for the existence of minimisers. When f is not quasiconvex, the infimum m in the minimisation problem (1.5) is not attained in general. However, in this context it is useful to define the *quasiconvex* *envelope of* f as the largest quasiconvex function below f. That is, we define for $\xi \in \mathbb{R}^{N\times n}$

$$Qf(\xi) := \sup\{g(\xi) : g \leq f \text{ and } g \text{ quasiconvex }\}. \tag{1.7}$$

In [32], Dacorogna showed that if f is locally bounded and Borel measurable, and there exists a quasiconvex function $h\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ such that $f(\xi) \geq h(\xi)$ for all $\xi \in \mathbb{R}^{N\times n}$, then Qf has the representation formula

$$Qf(\xi) = \inf\left\{\frac{1}{\mathscr{L}^n(E)}\int_E f(\xi + \nabla\phi(x))\,\mathrm{d}x : \phi \in W_0^{1,\infty}(E;\mathbb{R}^N)\right\}$$

for all $\xi \in \mathbb{R}^{N\times n}$, where E is a bounded, open set with $\mathscr{L}^n(\partial E) = 0$. In particular, this infimum is independent of the choice of E. It is now natural to replace the problem in (1.5) with the relaxed problem

$$m^{QC} := \inf\left\{\int_\Omega Qf(\nabla v(x))\,\mathrm{d}x : v \in W^{1,r}(\Omega;\mathbb{R}^N),\, v = g \text{ on } \partial\Omega\right\}. \tag{1.8}$$

We now have the following result.

Theorem 1.8. *Let Ω be a bounded, open subset of $\mathbb{R}^n$. Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a Borel measurable function satisfying growth condition* (1.4) *for $1 \leq r < \infty$. For $r = \infty$, suppose f is locally bounded. Let the quasiconvex envelope Qf of f be as defined in* (1.7).

1. *Let m and m^{QC} be the infima in the minimisation problems* (1.5) *and* (1.8) *respectively. Then*

   $$m = m^{QC}.$$

   More precisely, for every $u \in W^{1,r}(\Omega;\mathbb{R}^N)$ there exists a sequence $(u_j) \subset W^{1,r}(\Omega;\mathbb{R}^N)$ such that $u_j = u$ on $\partial\Omega$,

   $$u_j \to u \text{ in } L^r(\Omega;\mathbb{R}^N) \text{ as } j \to \infty,$$

   and

   $$\int_\Omega f(\nabla u_j)\,\mathrm{d}x \to \int_\Omega Qf(\nabla u)\,\mathrm{d}x \quad \text{as } j \to \infty.$$

2. *Now suppose, in addition to the assumptions in (1), that, if $1 < r < \infty$, there exist constants $c_1 \geq c_0 > 0$ and $\gamma \in \mathbb{R}$ such that*

   $$\gamma + c_0|\xi|^r \leq f(\xi) \leq c_1(1 + |\xi|^r)$$

   for all $\xi \in \mathbb{R}^{N\times n}$. Then, in addition to the conclusions of (1), we also have

   $$u_j \rightharpoonup u \text{ in } W^{1,r}(\Omega;\mathbb{R}^N) \text{ as } j \to \infty.$$

This result was established by Dacorogna in [32]. This shows us that even if (1.5) has no solution in $W^{1,r}(\Omega;\mathbb{R}^N)$, we can consider solutions of the relaxed problem (1.8) as generalised solutions of (1.5).

1.4 Lower semicontinuity in BV

As described during at the end of Section 1.2, several lower semicontinuity results have been obtained for integrals of linear growth specifically in the setting of the Sobolev Space $W^{1,1}(\Omega;\mathbb{R}^N)$. However, these are not satisfactory for most applications, since most existence theorems for functionals with linear coercivity conditions involve the space $\mathrm{BV}(\Omega;\mathbb{R}^N)$, of functions $u \in L^1(\Omega;\mathbb{R}^N)$ whose distributional derivative can be represented by a matrix-valued Radon measure in Ω. The main reason for this is because it has better compactness properties (see Theorem 2.5). In the next chapter, we collect some key properties of such functions, which will be of key importance in the context of this thesis. Here, we shall discuss only the lower semicontinuity results in this setting.

First, let us observe that it is not obvious how to define the functional $F(u;\Omega)$ when u is a function of Bounded Variation. Following the relaxation method from the previous section, we may consider the following Lebesgue-Serrin extension:

$$\mathscr{F}(u,\Omega) := \inf_{(u_j)} \left\{ \liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \;\middle|\; \begin{array}{l} (u_j) \subset W^{1,r}(\Omega,\mathbb{R}^N) \\ u_j \overset{*}{\rightharpoonup} u \text{ in } \mathrm{BV}(\Omega,\mathbb{R}^N) \end{array} \right\}. \tag{1.9}$$

Moreover, note that we may adapt the definition of $\mathscr{F}$ in (1.9) to include sequences (u_j) belonging to the local Sobolev Space $W^{1,r}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. That is, we define

$$\mathscr{F}_{\mathrm{loc}}(u,\Omega) := \inf_{(u_j)} \left\{ \liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \;\middle|\; \begin{array}{l} (u_j) \subset W^{1,r}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \\ u_j \overset{*}{\rightharpoonup} u \text{ in } \mathrm{BV}(\Omega,\mathbb{R}^N) \end{array} \right\}, \tag{1.10}$$

The properties of $\mathscr{F}_{\mathrm{loc}}$ in the case where f is quasiconvex, has linear growth, and $r=1$ have been studied extensively by Ambrosio and Dal Maso in [12]. Most notably they prove that for every open set $\Omega \subset \mathbb{R}^n$ and every $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ we have

$$\mathscr{F}_{\mathrm{loc}}(u,\Omega) = \int_\Omega f(\nabla u(x))\,\mathrm{d}x + \int_\Omega f_\infty\left(\frac{D^s u}{|D^s u|}\right)|D^s u|\,,$$

where ∇u is the density of the absolutely continuous part of the measure Du with respect to Lebesgue measure, $D^s u$ is the singular part of Du, $\frac{D^s u}{|D^s u|}$ is the Radon-Nikodým derivative of the measure $D^s u$ with respect to its variation $|D^s u|$, and f_∞

denotes the recession function of f, defined as

$$f_\infty(\xi) := \limsup_{t\to\infty} \frac{f(t\xi)}{t} \,. \tag{1.11}$$

In this connection see also Fonseca and Müller [54], where the case of general integrands $f = f(x, u, \nabla u)$ of linear growth is treated, and Rindler [82] for a proof that avoids the use of Alberti's rank-one theorem. This integral representation in the convex case was proved earlier by Goffman and Serrin in [60]. Other related material appears in work by Aviles and Giga [17], Ambrosio, Mortola and Tortorelli [15], Ambrosio and Pallara [16], and Fonseca and Rybka [55].

1.5 Outline of material presented in this work

The majority of previous results concerning lower semicontinuity in BV of quasiconvex integrals concern integrals f that satisfy linear growth conditions. In [64], Kristensen shows that when f is quasiconvex and satisfies the growth condition (1.4) for $1 \le r < \frac{n}{n-1}$, then the Lebesgue-Serrin extension $\mathscr{F}_{\mathrm{loc}}$ as defined in (1.10) satisfies the lower bound

$$\mathscr{F}_{\mathrm{loc}}(u, \Omega) \ge \int_\Omega f(\nabla u)\,\mathrm{d}x \,, \tag{1.12}$$

whenever $u \in \mathrm{BV}(\Omega; \mathbb{R}^N)$, where again ∇u is the density of the absolutely continuous part of the measure Du with respect to Lebesgue measure. In Chapter 4 of this thesis, we obtain the following lower semicontinuity result in the sequential weak* topology of BV for quasiconvex integrals of subquadratic growth.

Theorem 4.1 (Lower semicontinuity in BV for subquadratic growth)**.** *Let Ω be a bounded, open subset of $\mathbb{R}^n$. Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a quasiconvex function satisfying the growth condition* (1.4) *for some exponent $1 < r < 2$.*

Let (u_j) be a sequence in $W^{1,r}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$, where $p \ge 1$ and $p > \frac{r}{2}(n-1)$. Suppose

$$u_j \overset{*}{\rightharpoonup} u \textit{ in } BV_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$$

and

$$(u_j) \textit{ uniformly bounded in } L^q_{\mathrm{loc}}(\Omega; \mathbb{R}^n)\,,$$

where

$$q > \frac{r(n-1)}{2-r}\,.$$

Then

$$\liminf_{j\to\infty}\int_\Omega f(\nabla u_j)\,\mathrm{d}x \ge \int_\Omega f(\nabla u)\,\mathrm{d}x\,.$$

The result is to appear in [90]. The principal new distinction here compared to previous results is that we now have a lower semicontinuity result in the sequential weak* topology of BV in the case where the growth exponent r is greater than or equal to $\frac{n}{n-1}$ (but less than 2). However, we need to assume additionally that the maps (u_j) are bounded uniformly in L^q_{loc} for q suitably large, as well as, for technical reasons, an additional regularity requirement on the limit map u. A key new idea in its proof is a more careful blow-up technique that exploits a specific approximate differentiability property of (sufficiently regular) Sobolev maps.

Now let us establish some definitions: let μ be a Radon measure on $\bar{\Omega}$, where Ω is a bounded, open subset of $\mathbb{R}^n$. Then we say that μ (strongly) *represents* the Lebesgue-Serrin extension $\mathscr{F}(u,\cdot)$ if

$$\mu(U) = \mathscr{F}(u,U)$$

for all open sets $U \subset \Omega$. We say that μ *weakly represents* $\mathscr{F}(u,\cdot)$ if

$$\mu(U) \le \mathscr{F}(u,U) \le \mu(\bar{U})$$

for all open sets $U \subset \Omega$. In [50], in the setting of Sobolev spaces of exponent larger than one, Fonseca and Malý prove measure representations of appropriate Lebesgue-Serrin extensions defined on such spaces. In Chapter 5, essentially using their proof but now in the BV setting, we establish the following two main results.

Theorem 5.5 (Measure representation of $\mathscr{F}_{\mathrm{loc}}$)**.** *Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous function satisfying the growth condition* (1.4) *for some exponent $1 \le r < \frac{n}{n-1}$. Let $u \in BV(\Omega;\mathbb{R}^N)$ and $\mathscr{F}_{loc}$ be as defined in* (1.10)*. Then if $\mathscr{F}_{loc}(u,\Omega) < \infty$, then there exists a non-negative, finite Radon measure λ on Ω which represents $\mathscr{F}_{loc}$.*

Theorem 5.6 (Weak measure representation of $\mathscr{F}$)**.** *Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous function satisfying the growth condition* (1.4) *for some exponent $1 \le r < \frac{n}{n-1}$. Let $u \in BV(\Omega;\mathbb{R}^N)$ and $\mathscr{F}$ be as defined in* (1.9)*. Then if $\mathscr{F}(u,\Omega) < \infty$, then there exists a non-negative, finite Radon measure μ on $\bar{\Omega}$ which weakly represents $\mathscr{F}$.*

In Chapter 6, motivated by the integral representation result of Abrosio and Dal Maso described above, we extend the lower bound (1.12) established by Kristensen, provided we assume additionally that the recession function f_∞ is finite in certain rank-one directions. The main result in that chapter is as follows.

Theorem 6.2 (Lower semicontinuity in BV for superlinear growth)**.** *Let Ω be a bounded, open set in $\mathbb{R}^n$ and $u \in BV(\Omega;\mathbb{R}^N)$. Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a quasiconvex function satisfying the growth condition* (1.4) *for $r \in [1, \frac{n}{n-1})$. Let the recession function f_∞ be as defined in* (1.11)*, and suppose it is finite on rank-one matrices of the form $u(y) \otimes \nu$, $y \in \Omega$, $\nu \in \mathbb{R}^n$.*

Suppose (u_j) is a sequence in $W^{1,r}_{loc}(\Omega;\mathbb{R}^N)$ such that

$$u_j \overset{*}{\rightharpoonup} u \ \text{ in } BV(\Omega;\mathbb{R}^N)\,.$$

Then

$$\liminf_{j\to\infty} F(u_j;\Omega) \geq \int_\Omega f(\nabla u(x))\,\mathrm{d}x + \int_\Omega f_\infty\left(\frac{D^s u}{|D^s u|}\right)|D^s u|\,,$$

and hence the Lebesgue-Serrin extension $\mathscr{F}_{loc}$ as defined in (1.10) *satisfies*

$$\mathscr{F}_{loc}(u,\Omega) \geq \int_\Omega f(\nabla u(x))\,\mathrm{d}x + \int_\Omega f_\infty\left(\frac{D^s u}{|D^s u|}\right)|D^s u|\,.$$

The proof of this result involves adapting work by Kristensen [64], Ambrosio and Dal Maso [12], and Braides and Coscia [22]. It also makes use of a new technique, involving mollification, that provides an upper bound for $\mathscr{F}_{\mathrm{loc}}$ for specific types of functions of Special Bounded Variation, and a non-standard blow-up technique that exploits fine properties of BV maps. We also make frequent use of Theorem 5.5 from Chapter 5.

All of these results from Chapters 4, 5 and 6 rely, at some point, on various trace-preserving operators, which is the focus of Chapter 3. In this chapter, we first present a simplified version of a result by Kristensen in [64], where superlinear integral estimates are obtained for an extension operator that extends $W^{1,1}$ maps defined on $\mathbb{R}^{n-1}$ by mollification into the half-space $\mathbb{R}^n_+$, consisting of points in $\mathbb{R}^n$ whose n^{th} coordinate is non-negative. We then prove a new result to obtain subquadratic integral estimates for the same operator, but now taking our domain to be maps on $\mathbb{R}^{n-1}$ that are in both $W^{1,1}$ and L^q, for q sufficiently large. In the final section, we adapt a proof of Fonseca and Malý [50] to construct a linear operator T from $W^{1,1}$ into $W^{1,1}$ that preserves boundary values and improves the integrability of u and ∇u across a "layer" given by level sets of a real-valued smooth function defined on the domain.

As indicated previously, Chapter 2 is dedicated to providing an overview of key properties of functions of Bounded Variation, which feature frequently in Chapter 6.

Throughout the course of this thesis, we make use of standard well-known results from Real Analysis and the theory of Sobolev functions. Some key reference works in this area include [4, 23, 43, 45, 46, 47, 49, 67, 68, 73, 74, 84, 98].

We shall use C or c to denote positive constants that are not necessarily the same from line to line, and indicate what parameters these constants are dependent on when it is not clear.

Chapter 2

Functions of Bounded Variation

In this chapter, we collect some basic facts and definitions for functions of Bounded Variation that are essential in the context of the new results proved in this thesis. This includes the notion of weak* convergence in BV, and the decomposition of the derivative of a BV function into absolutely continuous and singular parts. We also state and prove some more particular results that will be used in later chapters, such as the behaviour of blow-up limits of BV functions. Aside from the final lemma, the results are well-known and hence stated without proofs. Much of what is presented here may be found in the monograph of Ambrosio, Fusco and Pallara [14]. Other important reference works include [47, 58, 74, 96, 98].

2.1 Basic definitions and properties

Throughout this chapter we denote by Ω a generic open set in $\mathbb{R}^n$. First, we begin with the most common definition of BV$(\Omega;\mathbb{R}^N)$, i.e. functions whose distributional derivative can be represented by a matrix-valued Radon measure in Ω.

Definition 2.1. We say that u is a *function of Bounded Variation* in Ω if u is in $L^1(\Omega;\mathbb{R}^N)$ and there exists a matrix-valued Radon measure

$$Du = (D_i u^j)_{1\le i\le n}^{1\le j\le N}\,,\ D_i u^j \text{ signed Radon measures on } \Omega$$

such that

$$\int_\Omega u^j \frac{\partial \phi}{\partial x_i}\,\mathrm{d}x = -\int_\Omega \phi\,\mathrm{d}D_i u^j\,. \tag{2.1}$$

for all $\phi \in C_c^\infty(\Omega)$, $1 \le i \le n$, $1 \le j \le N$. The vector space of all such functions of Bounded Variation in Ω is denoted BV$(\Omega;\mathbb{R}^N)$. We may also express (2.1) more concisely in a single formula by writing

$$\sum_{j=1}^{N}\int_{\Omega} u^j \mathrm{div}\phi^j \, \mathrm{d}x = -\sum_{j=1}^{N}\sum_{j=1}^{n}\int_{\Omega}\phi_i^j \, \mathrm{d}D_i u^j \quad \forall \phi \in C_c^1(\Omega;\mathbb{R}^N)\,. \tag{2.2}$$

Motivated perhaps by such an expression, we may now introduce the notion of *variation* $V(u,\Omega)$ of a function $u \in L^1(\Omega;\mathbb{R}^N)$. This is defined by

$$V(u,\Omega) := \sup\left\{ \sum_{j=1}^{N}\int_{\Omega} u^j \mathrm{div}\phi^j \, \mathrm{d}x \;:\; \phi \in C_c^1(\Omega;\mathbb{R}^{N\times n})\,, \|\phi\|_\infty \le 1 \right\}. \tag{2.3}$$

It is easy to verify that the mapping $u \mapsto V(u,\Omega) \in [0,\infty]$ is lower semicontinuous with respect to the $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ topology, since the map

$$u \mapsto \sum_{j=1}^{N}\int_{\Omega} u^j \mathrm{div}\phi^j \, \mathrm{d}x$$

is continuous in the $L^1(\Omega;\mathbb{R}^N)$ topology for any $\phi \in C_c^1(\Omega;\mathbb{R}^{N\times n})$. The following result, whose proof is a straightforward application of the expression (2.2) and the Riesz Theorem, provides a useful method of showing that a function $u \in L^1(\Omega;\mathbb{R}^N)$ belongs to $\mathrm{BV}(\Omega;\mathbb{R}^N)$. Recall that for an $\mathbb{R}^{N\times n}$-valued Radon measure μ in Ω, $|\mu|$ denotes the total variation of μ, defined for every Borel subset B of Ω by

$$|\mu|(B) := \sup\sum_{i\in I}|\mu(B_i)|\,,$$

where the supremum is taken over all finite or countable families $(B_i)_{i\in I}$ of pairwise disjoint Borel subsets of B which are relatively compact in Ω.

Proposition 2.2. *Let $u \in L^1(\Omega;\mathbb{R}^N)$. Then u is in BV$(\Omega;\mathbb{R}^N)$ if and only if*

$$V(u,\Omega) < \infty\,.$$

Moreover, $V(u,\Omega) = |Du|(\Omega;\mathbb{R}^N)$ and for $u \in$ BV$(\Omega;\mathbb{R}^N)$ the mapping $u \mapsto |Du|$ is lower semicontinuous with respect to the $L^1_{loc}(\Omega;\mathbb{R}^N)$ topology.

2.2 Weak* topology and compactness properties

Note that $\mathrm{BV}(\Omega;\mathbb{R}^N)$, when endowed with the norm

$$\|u\|_{\mathrm{BV}} := \int_{\Omega}|u|\,\mathrm{d}x + |Du|(\Omega)$$

is a Banach Space. However, the norm-topology is too strong for most applications. Hence we introduce the much weaker notion of *weak* convergence*, which is useful for its compactness properties and forms an important part of the main results in this thesis.

Definition 2.3. Let u, $u_j \in \mathrm{BV}(\Omega;\mathbb{R}^N)$. Then we say that (u_j) *weakly* converges to* u in $\mathrm{BV}(\Omega;\mathbb{R}^N)$ if $u_j \to u$ strongly in $L^1(\Omega;\mathbb{R}^N)$, and Du_j converges weakly* to Du in $\mathscr{M}(\Omega;\mathbb{R}^{N\times n})$, where $\mathscr{M}(\Omega;\mathbb{R}^{N\times n})$ is the space of $N \times n$ matrix-valued Borel measures on Ω. Since the set of signed measures $\mathscr{M}(\Omega)$ on Ω is isometrically isomorphic to the dual space of continuous functions on Ω, $[C_0(\Omega)]^*$, this means

$$\lim_{j\to\infty}\int_\Omega \phi \,\mathrm{d}Du_j = \int_\Omega \phi\,\mathrm{d}Du \quad \forall\phi \in C_0(\Omega)\,.$$

In fact, it is not difficult to deduce the following criterion for weak* convergence.

Proposition 2.4. *Suppose* $(u_j) \subset$ *BV*$(\Omega;\mathbb{R}^N)$. *Then* (u_j) *weakly* converges to* u *in BV*$(\Omega;\mathbb{R}^N)$ *if and only if the* (u_j) *are bounded in BV*$(\Omega;\mathbb{R}^N)$ *and* $u_j \to u$ *in* $L^1(\Omega;\mathbb{R}^N)$.

The proof of this is straightforward: we need only show that (Du_j) converges weakly* to Du in $\mathscr{M}(\Omega;\mathbb{R}^{N\times n})$. First we notice that (Du_j) is weakly* relatively compact, and then use the definition of distributional derivatives to show that Du coincides with any limit point μ of (Du_j). The converse implication follows from the Uniform Boundedness Theorem.

Lastly, we state the following compactness theorem for functions in BV. Since the Sobolev Space $W^{1,1}$ has no similar compactness property, this gives us good justification for the introduction of BV in the Calculus of Variations.

Theorem 2.5. *Let* (u_j) *be a sequence in BV*$(\Omega;\mathbb{R}^N)$ *satisfying*

$$\sup\left\{\int_A |u_j|\,\mathrm{d}x + |Du_j|(A) : j\in\mathbb{N}\right\} < \infty \quad \forall A \subset\subset \Omega \text{ open}\,.$$

Then there is a subsequence (u_{j_k}) *converging in* $L^1(\Omega;\mathbb{R}^N)$ *to* $u \in$ *BV*$(\Omega;\mathbb{R}^N)$. *If* Ω *has a compact Lipschitz boundary and* (u_j) *is bounded in BV*$(\Omega;\mathbb{R}^N)$*, then the subsequence converges weakly* in BV to* u.

We remark that we may generalise the last sentence of the theorem, requiring only that Ω is a *bounded extension domain*. This means that exists a linear and continuous extension operator that extends BV functions defined on Ω into $\mathbb{R}^n$ in a suitably "good" way (for further details, refer to [14]).

2.3 Mollification of BV functions

Let B denote the open unit ball in $\mathbb{R}^n$ and write B_ϵ to mean the ball of radius $\epsilon > 0$ centred at the origin. Let ϕ be a symmetric convolution kernel in $\mathbb{R}^n$. That is, it satisfies $\phi \in C_c^\infty(B)$, $\phi \geq 0$, $\int \phi = 1$, $\phi(x) = \phi(-x)$, and $\operatorname{supp}(\phi) \subset\subset B$. Now recall that by $(\phi_\epsilon)_{\epsilon>0}$ we denote the family of mollifiers $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$. Given μ, a $\mathbb{R}^N$-valued Radon measure on Ω, define the function

$$\mu * \phi_\epsilon(x) := \int_\Omega \phi_\epsilon(x-y)\,\mathrm{d}\mu(y) = \epsilon^{-n}\int_\Omega \phi\Big(\frac{x-y}{\epsilon}\Big)\,\mathrm{d}\mu(y)$$

whenever $x \in \Omega_\epsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}$.

It is often useful to approximate BV functions with smooth functions using mollification, for example in the proof of Lemma 2.10. The following proposition states some key properties in this context.

Proposition 2.6. *Let $u \in BV(\Omega; \mathbb{R}^N)$ and let $(\phi_\epsilon)_{\epsilon>0}$ be a family of mollifiers.*

(a) The following identity holds in Ω_ϵ

$$\nabla(u * \phi_\epsilon) = Du * \phi_\epsilon\,.$$

(b) If $U \subset\subset \Omega$ is such that $|Du|(\partial U) = 0$, then

$$\lim_{\epsilon \searrow 0} |D(u * \phi_\epsilon)|(U) = |Du|(U)\,.$$

(c) If $K \subset \Omega$ is a compact set, then for all $\epsilon \in (0, dist(K, \partial\Omega))$

$$\int_K |u * \phi_\epsilon - u|\,\mathrm{d}x \leq \epsilon |Du|(\Omega)\,.$$

2.4 Properties of the derivative of BV functions

2.4.1 Absolutely continuous and singular parts of Du

Recall that by the Radon-Nikodým Theorem, if μ is a positive, σ-finite measure and ν is a real or vector (or matrix) valued measure on a measure space, then there is a unique pair of measures ν^a, ν^s such that $\nu^a \ll \mu$, $\nu^s \perp \mu$, and $\nu = \nu^a + \nu^s$. Hence, for a given BV function u, we may decompose Du as $Du = D^a u + D^s u$, where $D^a u$ is the absolutely continuous part of Du with respect to the Lebesgue measure $\mathscr{L}^n$ and $D^s u$

is the singular part of Du with respect to $\mathscr{L}^n$. Moreover, by the Besicovitch Derivation Theorem, we may write $D^a u = \nabla u \lfloor \mathscr{L}^n$, where ∇u is the unique L^1 function given by

$$\nabla u(x) = \lim_{\varrho \to 0} \frac{Du(B(x,\varrho))}{\mathscr{L}^n(B(x,\varrho))}$$

at all points $x \in \Omega$ where this limit is finite. In fact, we do not need to take balls of radius ϱ in the above expression: if we have any bounded, convex, open set C containing the origin, and write $C(x,\varrho) := x + \varrho C$, then we also obtain the same limit when we replace all instances of $B(x,\varrho)$ with $C(x,\varrho)$. A proof of this may be found, for instance, in [12]. In particular, we will be interested in applying this when C is a cube in $\mathbb{R}^n$. For such points x where the above expression is finite, we say that u is *approximately differentiable at* x, and call the set of such points $\mathcal{D}_u$. By a result of Calderón and Zygmund, any function $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ is approximately differentiable at $\mathscr{L}^n$-almost every point of Ω. If we let S denote the set of points in Ω where u is not approximately differentiable, i.e.

$$S := \{x \in \Omega : \lim_{\varrho \searrow 0} \varrho^{-n}|Du|(B(x,\varrho)) = \infty\},$$

then $D^a u = Du\lfloor(\Omega \setminus S)$ and $D^s u = Du\lfloor S$.

2.4.2 Jump discontinuities

The notion of *jump discontinuities* of u plays a key role in our proof of Theorem 6.2, so we shall now examine in more detail this particular property of BV functions. To begin with let us note that, similar to the notion of approximate differentiability, we may also consider the larger set of points $x \in \Omega$ where u is *approximately continuous*. Namely, these are the points x satisfying

$$\lim_{\varrho \searrow 0} \fint_{B(x,\varrho)} |u(y) - z| \,\mathrm{d}y = 0$$

for some $z \in \mathbb{R}^N$ (which will be unique for each x). The set of points in Ω where this property does not hold is called the *approximate discontinuity set* and denoted S_u. It can be shown that $\mathcal{D}_u \subset \Omega \setminus S_u$, and indeed there are very many more interesting results that can be established about this set. For our present purposes let us specify, among these approximate discontinuity points, those that correspond to an approximate jump discontinuity between two values along a direction ν. To do this we introduce the notation

$$\begin{cases} B_\varrho^+(x,\nu) := \{y \in B(x,\varrho) : \langle y-x,\nu\rangle > 0\} \\ B_\varrho^-(x,\nu) := \{y \in B(x,\varrho) : \langle y-x,\nu\rangle < 0\} \end{cases}$$

to denote the two half balls contained in $B(x,\varrho)$ split by the hyperplane that passes through x and is orthogonal to ν. It is often also convenient to define

$$u_{a,b,\nu}(y) := \begin{cases} a & \text{if } \langle y,\nu\rangle > 0 \\ b & \text{if } \langle y,\nu\rangle < 0 \end{cases}$$

to be the function that jumps between a and b along the hyperplane orthogonal to ν. We may now give the following definition, recalling that by $\mathbb{S}^{n-1}$ we mean the sphere of radius 1 in $\mathbb{R}^n$.

Definition 2.7. Let $u \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $x \in \Omega$. Then x is an *approximate jump point of* u if there exist $a, b \in \mathbb{R}^N$ and $\nu \in \mathbb{S}^{n-1}$ such that $a \neq b$ and

$$\lim_{\varrho \searrow 0} \fint_{B_\varrho^+(x,\nu)} |u(y)-a|\,\mathrm{d}y = 0\,, \quad \lim_{\varrho \searrow 0} \fint_{B_\varrho^-(x,\nu)} |u(y)-b|\,\mathrm{d}y = 0\,. \tag{2.4}$$

The triplet (a,b,ν), uniquely determined by (2.4) up to a permutation of (a,b) and a change of sign of ν, is denoted by $(u^+(x),u^-(x),\nu_u(x))$. The set of approximate jump points of u is denoted J_u.

It can be shown that J_u is a Borel subset of S_u and that there exist Borel functions

$$\big(u^+(x),u^-(x),\nu_u(x)\big) : J_u \to \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{n-1}$$

such that (2.4) is satisfied at any $x \in J_u$. In fact, for $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$, S_u is a countably $\mathscr{H}^{n-1}$-rectifiable set and if we fix an orientation $\overline{\nu}$ of S_u, we have $\overline{\nu} = \nu_u(x)$ for all $x \in J_u$. This allows us to give a characterisation of Du at all points x of J_u, namely that it can be computed by difference of the one-sided limits $u^+(x)$ and $u^-(x)$ of u on either side of the jump set J_u along the normal vector $\nu_u(x)$. More precisely, we have the following result, attributable to Federer and Vol'pert.

Theorem 2.8. *Let $u \in BV(\Omega;\mathbb{R}^N)$. Then S_u is countably $\mathscr{H}^{n-1}$-rectifiable and $\mathscr{H}^{n-1}(S_u \setminus J_u) = 0$. Moreover, we have*

$$Du\lfloor J_u = (u^+ - u^-) \otimes \nu_u \mathscr{H}^{n-1}\lfloor J_u$$

In addition, for $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$, there is a countable sequence of C^1 hypersurfaces Γ_i, say, which covers $\mathscr{H}^{n-1}$-almost all of S_u, i.e.

$$\mathscr{H}^{n-1}\left(S_u \setminus \bigcup_{i=1}^{\infty} \Gamma_i\right) = 0\,.$$

2.4.3 Decomposition of $D^s u$ and rank-one properties

By these above definitions and results, we are now in a position to further split $D^s u$ into two parts: for any $u \in \mathrm{BV}(\Omega; \mathbb{R}^N)$, the measures

$$D^j u := D^s u \lfloor J_u \,, \quad D^c u := D^s u \lfloor (\Omega \setminus S_u)$$

are called respectively the *jump* part of the derivative and the *Cantor* part of the derivative. Hence we may now decompose Du as $Du = D^a u + D^j u + D^c u$. Notice that the above considerations about $D^a u$ and Theorem 2.8 imply

$$D^j u(B) = \int_{B \cap J_u} \big(u^+(x) - u^-(x)\big) \otimes \nu_u(x) \, \mathrm{d}\mathscr{H}^{n-1}(x)$$

and

$$D^a u(B) = \int_B \nabla u(x) \, \mathrm{d}\mathscr{L}^n(x)$$

for all Borel subsets B of Ω; for $|D^j|$ and $|D^a|$ we simply take the modulus of the integrands. The Lebesgue Differentiation Theorem implies that these two components of Du can be obtained by restrictions of Du to the points $x \in \Omega$ where $\varrho \mapsto |Du|(B(x, \varrho))$ is comparable with ϱ^n (for $D^a u$) and ϱ^{n-1} (for $D^j u$). The Cantor part of Du has intermediate behaviour and is trickier to characterise: unlike the absolutely continuous and jump parts of BV functions, the Cantor parts can only be seen as a measure and cannot be recovered by classical analysis of the pointwise behaviour of a functions. Indeed the Cantor-Vitali function (see Example 2.9) whose distributional derivative has no jump part and no absolutely continuous part, demonstrates that for general BV functions u, not all of $D^s u$ may be captured by $D^j u$. However, it is not too complicated to show that $D^c u$ vanishes on sets which are σ-finite with respect to $\mathscr{H}^{n-1}$. We define the proper subspace of $\mathrm{BV}(\Omega; \mathbb{R}^N)$, called functions of *Special Bounded Variation*, $\mathrm{SBV}(\Omega; \mathbb{R}^N)$, to be the space of BV functions where $D^s u = D^j u$ only. For an introduction to this space, we refer to [9, 10, 41].

Let us now turn our attention to the quantity $\xi = \frac{Du}{|Du|}$ i.e. the Radon-Nikodým derivative of the measure Du with respect to its variation $|Du|$ given by the expression

$$\xi(x) = \frac{Du}{|Du|}(x) = \lim_{\varrho \to 0} \frac{Du(B(x, \varrho))}{|Du|(B(x, \varrho))} \,, x \in \Omega.$$

Recalling that the support of a measure μ in Ω is defined by

$$\mathrm{supp}(\mu) = \{x \in \Omega : \mu(\Omega \cap B(x, \varrho)) > 0 \, \forall \varrho > 0\} \,,$$

we may consider $\xi(x)$ at the various parts of Ω in the support of the various components of Du. It follows straightforwardly from the basic properties described above that for $|D^a u|$-almost all $x \in \mathrm{supp}(|D^a|)$ we have

$$\xi(x) = \frac{\nabla u(x)}{|\nabla u(x)|},$$

and for $|D^j u|$-almost all $x \in \mathrm{supp}(|D^j|)$ (equivalently $x \in J_u$),

$$\xi(x) = \frac{u^+(x) - u^-(x)}{|u^+(x) - u^-(x)|} \otimes \nu_u(x).$$

Note that in this case, $\xi(x)$ is a rank-one matrix. It is much harder to establish properties of $\xi(x)$ for $x \in \mathrm{supp}(|D^c|)$. In [5], Alberti proved the famous result that $\xi(x)$ is also rank-one for $|D^c|$-almost every point. The proof of this property is very long and involved; a simpler proof based on the area formula and Reshetnyak continuity theorem is given in [6], but this proof only works for monotone BV functions. These properties of $\xi(x)$ are instrumental in the proof of Theorem 6.2, in particular in the context of the key blow-up lemma that the entirety of the final section of this chapter is devoted to.

Example 2.9 (The Cantor-Vitali function)**.** In this classical example we construct a BV function of one variable which is continuous and differentiable, with 0 derivative, almost everywhere. Recall that the Cantor middle third set C is defined to be $\cap_k C_k$, where $C_0 = [0,1]$ and C_{k+1} is obtained from C_k by splitting all intervals of C_k into three closed intervals of equal length, and removing the interior of the middle one. Note that the Cantor set is characterised by the property

$$C = \tfrac{1}{3}C \cup (\tfrac{1}{3}C - 2),$$

and C_k consists of 2^k pairwise disjoint intervals of length $(1/3)^k$, so $|C_k| = (2/3)^k$, which tends to 0 as $k \to \infty$; hence $|C| = 0$. Now define inductively a sequence of increasing, surjective functions $f_k \colon \mathbb{R} \to [0,1]$ by setting $f_0(t) = \mathrm{mid}\{0, t, 1\}$ and

$$f_{k+1}(t) = \frac{1}{2} \cdot \begin{cases} f_k(3t) & \text{if } t \in (-\infty, 1/3] \\ 1 & \text{if } t \in [1/3, 2/3] \\ 1 + f_k(3t-2) & \text{if } t \in [2/3, \infty) \end{cases} \quad \forall k \geq 0.$$

We may verify by induction that $f_k = 0$ for $t \leq 0$, $f_k = 1$ for $t \geq 1$, and for $l \geq k \geq 0$ $f_l = f_k$ is constant in any interval of $\mathbb{R} \setminus C_k$. Moreover, (f_k) is a Cauchy sequence in $C([0,1])$, so uniformly converges in $[0,1]$ to some continuous function f, which is still increasing and maps $[0,1]$ onto $[0,1]$. However, this function is constant in any connected component of $(0,1) \setminus C$, which has full Lebesgue measure in $(0,1)$. So $D^a f = 0$ $\mathscr{L}^1$-almost everywhere but Df has no jump part, as it is continuous. Hence we conclude that f is a Cantor function.

2.5 Sets of finite perimeter

Let E be a subset of Ω, and define the characteristic function $\mathbf{1}_E$ of E as

$$\mathbf{1}_E(x) = \begin{cases} 1 & \text{if } x \in E\,, \\ 0 & \text{if } x \in \Omega \setminus E\,. \end{cases}$$

We say that a set E is of *finite perimeter in* Ω if $\mathbf{1}_E \in \mathrm{BV}(\Omega;\mathbb{R}^N)$. Now define the *reduced boundary of* E, $(\partial^* E \cap \Omega)$, as

$$(\partial^* E \cap \Omega) = S_{\mathbf{1}_E}\,.$$

Note that $|D\mathbf{1}_E|(\Omega) = \mathscr{H}^{n-1}(\partial^* E \cap \Omega)$ for every E of finite perimeter. It is easy to verify that this notion of perimeter coincides with the elementary one, particularly when E is a polyhedron. In [42], De Giorgi shows that if E is a set of finite perimeter, then there exists a sequence of polyhedra (P_j) such that $|((P_j \setminus E) \cup (E \setminus P_j)) \cap \Omega| \to 0$, and

$$\mathscr{H}^{n-1}(\partial^* \{u > t\}) = \lim_{j\to\infty} \mathscr{H}^{n-1}(\partial P_j \cap \Omega)\,.$$

This shows that the measure-theoretic notion of perimeter is a sensible extension of the elementary one.

2.6 Properties of blow-up limits on the singular part

In this section we state and prove a result from [12] that is essential to our proof of Theorem 6.2. First let us note that when blowing up a function $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ at a point $x_0 \in \Omega$, we will need to use the following identities. Let Q denote the open unit cube $(-\frac{1}{2}, \frac{1}{2})^n$ in $\mathbb{R}^n$ and, consistent with the previous section, define

$$Q(x_0, \varrho) := \{\varrho y + x_0 : y \in Q\}\,.$$

Now for $y \in Q$ and ϱ sufficiently small, let

$$u_\varrho(y) := \varrho^{-1} u(x_0 + \varrho y)\,. \tag{2.5}$$

It follows from basic definitions that

$$Du_\varrho(Q) = \varrho^{-n} Du(Q(x_0,\varrho)), \quad \text{and} \quad |Du_\varrho|(Q) = \varrho^{-n}|Du|(Q(x_0,\varrho)). \tag{2.6}$$

Theorem 2.10. *Let $u \in BV(\Omega;\mathbb{R}^N)$, and let $\xi\colon \Omega \to \mathbb{R}^{N\times n}$ denote the density of Du with respect to $|Du|$. Then, for $|D^s u|$-almost all $x_0 \in \Omega$ we have $|\xi(x_0)| = 1$, rank$(\xi(x_0)) = 1$, and*

$$\lim_{\varrho\to 0^+} \frac{Du(Q(x_0,\varrho))}{|Du|(Q(x_0,\varrho))} = \xi(x_0), \quad \lim_{\varrho\to 0^+} \frac{Du(Q(x_0,\varrho))}{\varrho^n} = +\infty. \tag{2.7}$$

Let $x \in$ supp$(|Du|)$ with these properties, and write $\xi(x_0) = \eta\otimes\nu$ where $\eta \in \mathbb{R}^n$, $\nu \in \mathbb{R}^N$, $|\eta| = |\nu| = 1$. Now let

$$v_\varrho(y) = \frac{\varrho^n}{|Du|(Q(x_0,\varrho))}(u_\varrho(y) - m_\varrho), \tag{2.8}$$

where u_ϱ is defined in (2.5) *and m_ϱ is the mean value of u_ϱ on Q (with respect to Lebesgue measure). Then for ϱ sufficiently small and for every $0 < \sigma \le 1$ we have*

$$\int_Q v_\varrho \,\mathrm{d}y = 0, \quad |Dv_\varrho|(\sigma Q) = \frac{|Du|(Q(x_0,\sigma\varrho))}{|Du|(Q(x_0,\varrho))} \le 1. \tag{2.9}$$

Moreover, for every $0 < \sigma < 1$ there exists a decreasing sequence (ϱ_k) converging to 0 such that (v_{ϱ_k}) converges weakly in $BV(Q;\mathbb{R}^N)$ to a function $v \in BV(Q;\mathbb{R}^N)$ satisfying*

$$|Dv|(\sigma\bar{Q}) \ge \sigma^n,$$

and which can be represented as

$$v(y) = \psi(\langle y,\nu\rangle)\eta \tag{2.10}$$

for a suitable non-decreasing function $\psi\colon (a,b) \to \mathbb{R}$, where

$$a = \inf\{\langle y,\nu\rangle : y \in \mathbb{R}\}, \quad b = \sup\{\langle y,\nu\rangle : y\in\mathbb{R}\}.$$

Proof of Theorem 2.10. As has been stated earlier in this chapter, the rank-one properties of $\xi(x_0)$ are proved in [5]. For the properties in (2.7) refer to the discussion in Section 2.4.1 (or [12] for full details). Let $x_0 \in$ supp$(|D^s u|)$, $\eta\in\mathbb{R}^N$, $\nu\in\mathbb{R}^n$ and $0<\sigma<1$ be as in the statement of the Theorem. Now we show that

$$\limsup_{\varrho\to 0^+} \frac{|Du|(Q(x_0,\sigma\varrho))}{|Du|(Q(x_0,\varrho))} > \sigma^n. \tag{2.11}$$

If this were false, then there would exist $\varrho_0 > 0$ such that

$$|Du|(Q(x_0, \sigma\varrho)) \leq \sigma^n |Du|(Q(x_0, \varrho))$$

for all $0 < \varrho \leq \varrho_0$. Now note that for any $j \in \mathbb{N}$, since $0 < \sigma^j \varrho_0 < \varrho_0$,

$$\begin{aligned}
|Du|(Q(x_0, \sigma^j \varrho_0)) &= |Du|(Q(x_0, \sigma(\sigma^{j-1}\varrho_0))) \\
&\leq \sigma^n |Du|(Q(x_0, \sigma^{j-1}\varrho_0)) \\
&= \sigma^n |Du|(Q(x_0, \sigma(\sigma^{j-2}\varrho_0))) \\
&\leq \sigma^{2n} |Du|(Q(x_0, \sigma^{j-2}\varrho_0)) \\
&\dots \\
&\leq \sigma^{jn} |Du|(Q(x_0, \varrho_0))\,.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
|Du|(Q(x_0, \varrho_0)) &\geq \frac{|Du|(Q(x_0, \sigma^j \varrho_0))}{\sigma^{jn}} \\
&\to \infty \quad \text{as } j \to \infty\,,
\end{aligned}$$

which is a contradiction. Hence (2.11) holds and, using this and (2.9), there exists decreasing sequence (ϱ_k) such that $Q(x_0, \varrho_1) \subset \Omega$, converging to 0, with

$$\lim_{k \to \infty} |Dv_{\varrho_k}|(\sigma Q) > \sigma^n\,. \tag{2.12}$$

Let (v_k) denote the sequence (v_{ϱ_k}). By Theorem 2.5 and (2.9) we have, passing to a subsequence if necessary, that the sequence (v_k) converges weakly* in BV$(Q; \mathbb{R}^N)$ to some map $v \in$ BV$(Q; \mathbb{R}^N)$. Now note

$$\frac{Dv_k(sQ)}{|Dv_k|(sQ)} = \frac{Du(Q(x_0, s\varrho_k))}{|Du|(Q(x_0, s\varrho_k))} \to \xi(x_0) = \eta \otimes \nu \tag{2.13}$$

as $k \to \infty$ for any $s \in (0, 1)$. Passing to another subsequence if necessary, we can assume that the measures $|Dv_k|$ converge weakly* to a Radon measure μ in Q. Since the total variation is lower semicontinuous with respect to this convergence, we have $|Dv| \leq \mu$. Since (Dv_k) converge weakly* to Dv, for any $s \in (0, 1)$ with $\mu(\partial(sQ)) = 0$ we have

$$Dv_k(sQ) \to Dv(sQ)\,, \quad |Dv_k|(sQ) \to \mu(sQ)\,.$$

By (2.12), this implies $\mu(sQ) \geq \sigma^n$ if $s > \sigma$, and so $\mu(\sigma\bar{Q}) \geq \sigma^n$. By (2.13), we get

$$Dv(sQ) = (\eta \otimes \nu)\mu(sQ)$$

for any $s \in (\sigma, 1)$ with $\mu(\partial(sQ)) = 0$. By approximation, this also holds for $s \in (0, 1]$, hence

$$|Dv|(Q) \le \mu(Q) = |Dv(Q)| \le |Dv|(Q)\,,$$

and so $|Dv|(Q) = \mu(Q) = |Dv(Q)|$. Since $|Dv| \le \mu$, we in fact have $\mu = |Dv|$ and, in particular, $|Dv|(\sigma\bar{Q}) \ge \sigma^n$.

Let γ denote the density of Dv with respect to $|Dv|$. Recall that for a matrix $A \in \mathbb{R}^{N\times n}$, we can write

$$|A|^2 = \mathrm{tr}(AA^t)\,,$$

and hence, for A, $B \in \mathbb{R}^{N\times n}$, we have

$$|A - B|^2 = |A|^2 + |B|^2 - \mathrm{tr}(AB^t + BA^t)\,.$$

Therefore we get

$$\int_Q \left|\gamma - \frac{Dv(Q)}{|Dv|(Q)}\right|^2 |Dv| \\ = \int_Q |\gamma|^2 + \left|\frac{Dv(Q)}{|Dv|(Q)}\right|^2 - \mathrm{tr}\left(\gamma\frac{Dv(Q)^t}{|Dv|(Q)} + \frac{Dv(Q)}{|Dv|(Q)}\gamma^t\right)|Dv|\,.$$

Now note that $|\gamma| = 1$ $|Dv|$-almost everywhere, and certainly

$$\left|\frac{Dv(Q)}{|Dv|(Q)}\right|^2 = 1\,,$$

so

$$\int_Q |\gamma|^2 + \left|\frac{Dv(Q)}{|Dv|(Q)}\right|^2 |Dv| = 2|Dv|(Q)\,.$$

Next note that

$$\int_Q \gamma\,|Dv| = Dv(Q)\,,$$

so

$$\begin{aligned}
&\int_Q \mathrm{tr}\left(\gamma\frac{Dv(Q)^t}{|Dv|(Q)} + \frac{Dv(Q)}{|Dv|(Q)}\gamma^t\right)|Dv| \\
&\qquad = \mathrm{tr}\left(\int_Q \gamma\,|Dv|\,\frac{Dv(Q)^t}{|Dv|(Q)}\right) + \mathrm{tr}\left(\frac{Dv(Q)}{|Dv|(Q)}\left(\int_Q \gamma\,|Dv|\right)^t\right) \\
&\qquad = \mathrm{tr}\left(Dv(Q)\frac{Dv(Q)^t}{|Dv|(Q)}\right) + \mathrm{tr}\left(\frac{Dv(Q)}{|Dv|(Q)}Dv(Q)^t\right) \\
&\qquad = 2\frac{|Dv|(Q)^2}{|Dv|(Q)}
\end{aligned}$$

$$= 2|Dv|(Q)\,.$$

Thus

$$\int_Q \left|\gamma - \frac{Dv(Q)}{|Dv|(Q)}\right|^2 |Dv| = 0\,,$$

so

$$\gamma(x) = \frac{Dv(Q)}{|Dv|(Q)} = \eta \otimes \nu \quad \text{for } |Dv|\text{-a.e. } x \in Q\,. \tag{2.14}$$

Now we use a straightforward mollification argument to show that any function $v \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ satisfying (2.14) can be represented as in (2.10) for some non-decreasing real valued function ψ. By a suitable rotation of Ω and $\mathbb{R}^N$, we may assume without loss of generality that $\eta = \varepsilon_1$ and $\nu = e_1$, where $\{\varepsilon_1,\dots,\varepsilon_N\}$ and $\{e_1,\dots,e_n\}$ are the canonical bases of $\mathbb{R}^N$ and $\mathbb{R}^n$ respectively. From (2.14) we have

$$Dv = (\varepsilon_1 \otimes e_1)|Dv|\,.$$

Let $\epsilon > 0$ and ϕ_ϵ be a mollifier as in Section 2.3. Then by Proposition 2.6,

$$\nabla(v * \phi_\epsilon) = Du * \phi_\epsilon = (\varepsilon_1 \otimes e_1)(|Dv| * \phi_\epsilon)$$

on $Q_\epsilon = \{x \in Q : \mathrm{dist}(x,\partial Q) > \epsilon\}$. Hence, writing $v = (v^1,\dots,v^N)$, we have for all $x \in Q_\epsilon$

$$\frac{\partial}{\partial x_j}(v^i * \phi_\epsilon)(x) = 0 \quad \text{if } (i,j) \neq (1,1)$$

and

$$\frac{\partial}{\partial x_1}(v^1 * \phi_\epsilon)(x) = \alpha(x)$$

for some $\alpha(x) > 0$. Hence $v^i * \phi_\epsilon$ is some constant for all $i \geq 2$, and since from (2.9) we have $\int_Q v = 0$, it follows that in fact these $v^i * \phi_\epsilon$ are zero. Moreover, for $x = (x_1,\dots,x_n) \in Q_\epsilon$, $v^1 * \phi_\epsilon$ only depends on the variable x_1, and it is an increasing function of this variable. Thus we deduce that $v * \phi_\epsilon$ can be represented on Q_ϵ as

$$\begin{aligned} &v^1 * \phi_\epsilon(x) = g(x_1) \\ &v^i * \phi_\epsilon(x) = 0 \quad \text{if } 2 \leq i \leq N\,, \end{aligned}$$

for some non-increasing real-valued function g. We conclude by letting ϵ tend to 0. □

Chapter 3

Trace-preserving operators

The proofs of the main results of this thesis all involve, at some stage, obtaining higher integrability estimates for trace-preserving operators. In the first sections, we extend functions defined on $\mathbb{R}^{n-1}$ ($n \geq 2$) by mollification into the half-space $\mathbb{R}^n_+$ consisting of points in $\mathbb{R}^n$ whose n^{th} coordinate is non-negative. The definition of the operator is the same in all these cases, but we obtain different integral estimates depending on the domain we are considering. We first prove a result by Kristensen [64] and obtain superlinear integral estimates when extending functions in $W^{1,1}$, and then prove a new result to obtain subquadratic integral estimates when the functions are additionally assumed to be in L^q for q suitably large. These results are of crucial importance in our proofs of the main results in Chapters 4 and 6.

In the final section, we adapt a proof of Fonseca and Malý [50] to construct a linear operator Tu from $W^{1,1}$ into $W^{1,1}$ that preserves boundary values and improves the integrability of u and ∇u across a "layer" given by level sets of a real-valued smooth function defined on the domain. This result is of central importance in our proof of the measure representation of a Lebesgue-Serrin extension in Chapter 5, which in turn plays a role in parts of Chapter 6.

3.1 Preliminaries for extension operator

The operator used in Chapters 4 and 6 extends functions defined on the surface of the unit ball ∂B into the annulus $B_2 \setminus \bar{B}$, where B_2 is the ball of radius 2 centred at the origin. By a standard localisation argument (see, for example, [75]), it suffices to provide proofs in the context of extending functions defined on $\mathbb{R}^{n-1}$ into the half-space $\mathbb{R}^n_+$ defined as follows: denote points in $\mathbb{R}^n$ by (x, t), where $x \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}$, and let

$$\mathbb{R}^n_+ := \{(x,t) \in \mathbb{R}^n : t > 0\}\,.$$

A key component of all these proofs depends on the following lemma due to Greco, Iwaniec and Moscariello [61]. Before stating this lemma, we shall establish some definitions. For $g \in L^1_{\mathrm{loc}}(\mathbb{R}^m)$, recall that the Hardy-Littlewood maximal function is defined as:

$$(Mg)(x) := \sup_{\varrho>0} \fint_{\varrho\mathcal{B}^m} |g(x-y)|\,\mathrm{d}y\,, \quad x \in \mathbb{R}^m\,.$$

Let $\Theta\colon [0,\infty) \to [0,\infty)$ be a non-decreasing, right-continuous function and define the function $\Psi\colon [0,\infty) \to [0,\infty)$ as

$$\Psi(t) := \Theta(t) + t\int_0^t \frac{\Theta(s)}{s^2}\,\mathrm{d}s\,, \quad t \geq 0\,. \tag{3.1}$$

Lemma 3.1. [61] *Let Θ, $\Psi\colon [0,\infty) \to [0,\infty)$ be defined as above. If $g \in L^1(\mathbb{R}^m)$, then*

$$\int_{\mathbb{R}^m} \Theta(Mg)\,\mathrm{d}x \leq 2\cdot 5^m \int_{\mathbb{R}^m} \Psi(2|g|)\,\mathrm{d}x\,.$$

Further details on maximal functions may be found, for example, in [92]. The extension operator $\mathbf{E}$ is defined in the same way throughout and we get different integral estimates depending on the domain we are considering. In all cases, since for $r \geq 1$ the function $t \mapsto t^r$ is convex (or, in the final instance, the function Φ is convex), we need only consider the case where $N = 1$. Let $\mathcal{B}^{n-1}$ denote the open unit ball in $\mathbb{R}^{n-1}$. To define $\mathbf{E}$, we take a standard convolution kernel K in $\mathbb{R}^{n-1}$ supported in the $\mathcal{B}^{n-1}$ (i.e. K satisfies $K \in C^\infty_c(\mathbb{R}^{n-1})$, $K \geq 0$, $\int K = 1$, $\mathrm{supp}(K) \subset\subset \mathcal{B}^{n-1}$) and a function $\eta \in C^\infty_c(\mathbb{R})$ with $\eta(0) = 1$, $\eta(1) = 0$. Then we let

$$(\mathbf{E}h)(x,t) := \eta(t)(K_t * h)(x)\,, \quad (x,t) \in \mathbb{R}^n_+\,, \tag{3.2}$$

whenever $h \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$. A similar construction is used, for example, in [91]. It is immediately seen that $\mathbf{E}$ maps $L^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$ into $L^1_{\mathrm{loc}}(\mathbb{R}^n_+) \cap C^\infty(\mathbb{R}^n_+)$. In the subsequent sections of this chapter, we use this definition to obtain additional boundedness properties.

3.2 Extension into superlinear Sobolev spaces

In this section we state and prove a simplified version of a result of Kristensen which is used at various points in our proof of Theorem 6.2 in Chapter 6. Unlike the Lemmas in the next sections of this chapter, there is no requirement that functions in the domain also be in $L^q_{\mathrm{loc}}(\mathbb{R}^{n-1};\mathbb{R}^N)$ for q suitably large. The consequence of this, however, is that for $n \geq 3$ we get a weaker integral estimate for the extended functions.

Lemma 3.2. [64] *Let $r \in [1, \frac{n}{n-1})$. There exists a linear extension operator*

$$\mathbf{E}\colon W^{1,1}_{\mathrm{loc}}(\mathbb{R}^{n-1};\mathbb{R}^N) \to W^{1,r}_{\mathrm{loc}}(\mathbb{R}^n_+;\mathbb{R}^N)$$

with the following properties:

1. *If $h \in C^1(\mathbb{R}^{n-1};\mathbb{R}^N)$ then $(\mathbf{E}h)(x,0) := \lim_{t\to 0^+}(\mathbf{E}h)(x,t) = h(x)$ for all $x \in \mathbb{R}^{n-1}$.*
2. *If $(z_j) \subset C^\infty(\mathbb{R}^{n-1};\mathbb{R}^N)$ and $z_j \to 0$ in the sense of distributions, then for any multi-index α, $\partial^\alpha[\mathbf{E}z_j] \to 0$ locally uniformly in $\mathbb{R}^n_+$.*
3. *For all $R > 0$ there exist positive constants c_1, c_2, dependent on n, N, r, R, such that for all $h \in W^{1,1}(\mathbb{R}^{n-1};\mathbb{R}^N)$ with support contained in $\{x \in \mathbb{R}^{n-1} : |x| \leq R\}$ we have*

   (a)
   $$\int_{\mathbb{R}^n_+} |\mathbf{E}h|^r \,\mathrm{d}\mathscr{L}^n \leq c_1 \|h\|^r_{L^1(\mathbb{R}^{n-1})}$$

   (b)
   $$\int_{\mathbb{R}^n_+} |\nabla[\mathbf{E}h]|^r \mathscr{L}^n \leq \left(c_2 \int_{\mathbb{R}^{n-1}} |\nabla h| \,\mathrm{d}\mathscr{H}^{n-1}\right)^r.$$

As observed above, we may restrict our attention to the case $N = 1$, and define the operator $\mathbf{E}$ as in (3.2). It is straightforward to see that $\mathbf{E}$ satisfies properties (1) and (2) of Lemma 3.2. To prove that it maps the given domain into $W^{1,r}_{\mathrm{loc}}(\mathbb{R}^n_+)$, it suffices to prove 3(a) and (b). To show this, we first notice that

$$\begin{aligned}|\nabla(\mathbf{E}h)(x,t)| \leq |\eta(t)| &\left((K_t * |\nabla h|)(x) + \int_{\mathbb{R}^{n-1}} K(y)|y||\nabla h(x-ty)|\,\mathrm{d}y\right)\\ &+ |\eta'(t)|(K_t * |h|)(x)\,,\end{aligned}$$

and thus it suffices to prove the following lemma.

Lemma 3.3. *Let $K \in C^\infty(\mathbb{R}^{n-1})$ be non-negative and supported in the unit ball and let $\eta \in C^\infty(\mathbb{R})$ be non-negative and supported in $(-1,1)$. For a non-negative function $h \in L^1_{loc}(\mathbb{R}^{n-1})$ define the function $H(x,t) := \eta(t)(K_t * h)(x)$. Then for all $R > 0$ and $r \in [1, \frac{n}{n-1})$ there exists a constant c, depending on m, r and R, such that*

$$\int_{\mathbb{R}^n_+} H(x,t)^r \,\mathrm{d}(x,t) \leq \left(c \int_{\mathbb{R}^{n-1}} h(x)\,\mathrm{d}x \right)^r \tag{3.3}$$

for all non-negative functions h supported in $\{x \in \mathbb{R}^{n-1} : |x| \leq R\}$.

3.2.1 Non-tangential maximal functions

The proof of Lemma 3.3 depends on Lemma 3.1 and the following result that is due to Calderón and Torchinsky [27]: let $H = H(x,t)\colon \mathbb{R}^n_+ \to \mathbb{R}$ be a continuous function. Define for each $\alpha \geq 0$ the non-tangential maximal function associated with H

$$\Lambda_\alpha H(x) := \sup_{|x-y|\leq \alpha t} |H(y,t)| \,, \, x \in \mathbb{R}^{n-1} \,.$$

Clearly, $\Lambda_\alpha H\colon \mathbb{R}^{n-1} \to [0,+\infty]$ is lower semicontinuous, and hence in particular measurable.

Lemma 3.4. [27] *Suppose that $\alpha > 0$, $q > p > 0$ and that $\Lambda_\alpha H \in L^p(\mathbb{R}^{n-1})$. Then there exist constants $c_1 = c_1(\alpha, p)$ and $c_2 = c_2(\alpha, p, q)$, such that*

$$|H(x,t)| \leq c_1 t^{\frac{1-n}{p}} \|\Lambda_\alpha H\|_p \tag{3.4}$$

and

$$\left(\int_{\mathbb{R}^n_+} |H(x,t)|^q t^{(n-1)\frac{q}{p}-n} \,\mathrm{d}(x,t) \right)^{\frac{1}{q}} \leq c_2 \|\Lambda_\alpha H\|_p \,. \tag{3.5}$$

Proof of Lemma 3.4. First note that if $|x-y| \leq \alpha$ then

$$\Lambda_\alpha H(y) \geq |H(x,t)| \,.$$

Now raise this inequality to the power p and integrate with respect to y over $\{|x-y| \leq \alpha t\}$ to get

$$|\mathcal{B}^{n-1}| |H(x,t)|^p (\alpha t)^{n-1} \leq \|\Lambda_\alpha H\|_p^p \,,$$

which gives us (3.4).

Now put $b = c_1\|\Lambda_\alpha H\|_p$, where c_1 is the constant in (3.4), so

$$|H(x,t)| \leq bt^{\frac{1-n}{p}}\,.$$

Fix t and consider the set $\{|H| > s\}$. Clearly this is empty if $s \geq bt^{\frac{1-n}{p}}$, so, by the Layer-cake representation (see for example [68]) we have

$$\int_{\mathbb{R}^{n-1}\times\{t\}} |H(x,t)|^q \,\mathrm{d}x = q\int_0^{bt^{\frac{1-n}{p}}} |\{|H| > s\}| s^{q-1}\,\mathrm{d}s\,.$$

Moreover, since $|H(x,t)| \leq \Lambda_\alpha H(x)$ we have

$$|\{|H| > s\}| \leq |\{\Lambda_\alpha H > s\}|\,.$$

We can substitute this into the above expression to get

$$\int_{\mathbb{R}^{n-1}\times\{t\}} |H(x,t)|^q \,\mathrm{d}x \leq q\int_0^{bt^{(1-n)/p}} |\{\Lambda_\alpha H > s\}| s^{q-1}\,\mathrm{d}s\,.$$

Now we multiply both sides by $t^{(n-1)\frac{q}{p}-n}$ and integrate with respect to t, applying Fubini's Theorem on the right hand side to interchange the order of integration. This gives us

$$\begin{aligned}
\int_{\mathbb{R}^n_+} |H(x,t)|^q t^{(n-1)\frac{q}{p}-n}\,\mathrm{d}x\,\mathrm{d}t &\leq q\int_0^\infty t^{(n-1)\frac{q}{p}-n}\int_0^{bt^{(1-n)/p}} |\{\Lambda_\alpha H > s\}| s^{q-1}\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_0^\infty |\{\Lambda_\alpha H > s\}| s^{q-1}\int_0^{(b/s)^{\frac{p}{n-1}}} t^{(n-1)\frac{q}{p}-n}\,\mathrm{d}t\,\mathrm{d}s \\
&= \frac{qp}{(n-1)(q-p)} b^{q-p}\int_0^\infty |\{\Lambda_\alpha H > s\}| s^{p-1}\,\mathrm{d}s \\
&= c\|\Lambda_\alpha H\|_p^q\,,
\end{aligned}$$

which completes the proof. □

3.2.2 Proof of the main lemma

Proof of Lemma 3.3. Fix an $\alpha > 0$ and put $\Lambda = \Lambda_\alpha$. If $h \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$ and is non-negative, then note that

$$H(x,t) = \eta(t)\int_{t\mathcal{B}^{n-1}} t^{1-n} K(\tfrac{y}{t}) h(x-y)\,\mathrm{d}\mathscr{H}^{n-1}(y)$$

$$
\begin{aligned}
&\leq \|\eta\|_\infty \|K\|_\infty \int_{t\mathcal{B}^{n-1}} t^{1-n} h(x-y)\,\mathrm{d}y \\
&= \|\eta\|_\infty \|K\|_\infty |\mathcal{B}^{n-1}| ⨍_{B(x,t)} h(y)\,\mathrm{d}y\,,
\end{aligned}
$$

where $B(x,t)$ denotes the open ball in $\mathbb{R}^{n-1}$ with radius t. Hence

$$
\Lambda H(x) \leq c \sup_{|z-x|<\alpha t} ⨍_{B(z,t)} h(y)\,\mathrm{d}y\,.
$$

For a given $x \in \mathbb{R}^{n-1}$, and any $z \in \mathbb{R}^{n-1}$, $t > 0$ satisfying $|z-x| < \alpha t$, note that $B(z,t) \subset B(x, t+\alpha t)$, and hence

$$
\begin{aligned}
⨍_{B(z,t)} h(y)\,\mathrm{d}y &\leq \frac{1}{|t\mathcal{B}^{n-1}|} \int_{B(x,t+\alpha t)} h(y)\,\mathrm{d}y \\
&= (\alpha+1)^{n-1} ⨍_{B(x,t+\alpha t)} h(y)\,\mathrm{d}y \\
&\leq (\alpha+1)^{n-1} Mh(x)\,.
\end{aligned}
$$

Hence, taking the supremum over all such z, t, we have shown that

$$
\Lambda H(x) \leq cMh(x)\,,
$$

where $c = c(\alpha, \eta, K, n)$. Let $\Theta \colon [0,+\infty) \to [0,+\infty)$ be a non-decreasing, right-continuous function and apply Lemma 3.4 with $q = \frac{n}{n-1}$, $p = 1$ and $\Theta \circ H$ instead of H to deduce that

$$
\int_{\mathbb{R}^n_+} (\Theta(H(x,t)))^{\frac{n}{n-1}}\,\mathrm{d}(x,t) \leq \left(c \int_{\mathbb{R}^{n-1}} \Lambda(\Theta \circ H)\,\mathrm{d}x \right)^{\frac{n}{n-1}},
$$

provided $\Lambda(\Theta \circ H) \in L^1(\mathbb{R}^{n-1})$. Since

$$
\Lambda(\Theta \circ H)(x) \leq \Theta(\Lambda H(x)) \leq \Theta(cMh(x))\,,
$$

we get

$$
\int_{\mathbb{R}^n_+} (\Theta(H(x,t)))^{\frac{n}{n-1}}\,\mathrm{d}(x,t) \leq \left(c \int_{\mathbb{R}^{n-1}} \Theta(cMh(x))\,\mathrm{d}x \right)^{\frac{n}{n-1}}.
$$

Now use the Maximal Inequality from Lemma 3.1 on the right-hand side to get the following estimate

$$
\int_{\mathbb{R}^n_+} (\Theta(H(x,t)))^{\frac{n}{n-1}}\,\mathrm{d}(x,t) \leq \left(2 \cdot 5^{n-1} c \int_{\mathbb{R}^{n-1}} \Psi(2ch(x))\,\mathrm{d}x \right)^{\frac{n}{n-1}}. \tag{3.6}
$$

To obtain the required estimate for Lemma 3.3 let $\sigma = \frac{r(n-1)}{n} \in (0,1)$, and take

$$\Theta(t) = \begin{cases} t^\sigma\,, & t \geq 1 \\ t^2\,, & t \in [0,1)\,. \end{cases}$$

From (3.1) we have for all $t \geq 0$

$$\begin{aligned} \Psi(t) &= \Theta(t) + t\int_0^t \frac{\Theta(s)}{s^2}\,\mathrm{d}s \\ &\leq t + t\Big(\int_0^1 1\,\mathrm{d}s + \int_1^\infty s^{\sigma-2}\,\mathrm{d}s\Big) \\ &\leq (2 + \tfrac{1}{1-\sigma})t\,. \end{aligned}$$

Now note that $\Theta \circ H$ restricted to $\{H \geq 1\}$ is in $L^{\sigma^{-1}}(\mathbb{R}^n_+)$, so by the Hardy-Littlewood-Wiener Maximal Theorem $\Lambda(\Theta \circ H)$ is in $L^1(\mathbb{R}^{n-1})$ as required. Therefore, as a consequence of (3.6) we get

$$\begin{aligned} \int_{\{H\geq 1\}} H(x,t)^r\,\mathrm{d}(x,t) &= \int_{\{H\geq 1\}} (\Theta(H(x,t)))^{\frac{n}{n-1}}\,\mathrm{d}(x,t) \\ &\leq \left(2\cdot 5^{n-1}c\int_{\mathbb{R}^{n-1}} \Psi(2ch(x))\,\mathrm{d}x\right)^{\frac{n}{n-1}} \\ &\leq \left(2\cdot 5^{n-1}c\int_{\mathbb{R}^{n-1}} (2+\tfrac{1}{1-\sigma})(2ch(x))\,\mathrm{d}x\right)^{\frac{n}{n-1}} \\ &= \left(c\int_{\mathbb{R}^{n-1}} h(x)\,\mathrm{d}x\right)^{\frac{n}{n-1}}. \end{aligned}$$

If h is supported in $\{x : |x| \leq R\}$, then H is supported in $\{(x,t) : |x| \leq R+1\,, t \in (0,1)\}$. Hence we have

$$\int_{\mathbb{R}^n_+} H(x,t)^r\,\mathrm{d}(x,t) \leq \mathscr{L}^{n-1}(B(0,R+1)) + \left(c\int_{\mathbb{R}^{n-1}} h(x)\,\mathrm{d}x\right)^{\frac{n}{n-1}},$$

where $B(0,R+1)$ is the ball of radius $R+1$ in $\mathbb{R}^{n-1}$. Since $\mathbf{E}$ is a linear operator, the proof of the Lemma 3.3 easily follows by taking a larger constant c. $\square$

3.3 Extension into subquadratic Sobolev spaces

Now we show how the same extension operator from the previous section may also satisfy subquadratic integral estimates, provided the functions in the domain are also in $L^q_{\text{loc}}(\mathbb{R}^{n-1};\mathbb{R}^N)$ for sufficiently large q. This is a new result that involves adapting and generalising a result by Carozza, Kristensen and Passarelli di Napoli [29], and is key in the context of obtaining the proof of Theorem 4.1 in Chapter 4.

Lemma 3.5. *Let $1 < r < 2$. Then for $q \geq \frac{r(n-1)}{2-r}$ there exists a linear extension operator*

$$\mathbf{E}\colon (W^{1,1}_{\text{loc}} \cap L^q_{\text{loc}})(\mathbb{R}^{n-1};\mathbb{R}^N) \to W^{1,r}_{\text{loc}}(\mathbb{R}^n_+;\mathbb{R}^N)$$

with the following properties:

1. *If $h \in C^1(\mathbb{R}^{n-1};\mathbb{R}^N)$ then $(\mathbf{E}h)(x,0) := \lim_{t\to 0^+}(\mathbf{E}h)(x,t) = h(x)$ for all $x \in \mathbb{R}^{n-1}$.*
2. *If $(z_j) \subset C^\infty(\mathbb{R}^{n-1};\mathbb{R}^N)$ and $z_j \to 0$ in the sense of distributions, then for any multi-index α, $\partial^\alpha[\mathbf{E}z_j] \to 0$ locally uniformly in $\mathbb{R}^n_+$.*
3. *For all $R > 0$ there exist positive constants c_1, c_2, dependent on n, N, r, R, such that for all $h \in (W^{1,1} \cap L^q)(\mathbb{R}^{n-1};\mathbb{R}^N)$ with support contained in $\{x \in \mathbb{R}^{n-1} : |x| \leq R\}$ we have*

   (a)
   $$\int_{\mathbb{R}^n_+} |\mathbf{E}h|^r \,\mathrm{d}\mathscr{L}^n \leq c_1 \|h\|^r_{L^q(\mathbb{R}^{n-1})}$$

   (b)
   $$\int_{\mathbb{R}^n_+} |\nabla[\mathbf{E}h]|^r \mathscr{L}^n \leq c_2\, \|h\|^{\frac{r}{2}}_{L^q(\mathbb{R}^{n-1})} \cdot \int_{\mathbb{R}^{n-1}} |\nabla h| \,\mathrm{d}\mathscr{H}^{n-1}.$$

Once again, since the function $t \mapsto t^r$ is convex, it suffices to prove Lemma 3.5 in the case where $N = 1$. We define $\mathbf{E}$ as in (3.2), from where it is easily seen that $\mathbf{E}$ maps $(W^{1,1}_{\text{loc}} \cap L^q_{\text{loc}})(\mathbb{R}^m)$ into $C^\infty(\mathbb{R}^n_+)$ and that it satisfies properties (1) and (2). In order to show that it maps the given domain into $W^{1,r}_{\text{loc}}(\mathbb{R}^n_+)$, it suffices to prove 3(b). We shall prove it for the x derivative $\nabla_x(\mathbf{E}h(x,t))$ only, since proving it for the t derivative is entirely similar, concluding for $\nabla(\mathbf{E}h)$ using the convexity of $r \mapsto t^r$. For convenience, let $m = n-1$, and let $\mathcal{B}^m$ denote the open unit ball in $\mathbb{R}^m$. Throughout the proofs of 3(a), (b), we will use c to denote a constant, not always the same from line to line, that depends at most on n, r, η, K, R, q.

Proof of 3(a). Note that

$$(\mathbf{E}h)(x,t) = \eta(t)\int_{\mathcal{B}^m} t^{-m}K(\tfrac{y}{t})h(x-y)\,\mathrm{d}\mathscr{H}^m(y)$$

so

$$|\mathbf{E}h|^r \le \Big(c\|\eta\|_\infty\|K\|_\infty \int_{t\mathcal{B}^m} t^{-m}|h(x-y)|\,\mathrm{d}y\Big)^r.$$

Now we use Jensen's Inequality to obtain, for any $q \ge 1$,

$$\begin{aligned}
\int_{t\mathcal{B}^m} t^{-m}|h(x-y)|\,\mathrm{d}y &= |t\mathcal{B}^m|t^{-m} ⨍_{t\mathcal{B}^m} |h(x-y)|\,\mathrm{d}y \\
&= |\mathcal{B}^m|\left(\Big(⨍_{t\mathcal{B}^m}|h(x-y)|\,\mathrm{d}y\Big)^q\right)^{\frac{1}{q}} \\
&\le |\mathcal{B}^m|\Big(⨍_{t\mathcal{B}^m}|h(x-y)|^q\,\mathrm{d}y\Big)^{\frac{1}{q}} \\
&= t^{-\frac{m}{q}}\Big(\int_{t\mathcal{B}^m}|h(x-y)|^q\,\mathrm{d}y\Big)^{\frac{1}{q}} \\
&\le t^{-\frac{m}{q}}\|h\|_{L^q(\mathbb{R}^m)}\,.
\end{aligned}$$

Hence we have

$$|\mathbf{E}h|^r \le c\,\|h\|^r_{L^q(\mathbb{R}^m)}\,t^{-\frac{mr}{q}}\,.$$

Now note that $\int_0^1 t^{-\frac{mr}{q}}\,\mathrm{d}t$ is finite if and only if $\frac{mr}{q} > -1$, i.e. $q > rm$. Since $1 < r < 2$, this is certainly satisfied if $q \ge \frac{mr}{2-r}$.

Moreover, if h is compactly supported in $\{x \in \mathbb{R}^m : |x| \le R\}$, then $\mathbf{E}h$ is supported in $\{(x,t) \in \mathbb{R}^n_+ : |x| \le R+1,\, t \in (0,1)\}$. Hence, integrating $|\mathbf{E}h|^r$ first with respect to t, then x, we get, for $q > rm$,

$$\begin{aligned}
\int_{\mathbb{R}^n_+}|\mathbf{E}h|^r\,\mathrm{d}\mathscr{L}^n &= \int_{\{|x|\le R+1\}}\int_0^1 |\mathbf{E}h|^r\,\mathrm{d}t\,\mathrm{d}x \\
&\le c\,(R+1)^m\,|\mathcal{B}^m|\,\|h\|^r_{L^q(\mathbb{R}^m)}\cdot\int_0^1 t^{-\frac{rm}{q}}\,\mathrm{d}t \\
&\le c_1\,\|h\|^r_{L^q(\mathbb{R}^m)}\,.
\end{aligned}$$

□

Proof of 3(b). First observe that, by integration by parts and since K vanishes on $\partial\mathcal{B}^m$, we may write the derivative $\nabla_x(\mathbf{E}h)$ in both the following ways:

$$\nabla_x(\mathbf{E}h)(x,t) = \eta(t)\int_{\mathcal{B}^m}\nabla h(x-ty)K(y)\,\mathrm{d}y$$

$$= \eta(t) \int_{\mathcal{B}^m} t^{-1} h(x-ty) \nabla K(y)\, \mathrm{d}y\,.$$

Now fix $x \in \mathbb{R}^m$. Integrating $\nabla_x(\mathbf{E}h)$ first with respect to t over $(0,1)$, we get

$$\begin{aligned}
\int_0^1 |\nabla_x(\mathbf{E}h)(x,t)|^r\, \mathrm{d}t &= \int_0^\delta \Big(\Big| \eta(t) \int_{\mathcal{B}^m} \nabla h(x-ty) K(y)\, \mathrm{d}y \Big| \Big)^r \mathrm{d}t \\
&\quad \int_\delta^1 \Big(\Big| \eta(t) \int_{\mathcal{B}^m} t^{-1} h(x-ty) \nabla K(y)\, \mathrm{d}y \Big| \Big)^r \mathrm{d}t \\
&= I + II\,, \quad \text{say.}
\end{aligned} \tag{3.7}$$

Estimating I: We obtain the following bound on I:

$$\begin{aligned}
I &= \int_0^\delta \Big(\Big| \eta(t) \int_{\mathcal{B}^m} \nabla h(x-ty) K(y)\, \mathrm{d}y \Big| \Big)^r \mathrm{d}t \\
&\le \int_0^\delta \|\eta\|_\infty^r \|K\|_\infty^r \Big(\int_{\mathcal{B}^m} |\nabla h(x-ty)|\, \mathrm{d}y \Big)^r \mathrm{d}t \\
&\le \|\eta\|_\infty^r \|K\|_\infty^r \int_0^\delta M(\nabla h)(x)^r\, \mathrm{d}t \\
&\le c\, \delta\, (M \nabla h)(x)^r\,.
\end{aligned} \tag{3.8}$$

Estimating II: This is similar, albeit slightly more involved than, the proof of 3(a). First note that

$$\begin{aligned}
\Big(\Big| \eta(t) \int_{\mathcal{B}^m} t^{-1} h(x-ty) \nabla K(y)\, \mathrm{d}y \Big| \Big)^r &\le \Big(\|\eta\|_\infty \|\nabla K\|_\infty t^{-1} \int_{\mathcal{B}^m} |h(x-ty)|\, \mathrm{d}y \Big)^r \\
&\le c \left(t^{-1} \cdot t^{-\frac{m}{q}}\, \|h\|_{L^q(\mathbb{R}^m)} \right)^r \\
&= c\, t^{-\frac{(m+q)r}{q}}\, \|h\|_{L^q(\mathbb{R}^m)}^r\,.
\end{aligned} \tag{3.9}$$

Here we have used Jensen's Inequality just as in the proof of 3(a), but now we have an extra t^{-1} term to incorporate.

Now assume $0 < \delta < 1$ and consider $\int_\delta^1 t^{-\frac{(m+q)r}{q}}\, \mathrm{d}t$:

$$\begin{aligned}
\int_\delta^1 t^{-\frac{(m+q)r}{q}}\, \mathrm{d}t &= \tfrac{1}{1-\frac{(m+q)r}{q}} \left[t^{1-\frac{(m+q)r}{q}} \right]_\delta^1 \\
&= \tfrac{q}{(m+q)r-q} \left[t^{1-\frac{(m+q)r}{q}} \right]_1^\delta \\
&\le c\, \delta^{1-\frac{(m+q)r}{q}}\,.
\end{aligned}$$

Now note that $\delta^{1-\frac{(m+q)r}{q}} \le \delta^{-1}$ if and only if

$$1 - \frac{(m+q)r}{q} \ge -1\,,$$

i.e.

$$q \geq \frac{mr}{2-r}\,.$$

Therefore, from (3.9), and for such q, we get

$$\begin{aligned} II &\leq c \int_\delta^1 t^{-\frac{(m+q)r}{q}} \, \|h\|^r_{L^q(\mathbb{R}^m)} \, \mathrm{d}t \\ &\leq c\, \delta^{-1} \, \|h\|^r_{L^q(\mathbb{R}^m)}\,. \end{aligned} \tag{3.10}$$

Now note that even if $\delta \geq 1$, then $II \leq 0$, so clearly (3.10) is also true in this case.

Combining these estimates for I and II, we obtain

$$\int_0^1 |\nabla_x(\mathbf{E}h)(x,t)|^r \, \mathrm{d}t \leq c\,\delta\,(M\nabla h)(x)^r + c\,\delta^{-1}\,\|h\|^r_{L^q(\mathbb{R}^m)}\,. \tag{3.11}$$

If we take

$$\delta = \left(\frac{\|h\|^r_{L^q(\mathbb{R}^m)}}{(M\nabla h)(x)^r} \right)^{\frac{1}{2}},$$

then (3.11) becomes

$$\int_0^1 |\nabla_x(\mathbf{E}h)(x,t)|^r \, \mathrm{d}t \leq c\,(M\nabla h)(x)^{\frac{r}{2}} \, \|h\|^{\frac{r}{2}}_{L^q(\mathbb{R}^m)}\,. \tag{3.12}$$

Note that whereas the choice of δ to obtain (3.12) may depend on x, the constant c in (3.12) is independent of x. So (3.12) holds for all $x \in \mathbb{R}^m$. Define the function $\Theta\colon [0,\infty) \to [0,\infty)$ as follows:

$$\Theta(t) := \begin{cases} t^2 & \text{if } t \in [0,1)\,, \\ t^{\frac{r}{2}} & \text{if } t \geq 1\,. \end{cases}$$

Then Ψ as defined in (3.1) satisfies, for $t \geq 1$,

$$\begin{aligned} \Psi(t) &= t^{\frac{r}{2}} + t + t \int_1^t s^{\frac{r}{2}-2} \, \mathrm{d}t \\ &= t^{\frac{r}{2}} + t + \tfrac{2}{2-r}(1 - t^{\frac{r}{2}-1})\,t \\ &\leq ct\,. \end{aligned}$$

Now apply Lemma 3.1, noting that $\{|\nabla h| \geq 1\} \subseteq \{\mathbf{M}(\nabla h) \geq 1\}$:

$$\begin{aligned} \int_{\{|\nabla h| \geq 1\}} (M\nabla h)(x)^{\frac{r}{2}} \, \mathrm{d}x &\leq \int_{\{M\nabla h \geq 1\}} (M\nabla h)(x)^{\frac{r}{2}} \, \mathrm{d}x \\ &= \int_{\{M\nabla h \geq 1\}} \Theta\big((M\nabla h)(x)\big) \, \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\leq 2\cdot 5^m \int_{\{M\nabla h\geq 1\}} \Psi\big(2|(\nabla h)(x)|\big)\,\mathrm{d}x \\
&\leq 2\cdot 5^m\cdot 2c \int_{\{M\nabla h\geq 1\}} |(\nabla h)(x)|\,\mathrm{d}x \\
&\leq c\int_{\mathbb{R}^m} |(\nabla h)(x)|\,\mathrm{d}x\,. \qquad (3.13)
\end{aligned}
$$

Therefore, applying (3.13) to (3.12), we obtain

$$
\int_{\{|\nabla h|\geq 1\}}\int_0^1 |\nabla_x(\mathbf{E}h)(x,t)|^r\,\mathrm{d}t\,\mathrm{d}x \leq c\,\|h\|^{\frac{r}{2}}_{L^q(\mathbb{R}^m)}\cdot \int_{\mathbb{R}^m}|(\nabla h)(x)|\,\mathrm{d}x\,. \tag{3.14}
$$

As observed in the proof of 3(a), if h is compactly supported in $\{x : |x| \leq R\}$, then $\mathbf{E}h$ (and also $\nabla_x\mathbf{E}h$) is supported in $\{(x,t) : |x| \leq R+1\,,\, t\in(0,1)\}$. Therefore we have

$$
\int_{\{|\nabla h|<1\}}\int_0^1 |\nabla_x(\mathbf{E}h)(x,t)|^r\,\mathrm{d}t\,\mathrm{d}x \leq \big(\|\eta\|_\infty\|K\|_\infty|\mathcal{B}^m|\big)^r\cdot\mathscr{L}^m\big(\{x : |x|\leq R+1\}\big)\,. \tag{3.15}
$$

Combining (3.14) and (3.15) gives

$$
\int_{\mathbb{R}^n_+} |\nabla_x(\mathbf{E}h)(x,t)|^r\,\mathrm{d}(x,t) \leq c + c\,\|h\|^{\frac{r}{2}}_{L^q(\mathbb{R}^m)}\cdot\int_{\mathbb{R}^m}|(\nabla h)(x)|\,\mathrm{d}x\,. \tag{3.16}
$$

Since $\mathbf{E}$ is a linear operator, 3(b) easily follows from (3.16) by taking a larger constant c. □

3.4 Extension into subquadratic Orlicz-Sobolev spaces

We now state and prove the result of the previous section in the more general setting of Orlicz-Sobolev Spaces. This allows us to prove Theorem 4.5 in Chapter 4, which is a generalisation of Theorem 4.1. It is easy to see that this result in fact implies Lemma 3.5, and the proof is very similar, albeit slightly more involved.

Let $\Phi\colon [0,\infty)\to[0,\infty)$ be a convex, doubling, non-decreasing function such that $\Phi(0)=0$ and, for some $\sigma_\Phi>0$:

$$
t\mapsto \frac{\Phi(t)}{t^{2-\sigma_\Phi}} \text{ is non-increasing on } (0,\infty) \text{ and } \int_1^\infty \frac{\Phi(t)^{\frac{1}{2}}}{t^2}\,\mathrm{d}t<\infty\,. \tag{3.17}
$$

(Recall that Φ is doubling means that, for a fixed constant c, $\Phi(2t)\leq c\,\Phi(t)$ for all $t\geq 0$.)

Lemma 3.6. *Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a convex, doubling, non-decreasing map satisfying* (3.17) *for some $\sigma_\Phi > 0$. Then for*

$$q > \max\left\{2(n-1), \frac{(n-1)(2-\sigma_\Phi)}{\sigma_\Phi}\right\}, \tag{3.18}$$

there exists a linear extension operator

$$\mathbf{E}\colon (W^{1,1}_{loc} \cap L^q_{loc})(\mathbb{R}^{n-1};\mathbb{R}^N) \to W^{1,\Phi}_{loc}(\mathbb{R}^n_+;\mathbb{R}^N)$$

with the following properties:

1. *If $h \in C^1(\mathbb{R}^{n-1};\mathbb{R}^N)$ then $(\mathbf{E}h)(x,0) := \lim_{t\to 0^+}(\mathbf{E}h)(x,t) = h(x)$ for all $x \in \mathbb{R}^{n-1}$.*

2. *If $(z_j) \subset C^\infty(\mathbb{R}^{n-1};\mathbb{R}^N)$ and $z_j \to 0$ in the sense of distributions, then for any multi-index α, $\partial^\alpha[\mathbf{E}z_j] \to 0$ locally uniformly in $\mathbb{R}^n_+$.*

3. *For all $R > 0$ there exist positive constants c_1, c_2, dependent on n, N, Φ, R, such that for all $h \in (W^{1,1} \cap L^q)(\mathbb{R}^{n-1};\mathbb{R}^N)$ with support contained in $\{x \in \mathbb{R}^{n-1} : |x| \le R\}$ we have*

   (a)
   $$\int_{\mathbb{R}^n_+} \Phi(|\mathbf{E}h|)\,\mathrm{d}\mathscr{L}^n \le c_1\|h\|_{L^q(\mathbb{R}^{n-1})}$$

   (b)
   $$\int_{B_2\setminus B} \Phi(|\nabla[\mathbf{E}h]|)\mathscr{L}^n \le c_2\,\Phi(\|h\|_{L^q(\mathbb{R}^{n-1})})^{\frac{1}{2}} \cdot \int_{\mathbb{R}^{n-1}} |\nabla h|\,\mathrm{d}\mathscr{H}^{n-1}.$$

It is easily seen that we may replace condition (3.18) with simply

$$q > \frac{(n-1)(2-\sigma_\Phi)}{\sigma_\Phi}$$

if we further stipulate that $\sigma_\Phi \in (0, \frac{2}{3}]$. Since $t \mapsto \Phi(t)/t^{2-\sigma_\Phi}$ is non-increasing implies that so is $t \mapsto \Phi(t)/t^{2-\sigma'_\Phi}$ for any $0 < \sigma'_\Phi \le \sigma_\Phi$, we may always take σ_Φ sufficiently small for any Φ satisfying (3.17).

As stated earlier, since Φ is convex it suffices to prove Lemma 3.6 in the case where $N = 1$. We define $\mathbf{E}$ as in (3.2), from where it is easily seen that $\mathbf{E}$ maps $(W^{1,1}_{\mathrm{loc}} \cap L^q_{\mathrm{loc}})(\mathbb{R}^m)$ into $C^\infty(\mathbb{R}^n_+)$ and that it satisfies properties (1) and (2). As in the proof of the lemma in the setting of Sobolev spaces, in order to show that it maps the given domain into $W^{1,\Phi}_{\mathrm{loc}}(\mathbb{R}^n_+)$ it suffices to prove 3(b). We shall prove it for the x derivative $\nabla_x(\mathbf{E}h(x,t))$ only, since proving it for the t derivative is entirely similar, concluding for $\nabla(\mathbf{E}h)$ using the convexity of Φ. Again, we will use c to denote a constant, not always the same from line to line, that depends at most on n, Φ, η, K, R, q.

Proof of 3(a). As in the proof of Lemma 3.5, we first note that

$$(\mathbf{E}h)(x,t) = \eta(t)\int_{\mathcal{B}^m} t^{-m}K(\tfrac{y}{t})h(x-y)\,\mathrm{d}\mathscr{H}^m(y)\,,$$

so

$$\Phi(|\mathbf{E}h|) \le \Phi\Big(c\|\eta\|_\infty\|K\|_\infty\int_{t\mathcal{B}^m} t^{-m}|h(x-y)|\,\mathrm{d}y\Big)\,.$$

Recall that using Jensen's Inequality we obtain, for any $q \ge 1$,

$$\int_{t\mathcal{B}^m} t^{-m}|h(x-y)|\,\mathrm{d}y \le t^{-\frac{m}{q}}\|h\|_{L^q(\mathbb{R}^m)}\,,$$

and so, using the fact that Φ is doubling, we have

$$\Phi(|\mathbf{E}h|) \le c\,\Phi(\|h\|_{L^q(\mathbb{R}^m)})\,\Phi(t^{-\frac{m}{q}})\,.$$

Now consider for what q $\int_0^1\Phi(t^{-\frac{m}{q}})\,\mathrm{d}t$ is finite: if Φ satisfies (3.17) then certainly $\Phi(t) \le ct^2$ for some fixed constant c. Hence, if we take $q > 2m$, then

$$\int_0^1 \Phi(t^{-\frac{m}{q}})\,\mathrm{d}t \le c\int_0^1 t^{-\frac{2m}{q}}\,\mathrm{d}t < \infty\,.$$

If h is compactly supported in $\{x \in \mathbb{R}^m : |x| \le R\}$, then $\mathbf{E}h$ is supported in $\{(x,t) \in \mathbb{R}^n_+ : |x| \le R+1,\, t \in (0,1)\}$. Hence, integrating $\Phi(|\mathbf{E}h|)$ first with respect to t, then x, we get

$$\begin{aligned}\int_{\mathbb{R}^n_+}\Phi(|\mathbf{E}h|)\,\mathrm{d}\mathscr{L}^n &= \int_{\{|x|\le R+1\}}\int_0^1\Phi(|\mathbf{E}h|)\,\mathrm{d}t\,\mathrm{d}x\\ &\le c\,(R+1)^m\,|\mathcal{B}^m|\,\Phi(\|h\|_{L^q(\mathbb{R}^m)})\cdot\int_0^1\Phi(t^{-\frac{m}{q}})\,\mathrm{d}t\\ &\le c_1\,\Phi(\|h\|_{L^q(\mathbb{R}^m)})\,.\end{aligned}$$

$\square$

Proof of 3(b). As observed in the proof of Lemma 3.5, we may write the derivative $\nabla_x(\mathbf{E}h)$ in both the following ways:

$$\begin{aligned}\nabla_x(\mathbf{E}h)(x,t) &= \eta(t)\int_{\mathcal{B}^m}\nabla h(x-ty)K(y)\,\mathrm{d}y\\ &= \eta(t)\int_{\mathcal{B}^m} t^{-1}h(x-ty)\nabla K(y)\,\mathrm{d}y\,.\end{aligned}$$

Again, fixing $x\in\mathbb{R}^m$ and integrating $\nabla_x(\mathbf{E}h)$ first with respect to t over $(0,1)$, we get

$$\begin{aligned}\int_0^1\Phi(|\nabla_x(\mathbf{E}h)(x,t)|)\,\mathrm{d}t &= \int_0^\delta\Phi\Big(\,\Big|\,\eta(t)\int_{\mathcal{B}^m}\nabla h(x-ty)K(y)\,\mathrm{d}y\,\Big|\,\Big)\,\mathrm{d}t\\ &\qquad\int_\delta^1\Phi\Big(\,\Big|\,\eta(t)\int_{\mathcal{B}^m}t^{-1}h(x-ty)\nabla K(y)\,\mathrm{d}y\,\Big|\,\Big)\,\mathrm{d}t\\ &= I+II\,,\quad\text{say.}\end{aligned}\tag{3.19}$$

Estimating I: We obtain the following bound on I:

$$\begin{aligned}I &= \int_0^\delta\Phi\Big(\,\Big|\,\eta(t)\int_{\mathcal{B}^m}\nabla h(x-ty)K(y)\,\mathrm{d}y\,\Big|\,\Big)\,\mathrm{d}t\\ &\le \int_0^\delta\Phi\Big(\|\eta\|_\infty\|K\|_\infty\int_{\mathcal{B}^m}|\nabla h(x-ty)|\,\mathrm{d}y\Big)\,\mathrm{d}t\\ &\le \int_0^\delta\Phi\big(\|\eta\|_\infty\|K\|_\infty M(\nabla h)(x)\big)\,\mathrm{d}t\\ &\le c\,\delta\,\Phi\big((M\nabla h)(x)\big)\,,\end{aligned}\tag{3.20}$$

using the fact that Φ is doubling in the last line.

Estimating II: Just as in the proof of Lemma 3.5, we have, using Jensen's Inequality,

$$\begin{aligned}\Phi\Big(\,\Big|\,\eta(t)\int_{\mathcal{B}^m}t^{-1}h(x-ty)\nabla K(y)\,\mathrm{d}y\,\Big|\,\Big) &\le \Phi\Big(\|\eta\|_\infty\|\nabla K\|_\infty t^{-1}\int_{\mathcal{B}^m}|h(x-ty)|\,\mathrm{d}y\Big)\\ &\le c\,\Phi(t^{-1}\cdot t^{-\frac{m}{q}})\,\Phi(\|h\|_{L^q(\mathbb{R}^m)})\\ &= c\,\Phi(t^{-\frac{m+q}{q}})\,\Phi(\|h\|_{L^q(\mathbb{R}^m)})\,.\end{aligned}\tag{3.21}$$

Now consider $\int_\delta^1\Phi(t^{-\frac{m+q}{q}})\,\mathrm{d}t$: use the substitution $s=t^{-\frac{m+q}{q}}$ to get (where the σ_Φ comes from condition (3.17) on Φ):

$$\int_\delta^1\Phi(t^{-(\frac{m+q}{q})})\,\mathrm{d}t = \frac{q}{m+q}\int_1^{\delta^{-(\frac{m+q}{q})}}\Phi(s)s^{-(\frac{m+2q}{q})}\,\mathrm{d}s$$

$$
\begin{aligned}
&= \frac{q}{m+q}\int_1^{\delta^{-(\frac{m+q}{q})}} \frac{\Phi(s)}{s^{2-\sigma_\Phi}}\cdot s^{\frac{m}{m+q}-\sigma_\Phi}\,\mathrm{d}s \\
\text{since } \tfrac{\Phi(s)}{s^{2-\sigma_\Phi}} \text{ non-increasing} \quad &\le \frac{q}{m+q}\cdot\frac{\Phi(1)}{1^{2-\sigma_\Phi}}\int_1^{\delta^{-(\frac{m+q}{q})}} s^{\frac{m}{m+q}-\sigma_\Phi}\,\mathrm{d}s \\
&= c\left[s^{\frac{m}{m+q}-\sigma_\Phi+1}\right]_1^{\delta^{-(\frac{m+q}{q})}} .
\end{aligned}
$$

Now note that if we take $q > \frac{2-\sigma_\Phi}{\sigma_\Phi}\cdot m$, then $\frac{m}{m+q}-\sigma_\Phi < -\frac{m}{m+q}$. Hence, assuming that $0<\delta<1$, we get

$$
\begin{aligned}
\left[s^{\frac{m}{m+q}-\sigma_\Phi+1}\right]_1^{\delta^{-(\frac{m+q}{q})}} &\le c\left(\delta^{-(\frac{m+q}{q})}\right)^{\frac{m}{m+q}-\sigma_\Phi+1} \\
&= c\,\delta^{-(\frac{m+q}{q})}\cdot\left(\delta^{-(\frac{m+q}{q})}\right)^{\frac{m}{m+q}-\sigma_\Phi} \\
&\le c\,\delta^{-(\frac{m+q}{q})}\cdot\left(\delta^{-(\frac{m+q}{q})}\right)^{-(\frac{m}{m+q})} \\
&= c\,\delta^{-(\frac{m+q}{q})}\cdot\delta^{\frac{m}{q}} \\
&= c\,\delta^{-1}\,. \qquad (3.22)
\end{aligned}
$$

Therefore from (3.21) and (3.22), we get for q large enough:

$$
\begin{aligned}
II &\le c\int_\delta^1 \Phi(t^{-\frac{m+q}{q}})\,\Phi(\|h\|_{L^q(\mathbb{R}^m)})\,\mathrm{d}t \\
&\le c\,\delta^{-1}\,\Phi(\|h\|_{L^q(\mathbb{R}^m)})\,. \qquad (3.23)
\end{aligned}
$$

Note that we may only apply (3.22) if $0<\delta<1$. However, if $\delta\ge 1$, then $II\le 0$, so clearly (3.23) is also true in this case.

Now combine these estimates for I and II to obtain

$$
\int_0^1 \Phi(|\nabla_x(\mathbf{E}h)(x,t)|)\,\mathrm{d}t \le c\,\delta\,\Phi\big((M\nabla h)(x)\big) + c\,\delta^{-1}\,\Phi(\|h\|_{L^q(\mathbb{R}^m)})\,. \qquad (3.24)
$$

If we take

$$
\delta = \left(\frac{\Phi(\|h\|_{L^q(\mathbb{R}^m)})}{\Phi\big((M\nabla h)(x)\big)}\right)^{\frac{1}{2}},
$$

then (3.24) becomes

$$
\int_0^1 \Phi(|\nabla_x(\mathbf{E}h)(x,t)|)\,\mathrm{d}t \le c\,\Phi\big((M\nabla h)(x)\big)^{\frac{1}{2}}\,\Phi(\|h\|_{L^q(\mathbb{R}^m)})^{\frac{1}{2}}\,. \qquad (3.25)
$$

Again, note that whereas the choice of δ to obtain (3.25) may depend on x, the constant c in (3.25) is independent of x. So (3.25) holds for all $x\in\mathbb{R}^m$. Define the function

$\Theta\colon [0,\infty) \to [0,\infty)$ as follows:

$$\Theta(t) := \begin{cases} t^2 & \text{if } t \in [0,1)\,, \\ \Phi(t)^{\frac{1}{2}} & \text{if } t \geq 1\,. \end{cases}$$

Then Ψ as defined in (3.1) satisfies, for $t \geq 1$,

$$\begin{aligned} \Psi(t) &= \Phi(t)^{\frac{1}{2}} + t + t \int_1^t \frac{\Phi(s)^{\frac{1}{2}}}{s^2}\,\mathrm{d}s \\ &\leq \left(\Phi(1)^{\frac{1}{2}} + 1 + \int_1^\infty \frac{\Phi(s)^{\frac{1}{2}}}{s^2}\,\mathrm{d}s \right) \cdot t \\ &= Ct\,, \quad \text{say,} \end{aligned}$$

since, by (3.17), $\int_1^\infty \Phi(s)^{\frac{1}{2}} s^{-2}\,\mathrm{d}s < \infty$ and, for $t \geq 1$,

$$\frac{\Phi(t)}{t^2} \leq \frac{\Phi(t)}{t^2} t^{\sigma_\Phi} \leq \Phi(1)\,.$$

Now apply Lemma 3.1, noting that $\{|\nabla h| \geq 1\} \subseteq \{\mathbf{M}(\nabla h) \geq 1\}$:

$$\begin{aligned} \int_{\{|\nabla h|\geq 1\}} \Phi\big((M\nabla h)(x)\big)^{\frac{1}{2}}\,\mathrm{d}x &\leq \int_{\{M\nabla h \geq 1\}} \Phi\big((M\nabla h)(x)\big)^{\frac{1}{2}}\,\mathrm{d}x \\ &= \int_{\{M\nabla h \geq 1\}} \Theta\big((M\nabla h)(x)\big)\,\mathrm{d}x \\ &\leq 2\cdot 5^m \int_{\{M\nabla h \geq 1\}} \Psi\big(2|(\nabla h)(x)|\big)\,\mathrm{d}x \\ &\leq 2\cdot 5^m \cdot 2c \int_{\{M\nabla h \geq 1\}} |(\nabla h)(x)|\,\mathrm{d}x \\ &\leq c \int_{\mathbb{R}^m} |(\nabla h)(x)|\,\mathrm{d}x\,. \end{aligned} \tag{3.26}$$

Therefore, applying (3.26) to (3.25), we obtain

$$\int_{\{|\nabla h|\geq 1\}} \int_0^1 \Phi(|\nabla_x(\mathbf{E}h)(x,t)|)\,\mathrm{d}t\,\mathrm{d}x \leq c\,\Phi(\|h\|_{L^q(\mathbb{R}^m)})^{\frac{1}{2}} \cdot \int_{\mathbb{R}^m} |(\nabla h)(x)|\,\mathrm{d}x\,. \tag{3.27}$$

Once again, since h is compactly supported in $\{x : |x| \leq R\}$, $\mathbf{E}h$ (and also $\nabla_x \mathbf{E}h$) is supported in $\{(x,t) : |x| \leq R+1\,,\, t \in (0,1)\}$. Therefore we have

$$\int_{\{|\nabla h|<1\}} \int_0^1 \Phi(|\nabla_x(\mathbf{E}h)(x,t)|)\,\mathrm{d}t\,\mathrm{d}x \leq \Phi\big(\|\eta\|_\infty \|K\|_\infty |\mathcal{B}^m|\big) \mathscr{L}^m\big(\{|x| \leq R+1\}\big)\,. \tag{3.28}$$

Combining (3.27) and (3.28) gives

$$\int_{\mathbb{R}^n_+} \Phi(|\nabla_x(\mathbf{E}h)(x,t)|)\,\mathrm{d}(x,t) \leq c + c\,\Phi(\|h\|_{L^q(\mathbb{R}^m)})^{\frac{1}{2}} \cdot \int_{\mathbb{R}^m} |(\nabla h)(x)|\,\mathrm{d}x\,,$$

from where, since $\mathbf{E}$ is a linear operator, 3(b) easily follows. □

3.5 A trace-preserving linear operator

In this section we adapt a result of Fonseca and Malý [50] and construct a linear operator Tu from $W^{1,1}$ into itself that improves integrability over a "layer", allowing us to estimate the $W^{1,r}$ norm of Tu, for $r \in [1, \frac{n}{n-1})$, in terms of a special maximal function. This is used in Chapter 5 to "connect" two functions across a thin transition layer and estimate the increase of energy. In their paper, they are interested specifically in a linear operator from $W^{1,p}$ into $W^{1,p}$ for $p > 1$. However, we have observed that the proof also works for $p = 1$, which is what we require.

Let Ω be a bounded, open subset of $\mathbb{R}^n$. Let $\eta \in C_c^\infty(\Omega)$ be a non-negative function and $[t_1, t_2] \subset (0, \|\eta\|_\infty)$. Suppose also that $0 < |\nabla \eta| \leq A$ on $\{t_1 \leq \eta \leq t_2\}$. Given a subinterval $(a, b) \subset (t_1, t_2)$, let Z_a^b denote the set $\{a < \eta < b\}$, and for $t_0 \in (t_1, t_2)$, let Γ_{t_0} denote the level set $\{\eta = t_0\}$.

Fix $t_0 \in (t_1, t_2)$ and note that there exists a diffeomorphism G_{t_0} of $\Gamma_{t_0} \times [t_1, t_2]$ onto $\bar{Z}_{t_1}^{t_2}$ such that

$$\begin{cases} G_{t_0}(z, t_0) & = z \\ \eta(G_{t_0}(z, t)) & = t \end{cases} \tag{3.29}$$

for all $z \in G_{t_0}$, $t \in [t_1, t_2]$. To see this, consider the flow h_z verifying

$$\begin{cases} \frac{dh_z}{dt} & = \frac{\nabla \eta(h(t))}{|\nabla \eta(h(t))|^2} \\ h_z(t_0) & = z \end{cases}$$

and set $G_{t_0} := h_z(t)$. Note that the map G_{t_0} is bi-Lipschitz, and also that the Jacobians of G_{t_0} and $G_{t_0}^{-1}$ are bounded. This allows us to establish the following lemma.

Lemma 3.7. [50] *Let $s \in (t_1, t_2)$ and $\varrho > 0$ be such that $[s - \varrho, s + \varrho] \subset (t_1, t_2)$. Let h be a non-negative measurable function on Ω. Then*

$$\int_{\{\eta = s\}} \left(\int_{B(z, \frac{\varrho}{A})} h(y)\, \mathrm{d}y \right) \mathrm{d}\mathscr{H}^{n-1}(z) \leq C \varrho^{n-1} \int_{Z_{s-\varrho}^{s+\varrho}} h(y)\, \mathrm{d}y,$$

where C is a constant dependent on n, η, t_1 and t_2.

Proof of Lemma 3.7. First note that if $z \in \Gamma_s$, then $B(z, \frac{\varrho}{A}) \subset Z_{s-\varrho}^{s+\varrho}$. Hence, using the change of variables $y = G_s(z, t)$ and (3.29), we obtain

$$\begin{aligned} &\int_{\{\eta = s\}} \left(\int_{B(z, \frac{\varrho}{A})} h(y)\, \mathrm{d}y \right) \mathrm{d}\mathscr{H}^{n-1}(z) \\ &\quad \leq C \int_{\Gamma_s} \left(\int_{s-\varrho}^{s+\varrho} \left(\int_{\{\sigma \in \Gamma_s : |G_s(\sigma, t) - G_s(z, s)| < \frac{\varrho}{A}\}} h(G_s(\sigma, t))\, \mathrm{d}\mathscr{H}^{n-1}(\sigma) \right) \mathrm{d}t \right) \mathrm{d}\mathscr{H}^{n-1}(z) \end{aligned}$$

$$
\begin{aligned}
&= C\int_{\Gamma_s}\left(\int_{s-\varrho}^{s+\varrho}\left(\int_{\{z\in\Gamma_s:|G_s(\sigma,t)-G_s(z,s)|<\frac{\varrho}{A}\}} h(G_s(\sigma,t))\,\mathrm{d}\mathscr{H}^{n-1}(z)\right)\mathrm{d}t\right)\mathrm{d}\mathscr{H}^{n-1}(\sigma)\\
&\le C\int_{\Gamma_s\times(s-\varrho,s+\varrho)} \mathscr{H}^{n-1}\Big(\big\{z\in\Gamma_s:|G_s(\sigma,t)-G_s(z,s)|<\tfrac{\varrho}{A}\big\}\Big)\\
&\qquad\qquad \times h(G_s(\sigma,t))\,\mathrm{d}\mathscr{L}^n(\sigma,t)\\
&\le C\varrho^{n-1}\int_{Z_{s-\varrho}^{s+\varrho}} h(y)\,\mathrm{d}y\,,
\end{aligned}
$$

since, due to the Lipschitz continuity of G_s^{-1},

$$
\mathscr{H}^{n-1}\Big(\big\{z\in\Gamma_s:|G_s(\sigma,t)-G_s(z,s)|<\tfrac{\varrho}{A}\big\}\Big)\le C\varrho^{n-1}\,.
$$

□

We now state and prove the main result of this section.

Lemma 3.8. *Let $r\in[1,\frac{n}{n-1})$. Let $t_1<a<b<t_2$. There exists a linear operator $T\colon W^{1,1}(\Omega;\mathbb{R}^N)\to W^{1,1}(\Omega;\mathbb{R}^N)$ such that $Tu=u$ on $\Omega\setminus Z_a^b$ and*

$$
\begin{aligned}
\|Tu\|_{W^{1,r}(Z_a^b)} \le C(b-a)^{\frac{n}{r}-n+1}\Big(&\sup_{t\in(a,b)}(t-a)^{-1}\|u\|_{W^{1,1}(Z_a^t)}\\
&+\sup_{t\in(a,b)}(b-t)^{-1}\|u\|_{W^{1,1}(Z_t^b)}\Big)\,,
\end{aligned}
\tag{3.30}
$$

where C depends on n, r, η, t_1 and t_2.

Proof of Lemma 3.8. This proof is directly from [50], but we specifically consider a borderline case that is left out in that proof. Set

$$
Tu(x):=\fint_{B(0,1)} u(x+\theta(x)y)\,\mathrm{d}y\,,
$$

where

$$
\begin{aligned}
\theta(x):&=\frac{1}{2A}\max\{0,\min\{\eta(x)-a,b-\eta(x)\}\}\\
&=\begin{cases} 0 & \text{if } \eta(x)\ge b\\ \frac{b-\eta(x)}{2A} & \text{if } \frac{a+b}{2}<\eta(x)<b\\ \frac{\eta(x)-a}{2A} & \text{if } a<\eta(x)\le\frac{a+b}{2}\\ 0 & \text{if } \eta(x)\le a\,.\end{cases}
\end{aligned}
$$

It is clear to see that $Tu(x)=x$ if $x\notin Z_a^b$, and

$$
Tu(x)=\fint_{B(x,\theta(x))} u(z)\,\mathrm{d}z
$$

for $x \in Z_a^b$. Let $c := \frac{a+b}{2}$ and define

$$M_0 := \sup_{t\in(a,b)} (t-a)^{-1} \int_{Z_a^t} |u| \,\mathrm{d}y\,,$$
$$M_1 := \sup_{t\in(a,b)} (t-a)^{-1} \int_{Z_a^t} |u| + |\nabla u| \,\mathrm{d}y\,.$$

First assume u is smooth and fix $\alpha \geq 1$. If $\varrho \in (0, \frac{1}{4}(b-a))$ and if $z \in \{\eta = a+2\varrho\}$, then $\theta(z) = \frac{\varrho}{A}$ and $B(z,\theta(z)) \subset Z_{a+\varrho}^{a+3\varrho}$. Hence

$$\begin{aligned}
|Tu(z)|^\alpha &\leq C\varrho^{-n\alpha}\Big(\int_{B(z,\frac{\varrho}{A})} |u(y)| \,\mathrm{d}y\Big)^\alpha \\
&\leq C\varrho^{-n\alpha}\Big(\int_{Z_{a+\varrho}^{a+3\varrho}} |u(y)| \,\mathrm{d}y\Big)^{\alpha-1}\Big(\int_{B(z,\frac{\varrho}{A})} |u(y)| \,\mathrm{d}y\Big)\,.
\end{aligned}$$

Now use Lemma 3.7 to get

$$\begin{aligned}
&\int_{\{\eta=a+2\varrho\}} |Tu(z)|^\alpha \,\mathrm{d}\mathscr{H}^{n-1}(z) \\
&\leq C\varrho^{-n\alpha}\Big(\int_{Z_{a+\varrho}^{a+3\varrho}} |u(y)| \,\mathrm{d}y\Big)^{\alpha-1} \times \int_{\{\eta=a+2\varrho\}} \Big(\int_{B(z,\frac{\varrho}{A})} |u(y)| \,\mathrm{d}y\Big) \,\mathrm{d}\mathscr{H}^{n-1}(z) \\
&\leq C\varrho^{-n\alpha}\Big(\int_{Z_{a+\varrho}^{a+3\varrho}} |u(y)| \,\mathrm{d}y\Big)^{\alpha-1} \varrho^{n-1}\Big(\int_{Z_{a+\varrho}^{a+3\varrho}} |u(y)| \,\mathrm{d}y\Big) \\
&= C\varrho^{-n\alpha+n-1}\Big(\int_{Z_{a+\varrho}^{a+3\varrho}} |u(y)| \,\mathrm{d}y\Big)^{\alpha}\,.
\end{aligned} \tag{3.31}$$

By the co-area formula and (3.31) for $\alpha = r$, since $|\nabla\eta|$ is bounded away from zero, we get

$$\begin{aligned}
\int_{Z_a^c} |Tu(x)|^r \,\mathrm{d}x &\leq C\int_0^{\frac{1}{4}(b-a)} \Big(\int_{\{\eta=a+2\varrho\}} |Tu(z)|^r \,\mathrm{d}\mathscr{H}^{n-1}(z)\Big) \,\mathrm{d}\varrho \\
&\leq C\int_0^{\frac{1}{4}(b-a)} \varrho^{-nr+n-1}\Big(\int_{Z_{a+\varrho}^{a+3\varrho}} |u(y)| \,\mathrm{d}y\Big)^r \,\mathrm{d}\varrho\,.
\end{aligned} \tag{3.32}$$

We have shown that this inequality holds for u when u is smooth. Now we show that (3.32) holds for a general function $u \in L^1(\Omega;\mathbb{R}^N)$. By a standard approximation argument (for example, using mollification) there exists a sequence (u_j) of smooth functions such that $u_j \to u$ strongly in $L^1(\Omega;\mathbb{R}^N)$, and pointwise almost everywhere. Now we use this property and Fatou's Lemma to get:

$$\int_{Z_a^c} |Tu(x)|^r \,\mathrm{d}x = \int_{Z_a^c} \lim_{j\to\infty} |Tu_j(x)|^r \,\mathrm{d}x$$

$$
\begin{aligned}
&\leq \liminf_{j\to\infty} \int_{Z_a^c} |Tu_j(x)|^r \,\mathrm{d}x \\
&\leq \liminf_{j\to\infty} C \int_0^{\frac{1}{4}(b-a)} \varrho^{-nr+n-1} \left(\int_{Z_{a+\varrho}^{a+3\varrho}} |u_j(y)| \,\mathrm{d}y \right)^r \mathrm{d}\varrho \\
&= C \int_0^{\frac{1}{4}(b-a)} \varrho^{-nr+n-1} \left(\int_{Z_{a+\varrho}^{a+3\varrho}} |u(y)| \,\mathrm{d}y \right)^r \mathrm{d}\varrho \,,
\end{aligned}
$$

as required. Moreover, since

$$
\int_{Z_{a+\varrho}^{a+3\varrho}} |u(y)| \,\mathrm{d}y \leq CM_0\varrho \,,
$$

we have

$$
\begin{aligned}
\int_{Z_a^c} |Tu(x)|^r \,\mathrm{d}x &\leq CM_0^r \int_0^{\frac{1}{4}(b-a)} \varrho^{-nr+n-1+r} \,\mathrm{d}\varrho \\
&\leq CM_0^r (b-a)^{n-(n-1)r} \,. \qquad (3.33)
\end{aligned}
$$

We use an entirely similar argument to conclude that we also have

$$
\int_{Z_c^b} |Tu(x)|^r \,\mathrm{d}x \leq CM_0^r (b-a)^{n-(n-1)r} \,.
$$

Now note that we can also obtain the same estimates with the gradients ∇Tu and ∇u. This is because

$$
\frac{\partial Tu}{\partial x_i}(x) = \fint_{B(0,1)} \left(\frac{\partial u}{\partial x_i}(x+\theta(x)y) + \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x+\theta(x)y) y_j \frac{\partial \theta}{\partial x_i}(x) \right) \mathrm{d}y
$$

and so

$$
|\nabla Tu| \leq CT(|\nabla u|) \,. \qquad (3.34)
$$

Therefore the L^r estimate (3.33) also holds for derivatives, giving

$$
\begin{aligned}
\|Tu\|_{W^{1,r}(Z_a^b)} \leq C(b-a)^{\frac{n}{r}-n+1} \big(&\sup_{t\in(a,b)} (t-a)^{-1} \|u\|_{W^{1,1}(Z_a^t)} \\
&+ \sup_{t\in(a,b)} (b-t)^{-1} \|u\|_{W^{1,1}(Z_t^b)} \big) \,,
\end{aligned}
$$

as required.

It remains to show that T is a continuous linear operator. For u smooth, use the co-area formula, (3.31) with $\alpha = 1$, and (3.34) to get

$$
\int_{Z_a^c} (|Tu| + |\nabla Tu| \,\mathrm{d}y
$$

$$
\begin{aligned}
&\le C \int_0^{\frac14(b-a)} \left(\int_{\{\eta=a+2\varrho\}} |Tu(z)| + |\nabla Tu(z)| \,\mathrm{d}\mathscr{H}^{n-1}(z) \right) \mathrm{d}\varrho \\
&\le C \int_0^{\frac14(b-a)} \left(\int_{Z_{a+\varrho}^{a+3\varrho}} \varrho^{-1}(|u(y)| + |\nabla u(y)|) \,\mathrm{d}y \right) \mathrm{d}\varrho \\
&\le C \int_0^{\frac14(b-a)} \left(\int_{a+\varrho}^{a+3\varrho} \left(\int_{\{\eta=t\}} \varrho^{-1}(|u(z)| + |\nabla u(z)|) \,\mathrm{d}\mathscr{H}^{n-1}(z) \right) \mathrm{d}t \right) \mathrm{d}\varrho \\
&= C \int_a^b \left(\int_{\{\eta=t\}} \left(\int_{\frac{t-a}{3}}^{\min\{t-a,\frac{b-a}{4}\}} \varrho^{-1}(|u(z)| + |\nabla u(z)|) \,\mathrm{d}\varrho \right) \mathrm{d}\mathscr{H}^{n-1}(z) \right) \mathrm{d}t \\
&\le C \int_{Z_a^b} (|u(y)| + |\nabla u(y)|) \,\mathrm{d}y \,. \qquad (3.35)
\end{aligned}
$$

A similar bound holds for

$$\int_{Z_c^b} (|Tu| + |\nabla Tu| \,\mathrm{d}y \,.$$

For u smooth, it is easy to see that Tu is weakly differentiable and, by the above estimates, that $Tu \in W^{1,1}(\Omega;\mathbb{R}^N)$. For $u \in W^{1,1}(\Omega;\mathbb{R}^N)$, again let (u_j) be a sequence of smooth functions such that $u_j \to u$ strongly in $W^{1,1}(\Omega;\mathbb{R}^N)$, and pointwise almost everywhere. By (3.35) and Uniform Boundedness, (Tu_j) is bounded in $W^{1,1}(\Omega;\mathbb{R}^N)$, and hence there exists a subsequence that converges weakly* in $\mathrm{BV}(\Omega;\mathbb{R}^N)$ to Tu, so by (3.35) we have

$$\int_\Omega (|Tu| + |\nabla Tu|) \,\mathrm{d}y \le C \int_\Omega (|u(y)| + |\nabla u(y)|) \,\mathrm{d}y \,,$$

which establishes that indeed T is a linear continuous map from $W^{1,1}(\Omega;\mathbb{R}^N)$ into $W^{1,1}(\Omega;\mathbb{R}^N)$. This completes the proof. $\square$

Chapter 4

Lower semicontinuity in BV of integrals with subquadratic growth

This chapter is devoted to a proof of one of the main new theorems of this thesis, namely a lower semicontinuity result in BV for a quasiconvex integral with an integrand f of subquadratic growth at infinity. Recall that we are considering the variational integral

$$F(u;\Omega) := \int_\Omega f(\nabla u(x))\,\mathrm{d}x\,, \tag{4.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. We require that f satisfies the following growth condition for $1 < r < 2$:

$$0 \leq f(\xi) \leq L(|\xi|^r + 1) \tag{4.2}$$

for a fixed finite $L > 0$ and all $\xi \in \mathbb{R}^{N\times n}$. Note that (4.2) implies that F is defined and continuous on the Sobolev Space $W^{1,r}(\Omega;\mathbb{R}^N)$.

The structure of this chapter is as follows. First we give a statement of the theorem and note that the first step of our proof involves proving the result in the particular case where Ω is the unit ball in $\mathbb{R}^n$ and the limit is 0. We then prove the theorem for this special case: a key component of the proof here is Lemma 3.5 from Chapter 3, which involves obtaining higher integrability properties for an extension operator. We then establish a result concerning a pointwise approximate differentiability property of (sufficiently regular) Sobolev maps. This is because the standard "blow-up" technique used to establish the theorem in the general case from the particular case cannot be straightforwardly applied for our purposes. We require a more careful choice of blow-up functions, which involves exploiting this specific property of Sobolev functions. Lastly, we provide some additional remarks, including corollaries and extensions.

4.1 Main statements and preliminary remarks

The main result we intend to prove is as follows.

Theorem 4.1. *Let Ω be a bounded, open subset of $\mathbb{R}^n$. Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a quasiconvex function satisfying the growth condition* (4.2) *for some exponent $1 < r < 2$.*

Let (u_j) be a sequence in $W^{1,r}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, where $p \geq 1$ and $p > \frac{r}{2}(n-1)$. Suppose

$$u_j \overset{*}{\rightharpoonup} u \text{ in } BV_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \tag{4.3}$$

and

$$(u_j) \text{ uniformly bounded in } L^q_{\mathrm{loc}}(\Omega;\mathbb{R}^n)\,, \tag{4.4}$$

where

$$q > \frac{r(n-1)}{2-r}\,. \tag{4.5}$$

Then

$$\liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \geq \int_\Omega f(\nabla u)\,\mathrm{d}x\,. \tag{4.6}$$

The principal new distinction here compared to previous results is that we now have a lower semicontinuity result in the sequential weak* topology of BV in the case where the growth exponent r is greater than or equal to $\frac{n}{n-1}$ (but less than 2). However, we need to assume additionally that the maps (u_j) are bounded uniformly in L^q_{loc} for q suitably large. Also note that when $n \geq 3$, the conditions of this theorem require that the limit map u is more regular than the maps (u_j). When $n = 2$, however, we can take $u \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, so in this case u can be less regular than the u_j. In this case, however, there is a result by Kristensen [64] even for $u \in$ BV: see Corollary 6.6. By (4.4) we mean simply that for any compact set $K \subset \Omega$, the sequence (u_j) is uniformly bounded in $L^q(K;\mathbb{R}^N)$, i.e. $\sup_j \|u_j\|_{L^q(K)} \leq C(K)$, where $C(K)$ is a positive constant possibly depending on K. We may remark that this is a natural condition if, for example, we assume that the maps u_j and u are constrained to remain on a compact manifold (in which case we would infer the stronger condition that the (u_j) are uniformly bounded in $L^\infty(\Omega;\mathbb{R}^N)$. Indeed, many problems in materials science involve such constrained variational problems - see for instance [35]).

It is also worth briefly discussing more generally the regularity assumptions of the maps u_j, u in the main result. The increased regularity requirement on u, that it is in $W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ where $p \geq 1$ and $p > \frac{r}{2}(n-1)$, is required to make use of the fact that the (u_j) are uniformly bounded in $L^q_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ for q satisfying (4.5) when using the "blow-up argument" to obtain the proof of Theorem 4.1 from Lemma 4.2.

The proof of this Theorem 4.1 relies on the following lemma (which is essentially the theorem in the special case where the limit u is affine and where Ω is the open unit ball B in $\mathbb{R}^n$) combined with this aforementioned, precise blow-up technique that will be detailed later in this chapter.

Lemma 4.2. *Let B denote the open unit ball in $\mathbb{R}^n$. Suppose $(u_j) \subset W^{1,r}(B;\mathbb{R}^N)$, $1 < r < 2$, and $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is as above. Suppose the following conditions hold:*

(i)

$$u_j \to 0 \textit{ strongly in } L^1(B;\mathbb{R}^N) \tag{4.7}$$

(ii)

$$\sup_j \int_B |\nabla u_j| \,\mathrm{d}x < \infty \tag{4.8}$$

(iii) There exists a set $F \subset (0,1)$ such that for all $0 < \delta < 1$, $|F \cap (\delta,1)| > 0$ and

$$\sup_j \sup_{\varrho \in F} \|u_j\|_{L^q(\partial B_\varrho)} < \infty\,, \tag{4.9}$$

where

$$q > \frac{r(n-1)}{2-r}\,. \tag{4.10}$$

Then we have the following inequality:

$$\liminf_{j\to\infty} \int_B f(\nabla u_j)\,\mathrm{d}x \geq \mathscr{L}^n(B)\cdot f(0)\,. \tag{4.11}$$

The proof of this lemma relies on a technique originating in works by Malý, Meyers and Fonseca (see [50, 51, 69, 75]). A key step in this proof involves obtaining an integral estimate for a trace-preserving extension operator. The result, contained in the following lemma, involves adapting and generalising a result by Carozza, Kristensen and Passarelli di Napoli [29]. In the statement of this result, as well as subsequently, we denote by B_ϱ the open ball in $\mathbb{R}^n$ with centre 0, radius ϱ. For the proof of this lemma, refer to the proof of Lemma 3.5 in Chapter 3, where by a localisation argument it suffices to consider extending functions defined on $\mathbb{R}^{n-1}$ into the half space $\mathbb{R}^n_+$ consisting of points in $\mathbb{R}^n$ whose n^{th} coordinate is non-negative.

Lemma 4.3. *Let $1 < r < 2$. Then for $q \geq \frac{r(n-1)}{2-r}$ there exists a linear extension operator*

$$\mathbf{E}\colon (W^{1,1} \cap L^q)(\partial B; \mathbb{R}^N) \to W^{1,r}(B_2 \setminus \bar{B}; \mathbb{R}^N)$$

with the following properties:

1. *If $g \in C^1(\partial B; \mathbb{R}^N)$ then $\mathbf{E}(g) \in C^\infty(B_2 \setminus \bar{B})$ with $\mathbf{E}(g)|_{\partial B} = g$.*
2. *If $(z_j) \subset C^\infty(\partial B; \mathbb{R}^N)$ and $\lim_{j\to\infty} \int_{\partial B} z_j \cdot \phi \, \mathrm{d}\mathscr{H}^{n-1} = 0$ for all $\phi \in C^\infty(\partial B; \mathbb{R}^N)$, then for any multi-index α, $\partial^\alpha[\mathbf{E}z_j] \to 0$ locally uniformly in $B_2 \setminus \bar{B}$.*
3. *There exist positive constants c_1, c_2, dependent on n, N, r, such that:*

   (a)
$$\int_{B_2 \setminus B} |\mathbf{E}(g)|^r \leq c_1 \|g\|^r_{L^q(\partial B)}$$

   (b)
$$\int_{B_2 \setminus B} |\nabla \mathbf{E}(g)|^r \leq c_2 \, \|g\|^{\frac{r}{2}}_{L^q(\partial B)} \cdot \|\nabla g\|_{L^1(\partial B)}$$

   for all $g \in C^1(\partial B)$.

4.2 Proof of the main lemma

In this section we provide a proof of Lemma 4.2.

Proof of Lemma 4.2. By approximation we may assume $(u_j) \subset C^1(\bar{B}; \mathbb{R}^N)$. If the left hand side of (4.11) is infinite then there is nothing to prove, so suppose it is finite. Moreover, by extracting a subsequence if necessary, we can assume

$$l_0 := \liminf_{j\to\infty} \int_B f(\nabla u_j) \, \mathrm{d}x = \lim_{j\to\infty} \int_B f(\nabla u_j) \, \mathrm{d}x \, .$$

With reference to (4.9), write $M = \sup_j \sup_{\varrho \in F} \|u_j\|_{L^q(\partial B_\varrho)}$. From (4.7), by the Fubini-Tonelli theorem and the Rellich-Kondrachoff compactness theorem we have

$$\lim_{j\to\infty} \int_0^1 \int_{\partial B_\varrho} |u_j| \, \mathrm{d}\mathscr{H}^{n-1} \, \mathrm{d}\varrho = \lim_{j\to\infty} \int_B |u_j| \, \mathrm{d}x = 0 \, .$$

This implies there exists a subsequence $\{u_j\}_{j\in T}$ such that

$$\lim_{j\to\infty \, , \, j\in T} \int_{\partial B_\varrho} |u_j| \, \mathrm{d}\mathscr{H}^{n-1} = 0 \tag{4.12}$$

for almost all $\varrho \in (0,1)$. By Fatou's Lemma and (4.8) we have

$$\int_0^1 \liminf_{j\to\infty\,,\, j\in T} \int_{\partial B_\varrho} |\nabla u_j| \,\mathrm{d}\mathscr{H}^{n-1}\,\mathrm{d}\varrho \le \liminf_{j\to\infty\,,\, j\in T} \int_B |\nabla u_j|\,\mathrm{d}x < \infty\,.$$

Thus, for almost all $\varrho \in (0,1)$

$$\liminf_{j\to\infty\,,\, j\in T} \int_{\partial B_\varrho} |\nabla u_j| \,\mathrm{d}\mathscr{H}^{n-1} < \infty\,. \tag{4.13}$$

Now fix $0 < \delta < 1$. By (4.12), (4.13) and (4.9) we can choose $\varrho \in (\delta, 1)$ such that all the following hold:

1.
$$\lim_{j\to\infty\,,\, j\in T} \int_{\partial B_\varrho} |u_j| \,\mathrm{d}\mathscr{H}^{n-1} = 0$$

2.
$$\liminf_{j\to\infty\,,\, j\in T} \int_{\partial B_\varrho} |\nabla u_j| \,\mathrm{d}\mathscr{H}^{n-1} < \infty$$

3.
$$\sup_{j\in T} \|u_j\|_{L^q(\partial B_\varrho)} \le M\,.$$

Now take a further subsequence $\{u_j\}_{j\in S}$, where $S \subseteq T$, so that

$$\lim_{j\to\infty\,,\, j\in S} \int_{\partial B_\varrho} |\nabla u_j| \,\mathrm{d}\mathscr{H}^{n-1} = \liminf_{j\to\infty\,,\, j\in T} \int_{\partial B_\varrho} |\nabla u_j| \,\mathrm{d}\mathscr{H}^{n-1}\,.$$

Relabel the sequence (u_j) so that $S = \mathbb{N}$. Now define the sequence $(g_j) \subset W^{1,1}(\partial B;\mathbb{R}^N)$ as:

$$g_j(x) := u_j|_{\partial B_\varrho}(\varrho x) \quad \text{for } x \in \partial B\,.$$

Take a cut-off function $\eta \in C^1(B;\mathbb{R})$ such that $\mathbf{1}_{B_\varrho} \le \eta \le \mathbf{1}_B$, $|\nabla \eta| \le \frac{2}{1-\varrho}$, and define $(v_j) \subset W_0^{1,r}(B;\mathbb{R}^N)$ as:

$$v_j(x) := \begin{cases} \eta(x)\cdot(\mathbf{E}(g_j))(\frac{x}{\varrho}) & \text{if } |x| \ge \varrho\,, \\ u_j(x) & \text{if } |x| < \varrho\,, \end{cases}$$

where $\mathbf{E}$ is the extension operator from Lemma 4.3.

Since the function $t \mapsto t^r$ is convex, $(s+t)^r \le 2^{r-1}(s^r + t^r)$ for all $s, t \ge 0$. Hence from Lemma 4.3 we have

$$\int_{B\setminus B_\varrho} |\nabla v_j|^r \le \int_{B\setminus B_\varrho} \Big(\big|\, \nabla\eta \cdot \mathbf{E} g_j(\cdot/\varrho) \,\big| + \big|\, \eta \cdot \nabla[\mathbf{E} g_j(\cdot/\varrho)] \,\big| \Big)^r$$

$$\leq 2^{r-1}\int_{B\setminus B_\varrho} |\nabla\eta|^r \cdot \big|\,\mathbf{E}g_j(\cdot/\varrho)\,\big|^r + 2^{r-1}\int_{B\setminus B_\varrho} |\eta|^r \cdot \big|\,\nabla[\mathbf{E}g_j(\cdot/\varrho)]\,\big|^r$$

$$\leq C\int_{B\setminus B_\varrho} \big|\,\mathbf{E}g_j(\cdot/\varrho)\,\big|^r + C\int_{B\setminus B_\varrho} \big|\,\nabla[\mathbf{E}g_j(\cdot/\varrho)]\,\big|^r \tag{4.14}$$

for some constant C. We estimate the two terms in (4.14) using Lemma 4.3 (3) as follows:

$$\begin{aligned}\int_{B\setminus B_\varrho} \big|\,\nabla[\mathbf{E}g_j(\cdot/\varrho)]\,\big|^r &\leq c_2\,\|g_j\|_{L^q(\partial B)}^{\frac{r}{2}} \cdot \|\nabla g_j\|_{L^1(\partial B)} \\ \text{by (4.9)}\quad &\leq c_2\,M^{\frac{r}{2}} \cdot \|\nabla g_j\|_{L^1(\partial B)} \\ &= C\int_{\partial B_\varrho} |\nabla u_j|\,\mathrm{d}\mathscr{H}^{n-1}\,. \end{aligned}\tag{4.15}$$

for another constant C. Now note that we may obtain the same inequality (albeit for a different constant C) using Lemma 4.3 for any other r' such that $r < r' < 2$, and also (with reference to (4.10)) satisfying $q > \frac{r'(n-1)}{2-r'}$. Hence by (4.15) and Lemma 4.3, since

$$\sup_j \int_{\partial B_\varrho} |\nabla u_j|\,\mathrm{d}\mathscr{H}^{n-1} < \infty\,,$$

we can use the De la Vallée Poussin criterion to deduce that the sequence $|\nabla[\mathbf{E}g_j]|^r$ is equi-integrable on $B \setminus B_\varrho$. By Lemma 4.3, since

$$\sup_j \int_{\partial B_\varrho} |u_j|\,\mathrm{d}\mathscr{H}^{n-1} \to 0 \text{ as } j \to \infty\,,$$

$\nabla[\mathbf{E}g_j] \to 0$ locally uniformly on $B \setminus B_\varrho$, and hence so does $|\nabla[\mathbf{E}g_j]|^r$. Thus, by Vitali's Convergence Theorem,

$$\int_{B\setminus B_\varrho} \big|\,\nabla[\mathbf{E}g_j(\cdot/\varrho)]\,\big|^r \to 0 \quad \text{as } j \to \infty\,.$$

Similarly

$$\begin{aligned}\int_{B\setminus B_\varrho} \big|\,[\mathbf{E}g_j(\cdot/\varrho)]\,\big|^r &\leq c_1\,\|g_j\|_{L^q(\partial B)}^r \\ \text{by (4.9)}\quad &\leq c_1\,M^r\,,\end{aligned}$$

so $|\mathbf{E}g_j|^r$ is equi-integrable on $B \setminus B_\varrho$, and using Lemma 4.3 and Vitali,

$$\int_{B\setminus B_\varrho} \big|\,\mathbf{E}g_j(\cdot/\varrho)\,\big|^r \to 0 \quad \text{as } j \to \infty\,.$$

Combining these estimates in (4.14), we have

$$\limsup_{j\to\infty}\int_{B\setminus B_\varrho}|\nabla v_j|^r\,\mathrm{d}x = 0\,. \tag{4.16}$$

Now we use the quasiconvexity and non-negativity of f to obtain

$$\begin{aligned}\int_B f(\nabla u_j) \geq \int_{B_\varrho} f(\nabla u_j) = \int_B f(\nabla v_j) - \int_{B\setminus B_\varrho} f(\nabla v_j) \\ \geq \mathscr{L}^n(B)f(0) - \int_{B\setminus B_\varrho} f(\nabla v_j) \\ \geq \mathscr{L}^n(B)f(0) - L\int_{B\setminus B_\varrho}\big(1+|\nabla v_j|^r\big)\,.\end{aligned}$$

Let $j\to\infty$ to get, using (4.16),

$$l_0 \geq \mathscr{L}^n(B)f(0) - L\mathscr{L}^n(B\setminus B_\varrho)\,.$$

Recall $\varrho\in(\delta,1)$ for fixed $0<\delta<1$. Hence we conclude by taking δ arbitrarily close to 1, which completes the proof of the Lemma. □

4.3 Approximate differentiability of Sobolev maps

In order to obtain the proof of Theorem 4.1 from Lemma 4.2, we use a technique originating in work by Fonseca and Müller, which was further developed by Fonseca and Marcellini (see [53], [52]). However, this ”blow-up argument” still does not apply completely for our purposes. In order to use the fact that the sequence (u_j) in Theorem 4.1 is uniformly bounded in $L^q_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ for q satisfying (4.5), we need to be more careful in our choice of blow-up functions. This involves applying the following lemma.

Lemma 4.4. *Let $u\in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, where $1\leq p<n-1$. Then for almost all $x_0\in\Omega$ the following holds: there exists a set $E\subset(0,1)$ such that 0 is a point of right density one of E, and the difference quotient*

$$\int_{\partial B_\varrho}\left(\frac{\big|u(x_0+z)-u(x_0)-[\nabla u(x_0)]z\big|}{|z|}\right)^{\frac{(n-1)p}{n-1-p}}\mathrm{d}\mathscr{H}^{n-1}(z) \tag{4.17}$$

tends to 0 as $\varrho\to 0$ through the set E. Moreover, the set E has the following property: there exists a sequence $t_k\searrow 0$ and corresponding sets $E_{t_k}\subset[\frac{1}{2},1]$ such that

$$E = \bigcup_{i=1}^{\infty} t_i E_{t_i} \tag{4.18}$$

and, for any $\epsilon > 0$, we can choose t_k, E_{t_k} such that

$$\Big|\bigcap_{i=1}^{\infty} E_{t_i}\Big| > \tfrac{1}{2} - \epsilon\,.$$

Proof of Lemma 4.4. For x_0, $y \in \Omega$, $t > 0$, define

$$v(t,y) := \frac{u(x_0+ty) - u(x_0) - [\nabla u(x_0)](ty)}{t}\,.$$

It is clear that, provided $B(x_0,t) \subset \Omega$, $v(t,y) \in W^{1,p}(B;\mathbb{R}^N)$. Moreover, it is well known that $\int_B |v(t,y)|^p \,\mathrm{d}y \to 0$ as $t \to 0$ for almost all $x_0 \in \Omega$ (see, for example, [98]). In addition, by considering Lebesgue points of ∇u, we have

$$\begin{aligned}\int_B |\nabla v(t,y)|^p \,\mathrm{d}y &= \int_B \big|\nabla u(x_0+ty) - \nabla u(x_0)\big|^p \,\mathrm{d}y \\ &\longrightarrow 0 \ \text{ as } t \to 0\end{aligned}$$

for almost all $x_0 \in \Omega$. Fix such an x_0 and, for $0 < t < \operatorname{dist}(x_0, \partial\Omega)$, define

$$\gamma(t) := \int_{\partial B} |v(t,y)|^{\frac{(n-1)p}{n-1-p}} \,\mathrm{d}\mathscr{H}^{n-1}(y)\,,$$

and

$$\alpha(t) := \int_B \big(|v(t,y)|^p + |\nabla v(t,y)|^p\big) \,\mathrm{d}y\,.$$

Note that by our choice of x_0 we have $\alpha(t) \to 0$ as $t \to 0$. Since $v(t,y) \in W^{1,p}(B)$ for t sufficiently small, we have $v(t,y) \in W^{1,p}(\partial B_\varrho;\mathbb{R}^N)$ for almost all $\varrho \in (0,1)$. It then follows by the Rellich-Kondrachoff embedding theorem that in fact $v(t,y) \in L^{\frac{(n-1)p}{n-1-p}}(\partial B_\varrho;\mathbb{R}^N)$ for almost all $\varrho \in (0,1)$.

Now let

$$\phi_t(\varrho) := \int_{\partial B_\varrho} \big(|v(t,y)|^p + |\nabla v(t,y)|^p\big) \,\mathrm{d}\mathscr{H}^{n-1}(y)$$

and let

$$E_t := [\tfrac{1}{2}, 1] \cap \{\varrho : \phi_t(\varrho) < \alpha(t)^{\frac{1}{2}}\}\,. \tag{4.19}$$

Note

$$\begin{aligned}\alpha(t) = \int_0^1 \phi_t(\varrho)\,\mathrm{d}\varrho &\geq \int_{\frac{1}{2}}^1 \phi_t(\varrho)\,\mathrm{d}\varrho \\ &\geq \int_{[\frac{1}{2},1]\setminus E_t} \phi_t(\varrho)\,\mathrm{d}\varrho\end{aligned}$$

$$\geq |[\tfrac{1}{2},1] \setminus E_t| \cdot \alpha(t)^{\frac{1}{2}}\,,$$

so

$$|[\tfrac{1}{2},1] \setminus E_t| \leq \alpha(t)^{\frac{1}{2}}\,.$$

Next consider $\varrho \in E_t$. By the Sobolev Inequality we have for $\beta = \frac{(n-1)p}{n-1-p}$ and some constant $M = M(p,n)$:

$$\begin{aligned}\Big(\int_{\partial B_\varrho} |v(y,t)|^\beta \,\mathrm{d}\mathscr{H}^{n-1}(y)\Big)^{\frac{1}{\beta}} &\leq M\Big(\int_{\partial B_\varrho} |v(y,t)|^p + |\nabla v(t,y)|^p \,\mathrm{d}\mathscr{H}^{n-1}(y)\Big)^{\frac{1}{p}}\\ &= M\phi_t(\varrho)^{\frac{1}{p}}\\ &\leq M\alpha(t)^{\frac{1}{2p}}\,.\end{aligned}$$

Hence, again for $\varrho \in E_t$, we have

$$\begin{aligned}\gamma(\varrho t) &= \int_{\partial B}\left|\frac{u(x_0+\varrho t y) - u(x_0) - [\nabla u(x_0)](\varrho t y)}{\varrho t}\right|^\beta \mathrm{d}\mathscr{H}^{n-1}(y)\\ &= \varrho^{1-n-\beta}\int_{\partial B_\varrho} |v(t,y)|^\beta \,\mathrm{d}\mathscr{H}^{n-1}(y)\\ &\leq M\cdot 2^{n+\beta+1}\cdot \alpha(t)^{\frac{n-1}{2(n-1-p)}}\,.\end{aligned}$$

Now we may take any decreasing sequence $(t_i) \subset (0, \operatorname{dist}(x_0, \partial\Omega))$ such that $t_i \searrow 0$ and let E_{t_i} be defined as in (4.19). Note that we could also require $t_{i+1} < t_i/2$, so that the E_{t_i} are disjoint. Now define E as stated in (4.18). Thus 0 is a limit point of E, and $\gamma(\varrho) \to 0$ as $\varrho \to 0$, $\varrho \in E$, so the main statement of the lemma is proved.

It remains to show that we can choose (t_i) such that $\big|\bigcap_{i=1}^\infty E_{t_i}\big|$ is arbitrarily close to $\frac{1}{2}$. Write E_t^c for $[\frac{1}{2},1] \setminus E_t$. Since $E_t^c \searrow 0$ as $t \searrow 0$, for a given $\epsilon > 0$ we may choose $t_i \searrow 0$ such that $E_{t_i}^c < 2^{-i}\epsilon$ for all i. Hence

$$\Big|\bigcup_{i=1}^\infty E_{t_i}^c\Big| \leq \sum_{i=1}^\infty |E_{t_i}^c| < \epsilon$$

So

$$\Big|\bigcap_{i=1}^\infty E_{t_i}\Big| > \tfrac{1}{2} - \epsilon\,.$$

This completes the proof of the lemma. $\square$

Remark. If $u \in W^{1,p}_{\text{loc}}(\Omega;\mathbb{R}^N)$ for $p \geq n-1$, then obviously $u \in W^{1,p'}_{\text{loc}}(\Omega;\mathbb{R}^N)$ for any $1 \leq p' \leq p$, so we can still apply the above lemma for $1 \leq p' < n-1$ to prove Theorem 4.1. In fact, if $p > n-1$, then we have a stronger result: namely that u has a *regular approximate total differential* at almost all $x_0 \in \Omega$. This means that the difference quotient

$$\frac{\big|u(x_0+z) - u(x_0) - [\nabla u(x_0)]z\big|}{|z|}$$

tends to 0 uniformly for $z \in \partial B_\varrho$ as $\varrho \to 0$ through a set E for which 0 is a point of right density one. A scheme of a proof of this can be found in [98] (Chap. 3, Exercises), from which the proof of Lemma 4.4 has been adapted.

4.4 Proof of theorem in the general case

We are now in a position to prove Theorem 4.1, the main result of this chapter.

Proof of Theorem 4.1. Firstly we may assume that the left hand side of (4.6) is finite, as otherwise there is nothing to prove. Taking a subsequence if necessary, we can also assume that

$$\int_\Omega f(\nabla u_j) \to \liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,.$$

Since f is non-negative it suffices to prove the statement for any $\Omega' \subset\subset \Omega$. Hence without loss of generality we will assume that Ω is a bounded Lipschitz domain and that, by (4.3),

$$u_j \to u \quad \text{in } L^1(\Omega;\mathbb{R}^N) \tag{4.20}$$

$$\nabla u_j \overset{*}{\rightharpoonup} \nabla u \quad \text{in } \mathscr{M}(\Omega;\mathbb{R}^{N\times n})\,, \tag{4.21}$$

where $\mathscr{M}(\Omega;\mathbb{R}^{N\times n})$ is the space of $N\times n$ matrix-valued Borel measures on Ω. By (4.21) and the Uniform Boundedness Principle, $\sup_j \int_\Omega |\nabla u_j|\,\mathrm{d}\mathscr{L}^n < \infty$. Since $f(\nabla u_j)\mathscr{L}^n$ and $|\nabla u_j|\mathscr{L}^n$ are bounded in $\mathscr{M}(\bar\Omega)$, we that have for some subsequence (for convenience not relabelled) there exist measures μ and ν in $\bar\Omega$ such that

$$\left.\begin{array}{rrl} & f(\nabla u_j) & \overset{*}{\rightharpoonup} \mu \\ \text{and} & |\nabla u_j| & \overset{*}{\rightharpoonup} \nu \end{array}\right\} \quad \text{in } \mathscr{M}(\bar\Omega)\,.$$

Notice that, because $f \geq 0$, the proof of the Theorem follows if we can prove that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}(x) \geq f(\nabla u(x)) \tag{4.22}$$

holds for almost all $x \in \Omega$.

Let Ω_0 denote the set of points $x \in \Omega$ such that:

1. $$\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}(x) = \lim_{\varrho \to 0^+} \frac{\mu(\overline{B(x,\varrho)})}{\mathscr{L}^n(B(x,\varrho))} \quad \text{exists and is finite}$$

2. $$\frac{\mathrm{d}\nu}{\mathrm{d}\mathscr{L}^n}(x) = \lim_{\varrho \to 0^+} \frac{\nu(\overline{B(x,\varrho)})}{\mathscr{L}^n(B(x,\varrho))} \quad \text{exists and is finite}$$

3. $$\lim_{\varrho \to 0^+} \frac{1}{\varrho} \fint_{B(x,\varrho)} |u(y) - u(x) - [\nabla u(x)](x-y)| \,\mathrm{d}y = 0$$

4. Lemma 4.4 holds for u at x.

By standard results (see e.g. [84], [98]) and Lemma 4.4, Ω_0 has full measure in Ω. Fix $x_0 \in \Omega_0$. Let $(r_k) \subset (0, \mathrm{dist}(x_0, \Omega))$ be a sequence such that $r_k \searrow 0$ and define

$$v_{j,k}(y) := \frac{u_j(x_0 + r_k y) - u(x_0) - [\nabla u(x_0)](r_k y)}{r_k}, \quad y \in B. \tag{4.23}$$

Our aim is to pick a suitable sequence (r_k) so we may use $v_{j,k}$ to define a sequence $(z_k) \subset W^{1,r}(B;\mathbb{R}^N)$, say, that will enable us to apply Lemma 4.2 to obtain (4.22). In fact, we do not actually apply Lemma 4.2 for the same q as in Theorem 4.1, but for an arbitrarily smaller $q' < q$ that nevertheless satisfies (4.10).

However, also note that in order to apply Lemma 4.2 and Lemma 4.4, from (4.10) and (4.17) we need $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ for $1 \le p < n-1$ satisfying

$$\frac{p(n-1)}{n-1-p} > \frac{r(n-1)}{2-r}.$$

It is straightforward to verify that this holds if and only if $1 \le p < n-1$ also satisfies $p > \frac{r}{2}(n-1)$.

Recall from Lemma 4.4 that, for any $\epsilon > 0$, we can choose a sequence $t_k \searrow 0$, $t_k < \mathrm{dist}(x_0, \partial\Omega)$, such that $|\bigcap E_{t_k}| > \frac{1}{2} - \epsilon$, with E_{t_k} defined as in (4.19). By (4.4) and (4.20), using De la Vallée Poussin and Vitali, we have $u_j \to u$ in $L^{q'}_{\mathrm{loc}}(\Omega)$ for any $1 \le q' < q$. Hence, for any fixed k:

$$\lim_{j\to\infty} \frac{1}{t_k^{q'}} \int_B \left| u_j(x_0 + t_k y) - u(x_0 + t_k y) \right|^{q'} \mathrm{d}y = 0.$$

So, by Fubini-Tonelli,

$$\lim_{j\to\infty} \frac{1}{t_k^{q'}} \int_0^1 \int_{\partial B_\varrho} \big|u_j(x_0+t_k y) - u(x_0+t_k y)\big|^{q'} \,\mathrm{d}\mathscr{H}^{n-1}(y)\,\mathrm{d}\varrho = 0\,.$$

Hence, for every k, there exists a subsequence $(u_j)_{j\in S_k}$, $S_k \subseteq \mathbb{N}$, such that

$$\lim_{j\to\infty, j\in S_k} \frac{1}{t_k^{q'}} \int_{\partial B_\varrho} \big|u_j(x_0+t_k y) - u(x_0+t_k y)\big|^{q'} \,\mathrm{d}\mathscr{H}^{n-1}(y) = 0 \tag{4.24}$$

for almost all $\varrho \in (0,1)$. Now note that, by Egorov's Theorem, for a given $\epsilon > 0$, there exists a set $G_k \subset (0,1)$ such that $|(0,1) \setminus G_k| < \epsilon 2^{-k}$ and (4.24) holds uniformly for $\varrho \in G_k$. By discarding smaller elements of S_k if necessary, this implies that

$$\sup_{j\in S_k} \sup_{\varrho\in G_k} \frac{1}{t_k^{q'}} \int_{\partial B_\varrho} \big|u_j(x_0+t_k y) - u(x_0+t_k y)\big|^{q'} \,\mathrm{d}\mathscr{H}^{n-1}(y) < 1\,.$$

We can obtain such G_k and S_k for all $k \in \mathbb{N}$. Now note that we have, similarly to the Remark to Lemma 4.4, $|\bigcap G_k| > 1-\epsilon$, so $|(\frac{1}{2},1) \cap \bigcap G_k| > \frac{1}{2} - \epsilon$. Therefore

$$\big|\bigcap_{k=1}^{\infty} (G_k \cap E_{t_k})\big| > \tfrac{1}{2} - 2\epsilon > 0\,,$$

provided ϵ is small enough. This means that $\bigcap (G_k \cap E_{t_k})$ contains a point of left density one θ, say (so $\theta \in (\frac{1}{2},1)$). If we let

$$F = \theta^{-1} \bigcap_{k=1}^{\infty} (G_k \cap E_{t_k})\,,$$

then 1 is a point of left density one of F. Hence, for all $0 < \delta < 1$, $|(\delta,1) \cap F| > 0$.

Now let $r_k = \theta t_k$. So $\big(r_k\big) \subset \big(0, \mathrm{dist}(x_0,\Omega)\big)$, $r_k \searrow 0$. Note also that the set

$$\big\{\varrho \in \big(0, \mathrm{dist}(x_0,\Omega)\big) : (\mu+\nu)\big(\partial B(x_0,\varrho)\big) > 0\big\}$$

is at most countable, so has measure 0. Since there are uncountably many points of left density one, like θ, above, we may assume in addition that $(\mu+\nu)\big(\partial B(x_0,r_k)\big) = 0$ for all k for our choice of r_k. Define $v_{j,k}$ in (4.23) using this choice of r_k. Observe that we may also write $v_{j,k}$ as follows:

$$\begin{aligned} v_{j,k}(y) = \tfrac{1}{r_k}\Big(u(x_0+r_k y) - u(x_0) - [\nabla u(x_0)](r_k y)\Big) \\ + \tfrac{1}{r_k}\Big(u_j(x_0+r_k y) - u(x_0+r_k y)\Big) \end{aligned}$$

$$= I + II\,, \text{ say.}$$

We now consider I and II separately.

Estimating I: If $\varrho \in F$, then $\theta\varrho \in \bigcap E_{t_k}$, so $\varrho\theta t_k \in t_k E_{t_k}$ for all k. i.e. $\varrho r_k \in t_k E_{t_k}$ for all k. So, with reference to (4.18) in Lemma 4.4, we have that $\varrho r_k \in E$: so if $y \in \partial B$, then $\varrho r_k y \in A$. So by Lemma 4.4 we have

$$\sup_{\varrho \in F} \int_{\partial B} \left(\frac{\big|u(x_0 + \varrho r_k y) - u(x_0) - [\nabla u(x_0)](\varrho r_k y)\big|}{|\varrho r_k|} \right)^{\frac{(n-1)p}{n-1-p}} \mathrm{d}\mathscr{H}^{n-1}(y)$$
$$\leq C \cdot \alpha(t_k)^{\frac{n-1}{2(n-1-p)}} \longrightarrow 0 \text{ as } k \to 0\,.$$

This implies that

$$\sup_k \sup_{\varrho \in F} \left\| \tfrac{1}{r_k}\Big(u(x_0 + r_k \cdot) - u(x_0) - [\nabla u(x_0)](r_k \cdot) \Big) \right\|_{L^{q'}(\partial B_\varrho)} < \infty$$

for any $1 \leq q' \leq \frac{(n-1)p}{n-1-p}$. Hence, as noted above, if p satisfies the conditions in Theorem 4.1, then we can choose an appropriate q' satisfying (4.10).

Estimating II: Take a subsequence (u_{j_k}) of (u_j) such that $j_k \in S_k$ for all k. Hence we have for all $\varrho \in F$, $\theta\varrho \in \bigcap G_k$ (so indeed $\varrho\theta \in G_k$ for every k). So, for any $1 \leq q' < q$,

$$\begin{aligned}
&\frac{1}{r_k^{q'}} \int_{\partial B_\varrho} |u_{j_k}(x_0 + r_k y) - u(x_0 + r_k y)|^{q'} \,\mathrm{d}\mathscr{H}^{n-1}(y) \\
&\quad = \frac{1}{(\theta t_k)^{q'}} \int_{\partial B_\varrho} |u_{j_k}(x_0 + \theta t_k y) - u(x_0 + \theta t_k y)|^{q'} \,\mathrm{d}\mathscr{H}^{n-1}(y) \\
&\quad = \frac{1}{(\theta t_k)^{q'}} \int_{\partial B_{\varrho\theta}} |u_{j_k}(x_0 + t_k y) - u(x_0 + t_k y)|^{q'} \theta^{1-n} \,\mathrm{d}\mathscr{H}^{n-1}(y) \\
&\quad < \theta^{1-n-q'}
\end{aligned}$$

for every $k \in \mathbb{N}$, $\varrho \in F$. Hence

$$\sup_k \sup_{\varrho \in F} \left\| \tfrac{1}{r_k}\Big(u_{j_k}(x_0 + r_k \cdot) - u(x_0 + r_k \cdot) \Big) \right\|_{L^{q'}(\partial B_\varrho)} < \infty\,.$$

Therefore, combining these two estimates, we have shown that for this subsequence, $(v_{j_k,k})_{k \in \mathbb{N}}$ satisfies (4.9) of Lemma 4.2 (for q' in place of q).

Now note

$$\fint_B |\nabla v_{j,k}(y) + \nabla u(x_0)| \,\mathrm{d}y = \fint_B |\nabla u_j(x_0 + r_k y)| \,\mathrm{d}y$$

$$
\begin{aligned}
&= \frac{1}{|B(x_0, r_k)|} \int_{B(x_0,r_k)} |\nabla u_j(y)| \, \mathrm{d}y \\
&\longrightarrow \frac{\mathrm{d}\nu}{\mathrm{d}\mathscr{L}^n}(x_0) \quad \text{as } j, k \to \infty \,,
\end{aligned}
$$

and similarly we can take a subsequence (u_{j_k}) such that (by property (2) for Ω_0 above), (4.8) of Lemma 4.2 is satisfied.

In the same way,

$$
⨍_B f(\nabla v_{j,k}(y) + \nabla u(x_0)) \, \mathrm{d}y \longrightarrow \frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}(x_0) \quad \text{as } j, k \to \infty \,,
$$

and we can take a subsequence (u_{j_k}) so that this convergence happens as $k \to \infty$.

Thus, taking multiple subsequences, we can indeed create a sequence $(z_k) = (v_{j_k,k}) \subset W^{1,r}(B; \mathbb{R}^N)$ satisfying (4.7), (4.8) and (4.9). Hence, by Lemma 4.2 (applied to the function $\bar{f}(\xi) = f(\xi + \nabla u(x_0))$, say),

$$
\liminf_{k\to\infty} ⨍_B f\big(\nabla z_k(y) + \nabla u(x_0)\big) \, \mathrm{d}y \geq f\big(\nabla u(x_0)\big) \,,
$$

i.e.

$$
\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}(x_0) \geq f\big(\nabla u(x_0)\big) \,.
$$

This completes the proof of Theorem 4.1. □

4.5 Additional Remarks on the Result

4.5.1 Generalisation for Orlicz-Sobolev spaces

The results of this chapter can actually be stated with the following more general subquadratic growth condition:

Let $\Phi \colon [0, \infty) \to [0, \infty)$ be a convex, doubling, non-decreasing function such that $\Phi(0) = 0$ and, for some $\sigma_\Phi > 0$:

$$
t \mapsto \frac{\Phi(t)}{t^{2-\sigma_\Phi}} \text{ is non-increasing on } (0, \infty) \text{ and } \int_1^\infty \frac{\Phi(t)^{\frac{1}{2}}}{t^2} \, \mathrm{d}t < \infty \,. \tag{4.25}
$$

(Recall that Φ is doubling means that, for a fixed constant c, $\Phi(2t) \leq c\,\Phi(t)$ for all $t \geq 0$.)

Assume that f satisfies

$$
0 \leq f(\xi) \leq L(\Phi(|\xi|) + 1) \tag{4.26}
$$

for a fixed finite $L > 0$ and all $\xi \in \mathbb{R}^{N\times n}$.

This growth condition implies that F is defined and continuous on the generalised Orlicz-Sobolev Space $W^{1,\Phi}(\Omega;\mathbb{R}^N)$. For further information on the subject of such spaces, we refer to the book of Iwaniec and Martin [63], and also that of Rao and Ren [80]. With this general growth condition, we have the following result:

Theorem 4.5. *Let Ω be a bounded, open subset of $\mathbb{R}^n$. Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a quasiconvex function satisfying the growth condition* (4.26) *for some non-decreasing, doubling, convex $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$, satisfying* (4.25).

Suppose (u_j) is a sequence in $W^{1,\Phi}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, where $p \geq 1$ and $p > \frac{r}{2}(n-1)$. Suppose

$$u_j \overset{*}{\rightharpoonup} u \textit{ in } BV_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$$

and

$$(u_j) \textit{ uniformly bounded in } L^q_{\mathrm{loc}}(\Omega;\mathbb{R}^n)\,,$$

where

$$q > \max\left\{2(n-1), \frac{(n-1)(2-\sigma_\Phi)}{\sigma_\Phi}\right\}.$$

Then

$$\liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \geq \int_\Omega f(\nabla u)\,\mathrm{d}x\,.$$

We have focused in particular on the case where Φ is of the form $\Phi(t) = t^r$, which just puts us in the more familiar setting of the Sobolev Space $W^{1,r}(\Omega;\mathbb{R}^N)$, noting that for such Φ, (4.25) is satisfied whenever $1 < r < 2$. In order to prove this general result, we use Lemma 3.6 instead of Lemma 3.5.

4.5.2 Possible extensions and counterexamples

Our result for subquadratic quasiconvex integrands hinges on a result by Greco, Iwaniec and Moscariello in [61], concerning integral estimates of the Hardy-Littlewood Maximal function (see Chapter 3). In this connection the condition (4.25) is sharp and the proof provided cannot be weakened to include integrands of quadratic growth at infinity. Indeed, we do not know if the main theorem is true for f satisfying (4.2) for $r = 2$, even when $\Omega = B$, $u = 0$, $u_j \overset{*}{\rightharpoonup} 0$ in $\mathrm{BV}(B;\mathbb{R}^N)$ and $u_j \to 0$ in $L^\infty(B;\mathbb{R}^N)$. However, what is clear is that the proof of such a result, if it is true, needs to proceed by a different means. We may deduce from Counterexample 1.7, established by Malý in [70], that the result is certainly not true when f has at least cubic growth in some directions (and $n, N \geq 3$). A suitable counterexample for our purposes immediately follows by taking $n = 3$ (or, if we want a result in higher dimensions we can simply consider a suitable 3×3 minor of the Jacobian), $\Omega = Q$, and $f(\xi) = |\det \xi|$. f is polyconvex, hence quasiconvex, and satisfies the growth condition

$$0 \leq f(\xi) \leq L(|\xi|^3 + 1)\,.$$

Moreover the u_j, being diffeomorphisms of Q onto Q, are clearly uniformly bounded in $L^q(Q, \mathbb{R}^n)$ for any $1 \leq q \leq \infty$, and weak convergence in $W^{1,r}$ for $1 < r < 2$ obviously implies weak* convergence in BV. And if u is the identity map on Q, then $\int_Q \det\nabla u \,\mathrm{d}x = 1$. So all the conditions of Theorem 4.1 except the growth condition are satisfied, but lower semicontinuity does not obtain.

Another issue is that if we just assume that f is quasiconvex in the sense of Morrey, then we could consider whether lower semicontinuity still obtains if the maps (u_j) are less regular than $W^{1,r}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$. Even though it is still an open question whether lower semicontinuity obtains when f has quadratic growth, Counterexample 1.6, provided by Ball and Murat in [20], demonstrates (if we take $n = 2$) that in this case we would certainly require at least that the (u_j) are in $W^{1,2}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$.

4.5.3 $W^{1,1}$-quasiconvexity case

It is interesting to note that we can also set the main result of this paper in the context of $W^{1,1}$-quasiconvexity. Recall from Definition 1.4 that a Borel measurable integrand $f\colon \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ is said to be $W^{1,r}$-quasiconvex (where $1 \leq r \leq \infty$) if it is bounded below and satisfies

$$\int_E f(\xi + \nabla\phi(x))\,\mathrm{d}x \geq \mathscr{L}^n(E) f(\xi)$$

for every bounded open set $E \subset \mathbb{R}^n$ with $\mathscr{L}^n(\partial E) = 0$, for all $\xi \in \mathbb{R}^{N\times n}$, and all test functions $\phi \in W^{1,r}_0(E; \mathbb{R}^N)$. For $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ bounded below and locally bounded, $W^{1,\infty}$-quasiconvexity is just the usual definition of quasiconvexity. In the proof of Lemma 4.2, we use the well known result that the conditions on f in Theorem 4.1, in particular its continuity, quasiconvexity and growth condition (4.2), imply that it is $W^{1,r}$-quasiconvex. Crucially, this is why we need the maps (u_j) in Theorem 4.1

to be in $W^{1,r}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. However, if we assume the stronger condition that f is $W^{1,1}$-quasiconvex, we only require the (u_j) to be in $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. That is, we have the following result, which has virtually the same proof as Theorem 4.1.

Theorem 4.6. *Let Ω be a bounded, open subset of $\mathbb{R}^n$. Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a $W^{1,1}$-quasiconvex function satisfying the growth condition* (4.2) *for some exponent $1 < r < 2$.*

Let (u_j) be a sequence in $W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, where $p \geq 1$ and $p > \frac{r}{2}(n-1)$. Suppose

$$u_j \overset{*}{\rightharpoonup} u \textit{ in } BV_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$$

and

$$(u_j) \textit{ uniformly bounded in } L^q_{\mathrm{loc}}(\Omega;\mathbb{R}^n)\,,$$

where

$$q > \frac{r(n-1)}{2-r}\,.$$

Then

$$\liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \geq \int_\Omega f(\nabla u)\,\mathrm{d}x\,.$$

Further results concerning lower semicontinuity and relaxation for $W^{1,r}$-quasiconvex functions ($1 < r < \infty$) may be found in work by Kristensen [65].

4.5.4 Properties of the Lebesgue-Serrin extension

Let us now consider properties of a suitable Lebesgue-Serrin extension, and introduce the functional (for $1 < r < 2$ and q satisfying (4.5))

$$\mathscr{F}(u,\Omega) := \inf_{(u_j)} \left\{ \liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \;\middle|\; \begin{array}{l} (u_j) \subset W^{1,r}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \\ (u_j) \text{ uniformly bounded in } L^q_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \\ u_j \overset{*}{\rightharpoonup} u \text{ weakly}^* \text{ in } \mathrm{BV}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \end{array} \right\}.$$

Note that Theorem 4.1 implies the following result:

Corollary 4.7. *Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ satisfy the conditions in Theorem 4.1. Then*

- *If $n \geq 3$, $p > \frac{r}{2}(n-1)$, and q satisfies* (4.5)*, $\mathscr{F}(u;\Omega) = F(u;\Omega)$ for all $u \in (W^{1,p}_{\mathrm{loc}} \cap L^q_{\mathrm{loc}})(\Omega;\mathbb{R}^N)$.*

- *If $n = 2$, then this equality holds for all $u \in (W^{1,r}_{\mathrm{loc}} \cap L^q_{\mathrm{loc}})(\Omega;\mathbb{R}^N)$.*

Proof. For any n, Theorem 4.1 tells us that if $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ for $p > \frac{r}{2}(n-1)$, $p \geq 1$, and (u_j) is a sequence satisfying the conditions given in the definition of $\mathscr{F}(u;\Omega)$, then

$$\liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \geq F(u;\Omega)\,.$$

Taking the infimum of all such (u_j), we get

$$\mathscr{F}(u,\Omega) \geq F(u;\Omega) \text{ when } u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)\,.$$

Now note that if $u \in (W^{1,r}_{\mathrm{loc}} \cap L^q_{\mathrm{loc}})(\Omega;\mathbb{R}^N)$, then by simply taking $u_j = u$ for all j, we get a sequence satisfying the conditions for $\mathscr{F}(u,\Omega)$, so certainly

$$F(u;\Omega) \geq \mathscr{F}(u,\Omega) \text{ when } u \in (W^{1,r}_{\mathrm{loc}} \cap L^q_{\mathrm{loc}})(\Omega;\mathbb{R}^N)\,.$$

We conclude by noting that since $1 < r < 2$, for $n \geq 3$ we have

$$W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \subset W^{1,r}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)\,,$$

and for $n = 2$, since in this case we can take $p = 1$,

$$W^{1,r}_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \subset W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)\,.$$

□

If we wish to describe $\mathscr{F}$ for an even wider class of functions u, things can be more difficult. Certainly, if $u \notin L^q_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ then there can be no sequence (u_j) uniformly bounded in $L^q_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ satisfying $u_j \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ (or even just strongly in L^1_{loc}), since this would imply that u is itself in $L^q_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$. Hence we have

$$\mathscr{F}(u,\Omega) = \inf\emptyset = +\infty\,.$$

Results by Bouchitté, Fonseca and Malý [21, 50, 51], as well as those of the next chapter, indicate that, even for $n \geq 3$, a measure representation for $\mathscr{F}$ should exist for $u \in (W^{1,r}_{\mathrm{loc}} \cap L^q_{\mathrm{loc}})(\Omega;\mathbb{R}^N)$, but we have been unable to prove this yet. A counterexample due to Acerbi and Dal Maso [2] shows that if $r = n = N = 2$ and $u \in (\mathrm{BV}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}})(\Omega;\mathbb{R}^N)$, then a measure representation does not exist at all. Although their conditions are slightly different from ours, it is not difficult to see from their paper that their counterexample also applies to our case. In fact, they present an example where the set function $\omega \mapsto \mathscr{F}(u,\omega)$ is not even subadditive (for an alternative proof, see [31]). It is also possible, under certain conditions, for the Lebesgue-Serrin extension to have atoms - see [52].

Chapter 5

Relaxation in BV of integrals with superlinear growth

In this chapter, we study properties of the Lebesgue-Serrin extensions

$$\mathscr{F}(u,\Omega) := \inf_{(u_j)} \left\{ \liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \;\middle|\; \begin{array}{l} (u_j) \subset W^{1,r}(\Omega,\mathbb{R}^N) \\ u_j \overset{*}{\rightharpoonup} u \text{ in } \mathrm{BV}(\Omega,\mathbb{R}^N) \end{array} \right\} \tag{5.1}$$

and

$$\mathscr{F}_{\mathrm{loc}}(u,\Omega) := \inf_{(u_j)} \left\{ \liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \;\middle|\; \begin{array}{l} (u_j) \subset W^{1,r}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \\ u_j \overset{*}{\rightharpoonup} u \text{ in } \mathrm{BV}(\Omega,\mathbb{R}^N) \end{array} \right\}, \tag{5.2}$$

where Ω is a bounded, open subset of $\mathbb{R}^n$, $n \geq 2$, and f is a continuous integrand satisfying the growth condition

$$0 \leq f(\xi) \leq L(|\xi|^r + 1) \tag{5.3}$$

for a fixed finite $L > 0$ and all $\xi \in \mathbb{R}^{N\times n}$, where $r \in [1, \frac{n}{n-1})$. We first establish some basic properties of these functionals, and then use the trace-preserving operator of Chapter 3 to show that they are representable by finite Radon measures on Ω. This essentially comes directly from the work of Fonseca and Malý in [50], where measure representation is obtained for Lebesgue-Serrin extensions in the context of Sobolev Spaces of exponent larger than one (in fact they consider more general integrands of the form $f = f(x, u, \nabla u)$). These results will also be important in the context of the next chapter, where we obtain a lower semicontinuity result in the case where f is assumed additionally to be quasiconvex and have at most linear growth in certain directions.

5.1 Some basic properties

In this section we collect some elementary facts of the Lebesgue-Serrin extensions defined above. A key reference for general properties is [25]. For every $z \in \mathbb{R}^n$ define the translation operator T_z by $(T_z u)(x) = u(x-z)$ and $T_z\Omega = \{x \in \mathbb{R}^n : x-z \in \Omega\} = z+\Omega$. For every $\varrho > 0$, define the homothety operator θ_ϱ by $(\theta_\varrho u)(x) = (1/\varrho)(u(\varrho x))$ and $\theta_\varrho\Omega = \{x \in \mathbb{R}^n : \varrho x \in \Omega\} = (1/\varrho)\Omega$. The following proposition states some important facts about $\mathscr{F}$ and $\mathscr{F}_{\mathrm{loc}}$ that come directly from their definitions.

Proposition 5.1. *Let Ω be an open subset of $\mathbb{R}^n$ and $u \in BV(\Omega;\mathbb{R}^N)$. Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous function satisfying the growth condition* (5.3) *for some exponent $1 \le r < \frac{n}{n-1}$. Then $\mathscr{F}$ as defined in* (5.1) *satisfies the following properties:*

(a) $\mathscr{F}(T_z u, T_z\Omega) = \mathscr{F}(u,\Omega)$ *for every $z \in \mathbb{R}^n$,*

(b) $\mathscr{F}(u+\eta,\Omega) = \mathscr{F}(u,\Omega)$ *for every $\eta \in \mathbb{R}^N$,*

(c) $\mathscr{F}(\theta_\varrho u, \theta_\varrho\Omega) = \varrho^{-n}\mathscr{F}(u,\Omega)$ *for every $\varrho > 0$.*

Identical statements hold true for $\mathscr{F}_{loc}$ as defined in (5.2).

The next proposition shows that, provided we assume that f is coercive, then by a straightforward diagonalisation argument and compactness properties in BV we have that $\mathscr{F}$ and $\mathscr{F}_{\mathrm{loc}}$ are attained and are lower semicontinuous in the weak* topology of BV.

Proposition 5.2. *Let Ω and f be as in Proposition 5.1. Assume in addition that f satisfies, for some constant $c_0 > 0$,*

$$f(\xi) \ge c_0|\xi| \tag{5.4}$$

for all $\xi \in \mathbb{R}^{N\times n}$. Let $\mathscr{F}$ be as defined in (5.1). *Then*

(a) If $\mathscr{F}(u,\Omega) < \infty$, then it is attained. That is, there exists a sequence (u_j) in $W^{1,r}(\Omega;\mathbb{R}^N)$ such that $u_j \xrightarrow{*} u$ in $BV(\Omega;\mathbb{R}^N)$ and

$$\lim_{j\to\infty} \int_\Omega f(\nabla u)\,\mathrm{d}x = \mathscr{F}(u,\Omega)\,.$$

(b) If (u_j) is a sequence in BV$(\Omega;\mathbb{R}^N)$ converging weakly in BV to $u \in$ BV$(\Omega;\mathbb{R}^N)$, and $\mathscr{F}(u_j,\Omega) < \infty$ for all j, then*

$$\liminf_{j\to\infty} \mathscr{F}(u_j,\Omega) \geq \mathscr{F}(u,\Omega)\,.$$

Identical statements hold true for $\mathscr{F}_{loc}$ as defined in (5.2).

Proof of Proposition 5.2. To prove (a) first note that by the definition of $\mathscr{F}$, for each $j \in \mathbb{N}$, there exists a sequence $(u_{j,k})_{k\in\mathbb{N}}$ in $W^{1,r}(\Omega;\mathbb{R}^N)$ such that $u_{j,k} \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^N)$ as $k \to \infty$ and

$$\liminf_{k\to\infty} \int_\Omega f(\nabla u_{j,k})\,\mathrm{d}x < \mathscr{F}(u,\Omega) + 1/j\,.$$

By taking a subsequence of $(u_{j,k})_{k\in\mathbb{N}}$ if necessary for each j, we may assume

$$\lim_{k\to\infty} \int_\Omega f(\nabla u_{j,k})\,\mathrm{d}x = \liminf_{k\to\infty} \int_\Omega f(\nabla u_{j,k})\,\mathrm{d}x\,.$$

Hence for each j, there exists k_j^1 such that

$$\int_\Omega f(\nabla u_{j,k})\,\mathrm{d}x < \mathscr{F}(u,\Omega) + 1/j \tag{5.5}$$

for all $k \geq k_j^1$. Moreover, since $u_{j,k} \to u$ strongly in $L^1(\Omega;\mathbb{R}^N)$ as k tends to infinity for each j, there exists k_j^2 such that

$$\|u_{j,k} - u\|_{L^1(\Omega;\mathbb{R}^N)} < 1/j$$

for all $k \geq k_j^2$. Hence we may take an increasing sequence k_j in $\mathbb{N}$ such that $k_j \geq \max\{k_j^1, k_j^2\}$ for all $j \in \mathbb{N}$. Then (u_{j,k_j}) is a sequence in $W^{1,r}(\Omega;\mathbb{R}^N)$ satisfying

$$\|u_{j,k_j} - u\|_{L^1(\Omega;\mathbb{R}^N)} < 1/j \to 0 \quad \text{as } j \to \infty\,.$$

Hence by (5.4) and Proposition 2.4 $u_{j,k_j} \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^N)$ as j tends to infinity. Consequently, by the definition of $\mathscr{F}$ and (5.5),

$$\liminf_{j\to\infty} \int_\Omega f(\nabla u_{j,k_j})\,\mathrm{d}x = \mathscr{F}(u,\Omega),$$

which, taking a further subsequence if necessary, establishes property (a).

Now we prove (b): let (u_j), $u \subset \mathrm{BV}(\Omega;\mathbb{R}^N)$ with u_j converging weakly* to u. By (a), we have for each j there exist sequences $(u_{j,k})$ in $W^{1,r}(\Omega;\mathbb{R}^N)$ such that $u_{j,k} \overset{*}{\rightharpoonup} u_j$ in $\mathrm{BV}(\Omega;\mathbb{R}^N)$ and

$$\lim_{k\to\infty}\int_\Omega f(\nabla u_{j,k})\,\mathrm{d}x = \mathscr{F}(u_j,\Omega).$$

We may clearly also assume (by taking a subsequence) that for each j we have

$$\int_\Omega f(\nabla u_{j,k})\,\mathrm{d}x < \mathscr{F}(u_j,\Omega) + 1/k. \tag{5.6}$$

Since $u_{j,k} \to u_j$ in $L^1(\Omega;\mathbb{R}^N)$ as $k\to\infty$ and $u_j \to u$ in $L^1(\Omega;\mathbb{R}^N)$ as $j\to\infty$, by a standard diagonalisation argument there exists a sequence (u_{j,k_j}) converging to u in $L^1(\Omega;\mathbb{R}^N)$ as $j\to\infty$. As before, by the coercivity property (5.4) and Proposition 2.4 u_{j,k_j} converges weakly* to u in $\mathrm{BV}(\Omega;\mathbb{R}^N)$. Thus by (5.6) and the definition of $\mathscr{F}$,

$$\begin{aligned}\liminf_{j\to\infty}\mathscr{F}(u_j,\Omega) &\geq \liminf_{j\to\infty}\Big(\int_\Omega f(\nabla u_{j,k_j})\,\mathrm{d}x - 1/k_j\Big)\\ &\geq \mathscr{F}(u,\Omega)\,,\end{aligned}$$

as required. The proof for $\mathscr{F}_{\mathrm{loc}}$ is identical. □

5.2 Technical Preliminaries

In this section we establish some results that are key to proving the main result of this chapter. First, we have the following elementary lemma.

Lemma 5.3. *Let ψ be a continuous non-decreasing function on an interval $[a,b]$, $a < b$. Then there exist $a' \in [a, a+\frac{1}{3}(b-a)]$ and $b' \in [b-\frac{1}{3}(b-a), b]$ such that $a \leq a' < b' \leq b$, and*

$$\begin{cases} \dfrac{\psi(t)-\psi(a')}{t-a'} & \leq 3\dfrac{\psi(b)-\psi(a)}{b-a} \\[2ex] \dfrac{\psi(b')-\psi(t)}{b'-t} & \leq 3\dfrac{\psi(b)-\psi(a)}{b-a} \end{cases} \tag{5.7}$$

for all $t \in (a',b')$.

Proof of Lemma 5.3. Without loss of generality we may assume $a = 0$ and $\psi(a) = 0$. Define

$$\phi(t) := \psi(t) - 3t\frac{\psi(b)}{b}\,.$$

Let a' be the point in $[0,b]$ where ϕ attains its maximum and let b' be the point where ϕ attains its minimum. It follows clearly that (5.7) holds from this choice of a' and b': note that when $t > \frac{b}{3}$, since ψ is non-decreasing, $3t\frac{\psi(b)}{b} > \psi(b) \geq \psi(b)$. Hence we have $\phi(0) = 0$ and $\phi(t) < 0$, so it follows that $a' \leq \frac{b}{3}$. We argue in a similar way to show that $b' \geq b - \frac{1}{3}b$. $\square$

We now apply this result to establish the following.

Lemma 5.4. *Let $V \subset\subset \Omega$ and $W \subset \Omega$ be open sets satisfying $\Omega = V \cup W$. Let $v \in W^{1,r}(V)$ and $w \in W^{1,r}(W)$ for $r \in [1, \frac{n}{n-1})$. Let $k \in \mathbb{N}$. Then there exists a function $z \in W^{1,r}_{loc}(\Omega)$ and open sets $V' \subset V$ and $W' \subset W$, such that $V' \cup W' = \Omega$, $z = v$ on $\Omega \setminus W'$, $z = w$ on $\Omega \setminus V'$,*

$$\mathscr{L}^n(V' \cap W') \leq Ck^{-1} \tag{5.8}$$

and

$$\|z\|_{W^{1,r}(V'\cap W')} \leq Ck^{n-1-\frac{n}{r}}\Big(\|v\|_{W^{1,1}(V\cap W)} + \|w\|_{W^{1,1}(V\cap W)} + k\|w-v\|_{L^1(V\cap W)}\Big), \tag{5.9}$$

where C is a constant dependent on r, V and W.

Proof of Lemma 5.4. Let $\eta \in C^\infty_c(\Omega)$ be such that

$$\eta = 0 \quad \text{on } \Omega \setminus V \quad \text{and} \quad \eta = 1 \quad \text{on } \Omega \setminus W\,. \tag{5.10}$$

By Sard's Lemma, the image of the set of all critical points of η is a closed set of measure zero. Hence, there exists a nondegenerate interval $[a.b] \subset (0,1) \setminus \eta(\{\nabla\eta = 0\})$. Take $k \in \mathbb{N}$ and define

$$f := 1 + |v| + |w| + |\nabla v| + |\nabla w| + k|w - v|\,.$$

Since $\{a < \eta < b\} \subset V \cap W$, we may find $j \in \{1, \ldots, k\}$ such that

$$\int_{\{a_j<\eta<b_j\}} f \,\mathrm{d}x \leq \frac{1}{k}\int_{V\cap W} f\,\mathrm{d}x\,, \tag{5.11}$$

where $a_j := a + \frac{(j-1)(b-a)}{k}$ and $b_j := a + \frac{j(b-a)}{k}$. Now apply Lemma 5.3 with

$$\psi(t) := \int_{\{\eta<t\}} f\,\mathrm{d}x\,,$$

to find $[a', b'] \subset [a_j, b_j]$ such that $b' - a' \geq \frac{1}{3}(b_j - a_j)$, and

$$\int_{\{a'<\eta<t\}} f \, \mathrm{d}x \leq 3\frac{t-a'}{b'-a'} \int_{\{a'<\eta<b'\}} f \, \mathrm{d}x \,,$$
$$\int_{\{t<\eta<b'\}} f \, \mathrm{d}x \leq 3\frac{b'-t}{b'-a'} \int_{\{a'<\eta<b'\}} f \, \mathrm{d}x \tag{5.12}$$

for all $t \in (a', b')$. Now set

$$V' := \Omega \cap \{\eta > a'\} \,, \qquad W' := \Omega \cap \{\eta < b'\} \,,$$

and

$$u := \begin{cases} v & \text{on } \{\eta \geq b'\} \,, \\ \dfrac{(\eta - a')v + (b' - \eta)w}{b' - a'} & \text{on } \{a' \leq \eta \leq b'\} \,, \\ w & \text{on } \{\eta \leq a'\} \,. \end{cases}$$

By (5.10), it is clear that $V' \subset V$, $W' \subset W$, and $V' \cup W' = \Omega$. Moreover, (5.8) holds as $|\nabla \eta|$ is bounded away from zero on $\{a < \eta < b\}$ and $b' - a' \leq \frac{b-a}{k}$. It is easy to verify that on $\{a' < \eta < b'\}$ we have

$$|u| + |\nabla u| \leq Cf \,.$$

Now use (5.11), (5.12) and Lemma 3.8 to find a function $z \in W^{1,1}(\Omega)$ such that $z = u = v$ on $\{\eta \geq b'\} = \Omega \setminus W'$, $z = u = w$ on $\{\eta \leq a'\} = \Omega \setminus V'$, and (5.9) is satisfied. □

5.3 Proof of measure representation

We first establish some key definitions. Let μ be a Radon measure on $\bar{\Omega}$, where Ω is a bounded, open subset of $\mathbb{R}^n$. Then we say that μ (strongly) *represents* $\mathscr{F}(u, \cdot)$ if

$$\mu(U) = \mathscr{F}(u, U)$$

for all open sets $U \subset \Omega$. We say that μ *weakly represents* $\mathscr{F}(u, \cdot)$ if

$$\mu(U) \leq \mathscr{F}(u, U) \leq \mu(\bar{U})$$

for all open sets $U \subset \Omega$. The following two theorems are the main results of this chapter.

Theorem 5.5. *Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous function satisfying the growth condition* (5.3) *for some exponent $1 \le r < \frac{n}{n-1}$. Let $u \in BV(\Omega;\mathbb{R}^N)$ and $\mathscr{F}_{loc}$ be as defined in* (5.2). *Then if $\mathscr{F}_{loc}(u,\Omega) < \infty$, then there exists a non-negative, finite Radon measure λ on Ω which represents $\mathscr{F}_{loc}$.*

Theorem 5.6. *Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous function satisfying the growth condition* (5.3) *for some exponent $1 \le r < \frac{n}{n-1}$. Let $u \in BV(\Omega;\mathbb{R}^N)$ and $\mathscr{F}$ be as defined in* (5.1). *Then if $\mathscr{F}(u,\Omega) < \infty$, then there exists a non-negative, finite Radon measure μ on $\bar{\Omega}$ which weakly represents $\mathscr{F}$.*

The following lemma is instrumental in our proofs of these two theorems.

Lemma 5.7. *Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous function satisfying the growth condition* (5.3) *for some exponent $1 \le r < \frac{n}{n-1}$. Let V, $W \subset \Omega$ be open sets, $V \subset\subset \Omega$ and $\Omega = V \cup W$, and let $u \in W^{1,1}(\Omega;\mathbb{R}^N)$. Let $\mathscr{F}$ be as defined in* (5.1). *Then*

$$\mathscr{F}(u,\Omega) \le \mathscr{F}(u,V) + \mathscr{F}(u,W)\,.$$

An identical assertion holds for $\mathscr{F}_{loc}$ as defined in (5.2).

Proof of Lemma 5.7. Let $\epsilon > 0$. By the definition of $\mathscr{F}$, there exist sequences $(v_k) \subset W^{1,r}(V;\mathbb{R}^N)$ and $(w_k) \subset W^{1,r}(W;\mathbb{R}^N)$ such that

$$v_k \overset{*}{\rightharpoonup} u \text{ weakly* in } \mathrm{BV}(V;\mathbb{R}^N) \quad \text{and} \quad w_k \overset{*}{\rightharpoonup} u \text{ weakly* in } \mathrm{BV}(W;\mathbb{R}^N)\,,$$

and (by eliminating the first terms of the sequences if necessary),

$$\int_V f(\nabla v_k)\,\mathrm{d}x \le \mathscr{F}(u,V) + \epsilon\,,$$
$$\int_W f(\nabla w_k)\,\mathrm{d}x \le \mathscr{F}(u,W) + \epsilon\,.$$

Moreover, by taking subsequences if necessary, we can ensure

$$\|v_k - u\|_{L^1(V\cap W)} \le \frac{1}{k} \quad \text{and} \quad \|w_k - u\|_{L^1(V\cap W)} \le \frac{1}{k} \tag{5.13}$$

for all k. Using Lemma 5.4, for each k we can find open sets $V_k \subset V$, $W_k \subset W$, and functions $(z_k) \subset W^{1,r}(\Omega;\mathbb{R}^N)$, such that $V_k \cup W_k = \Omega$, $z_k = v_k$ on $\Omega \setminus W_k$, and $z_k = w_k$ on $\Omega \setminus V_k$. Moreover, by growth condition (5.3), (5.13), and since by the Uniform Boundedness Principle the sequences (v_k), (w_k) are bounded in $W^{1,1}(V;\mathbb{R}^N)$, $W^{1,1}(W;\mathbb{R}^N)$ respectively,

$$\int_{V_k\cap W_k} f(\nabla z_k)\,\mathrm{d}x \le L\int_{V_k\cap W_k} (1 + |\nabla z_k|^r)\,\mathrm{d}x$$

$$\leq Ck^{-1} + Ck^{r(n-1)-n}\Big(\|v\|_{W^{1,1}(V\cap W)} + \|w\|_{W^{1,1}(V\cap W)} + k\|w-v\|_{L^1(V\cap W)}\Big)^r$$
$$\leq Ck^{r(n-1)-n}\,. \tag{5.14}$$

Therefore

$$\int_\Omega f(\nabla z_k)\,\mathrm{d}x \leq \int_V f(\nabla v_k)\,\mathrm{d}x + \int_W f(\nabla w_k)\,\mathrm{d}x + Ck^{r(n-1)-n}\,. \tag{5.15}$$

Now we show that $z_k \stackrel{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^N)$. Certainly, since $z_k = v_k$ on $\Omega\backslash W_k$, $z_k = w_k$ on $\Omega\backslash V_k$, and by (5.14) the $W^{1,r}$-norm of each z_k is bounded on $V_k\cap W_k$, the sequence is bounded in $W^{1,1}(\Omega;\mathbb{R}^N)$. Moreover, using the fact that $\mathscr{L}^n(V_k\cap W_k)\to 0$ and Rellich-Kondrachoff, we have that each subsequence of (z_k) has a sub-subsequence converging in $L^1(\Omega;\mathbb{R}^N)$ to u. Therefore it follows by Proposition 2.4 that $z_k \stackrel{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^N)$ as required. Hence by the definition of $\mathscr{F}$ and (5.15)

$$\mathscr{F}(u,\Omega) \leq \liminf_{k\to\infty}\int_\Omega f(\nabla z_k)\,\mathrm{d}x \leq \mathscr{F}(u,V) + \mathscr{F}(u,W) + 2\epsilon\,,$$

which concludes the proof. The proof for $\mathscr{F}_{\mathrm{loc}}$ is essentially the same. □

Proof of Theorem 5.6. First we assume in addition that f satisfies the coercivity condition

$$f(\xi) \geq c_0|\xi| \tag{5.16}$$

for some constant $c_0 > 0$, for all $\xi\in\mathbb{R}^{N\times n}$. Using Proposition 5.2, let $(u_k)\subset W^{1,r}(\Omega;\mathbb{R}^N)$ be a minimising sequence for $\mathscr{F}(u,\Omega)$, i.e. $u_k \stackrel{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^N)$ and

$$\lim_{k\to\infty}\int_\Omega f(\nabla u_k)\,\mathrm{d}x = \mathscr{F}(u,\Omega)\,.$$

Note that since the sequence $f(\nabla u_j)\mathscr{L}^n$ is bounded in $\mathscr{M}(\bar\Omega)$, we that have for some subsequence (for convenience not relabelled) there exists a measure μ in $\bar\Omega$ such that

$$f(\nabla u_j) \stackrel{*}{\rightharpoonup} \mu \text{ in } \mathscr{M}(\bar\Omega)\,.$$

Clearly, since f is non-negative, μ must also be a non-negative measure on $\bar\Omega$. In particular, we have

$$\mu(\bar\Omega) = \mathscr{F}(u,\Omega) \tag{5.17}$$

and for every open set $V \subset \Omega$

$$\mathscr{F}(u,V) \leq \liminf_{k\to\infty} \int_V f(\nabla u_k)\,\mathrm{d}x \leq \mu(\bar{V})\,. \tag{5.18}$$

Now let $V \subset \Omega$ be an open set and fix $\epsilon > 0$. Take an open set $Z \subset\subset V$ such that

$$\mu(V) - \mu(Z) < \epsilon\,.$$

Now use Lemma 5.7, (5.17) and (5.18) to get

$$\begin{aligned} \mu(V) \leq \mu(Z) + \epsilon &= \mu(\bar{\Omega}) - \mu(\bar{\Omega}\setminus Z) + \epsilon \\ &\leq \mathscr{F}(u,\Omega) - \mathscr{F}(u,\bar{\Omega}\setminus Z) + \epsilon \\ &\leq \mathscr{F}(u,V) + \epsilon\,. \end{aligned}$$

Let $\epsilon \to 0$ to obtain

$$\mu(V) \leq \mathscr{F}(u,V).$$

Now we show how the coercivity assumption (5.16) may be removed. Define $f^\epsilon \colon \mathbb{R}^{N\times n} \to \mathbb{R}$ as

$$f^\epsilon(\xi) := f(\xi) + \epsilon|\xi|\,,$$

for all $\xi \in \mathbb{R}^{N\times n}$, for some $\epsilon > 0$. Define $\mathscr{F}^\epsilon$ to be the corresponding Lebesgue-Serrin extension of f^ϵ as in (5.1). By the above part of the proof, we obtain a measure μ^ϵ weakly representing $\mathscr{F}^\epsilon$. Letting $(u_k) \subset W^{1,r}(\Omega;\mathbb{R}^N)$ be a minimising sequence for $\mathscr{F}^\epsilon(u,\Omega)$, we have

$$\mu^\epsilon(\bar{\Omega}) = \mathscr{F}^\epsilon(u,\Omega) \leq \mathscr{F}(u,\Omega) + \epsilon \sup_k \|u_k\|_{W^{1,1}} \leq C\,. \tag{5.19}$$

Moreover, if $U \subset \Omega$ is open, then clearly

$$\mathscr{F}(u,U) \leq \liminf_{k\to\infty} \int_U f(\nabla u_k)\,\mathrm{d}x \leq \liminf_{k\to\infty} \int_U f^\epsilon(\nabla u_k)\,\mathrm{d}x \leq \mu^\epsilon(\bar{U})\,. \tag{5.20}$$

Hence, by (5.19), we may select $\epsilon_j \to 0$ such that the sequence μ^{ϵ_j} converges weakly* in the sense of measures to a finite, non-negative, Radon measure μ. Then, by (5.20),

$$\mathscr{F}(u,U) \leq \mu^{\epsilon_j}(\bar{U})\,,$$

and passing to the weak* limit,

$$\mathscr{F}(u,U) \leq \mu(\bar{U})\,.$$

Conversely, let $\epsilon' > 0$, and take a sequence $(v_k) \subset W^{1,r}(U;\mathbb{R}^N)$ satisfying $v_k \overset{*}{\rightharpoonup} u$ weakly* in $\mathrm{BV}(U;\mathbb{R}^N)$ and

$$\int_U f(\nabla v_k)\,\mathrm{d}x \le \mathscr{F}(u,U) + \epsilon'$$

for all k. Then, for j large enough, we have

$$\int_U f^{\epsilon_j}(\nabla v_k)\,\mathrm{d}x = \int_U \big(f(\nabla v_k) + \epsilon_j|v_k| + \epsilon_j|\nabla v_k|\big)\,\mathrm{d}x \le \mathscr{F}(u,U) + 2\epsilon'\,,$$

and so

$$\mu^{\epsilon_j}(U) \le \mathscr{F}^{\epsilon_j}(u,U) \le \liminf_{k\to\infty} \int_U f^{\epsilon_j}(\nabla v_k)\,\mathrm{d}x \le \mathscr{F}(u,U) + 2\epsilon'\,.$$

Now we pass to the weak* limit and let $\epsilon' \to 0$ to conclude the proof. □

Now we show that we also have strong measure representation for $\mathscr{F}$ if certain technical conditions are satisfied. First we establish the following lemma, which will also play a part in our proof of Theorem 5.5.

Lemma 5.8. *Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous function satisfying the growth condition* (5.3) *for some exponent $1 \le r < \frac{n}{n-1}$. Let $u \in BV(\Omega;\mathbb{R}^N)$ and $\mathscr{F}$ be as defined in* (5.1)*. Let U be an open subset of Ω. If μ is a Radon measure on $\bar{\Omega}$ weakly representing $\mathscr{F}(u,\cdot)$ and*

$$\inf_K\{\mathscr{F}(u,U\setminus K) : K \subset U \text{ is compact }\} = 0\,, \tag{5.21}$$

then

$$\mu(U) = \mathscr{F}(u,U)\,.$$

An identical statement holds for $\mathscr{F}_{loc}$ as defined in (5.2)*.*

Proof of Lemma 5.8. We need to show $\mathscr{F}(u,U) \le \mu(U)$. Let $\epsilon > 0$ and, using (5.21), let $K \subset U$ be a compact set such that

$$\mathscr{F}(u,U\setminus K) < \epsilon\,.$$

Now take an open set W such that $K \subset W \subset\subset U$ and apply Lemma 5.7 to get

$$\begin{aligned}\mathscr{F}(u,U) &\le \mathscr{F}(u,W) + \mathscr{F}(u,U\setminus K)\\ &\le \mathscr{F}(u,W) + \epsilon\\ &\le \mu(\bar{W}) + \epsilon\\ &\le \mu(U) + \epsilon\,.\end{aligned}$$

Take $\epsilon \to 0$ to complete the proof. The proof for $\mathscr{F}_{\mathrm{loc}}$ is the same. □

This allows us to deduce

Corollary 5.9. *Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous function satisfying the growth condition* (5.3) *for some exponent $1 \le r < \frac{n}{n-1}$. Let $u \in BV(\Omega;\mathbb{R}^N)$ and $\mathscr{F}$ be as defined in* (5.1). *If μ is a finite Radon measure on $\bar{\Omega}$ weakly representing $\mathscr{F}(u,\cdot)$, then μ represents $\mathscr{F}(u,\cdot)$ if and only if there exists a Radon measure ν such that*

$$\mathscr{F}(u,U) \le \nu(U) \tag{5.22}$$

for all open subsets $U \subset \Omega$.

Proof of Corollary 5.9. If (5.22) is satisfied, then clearly (5.21) holds for any open set $U \subset \Omega$ so, by Lemma 5.8, μ represents $\mathscr{F}(u,\cdot)$. The converse implication is trivial, taking $\nu = \mu$. $\square$

We are now in a position to prove the remaining main theorem of this chapter.

Proof of Theorem 5.5. Again, assume first that the coercivity condition (5.16) is satisfied. Using the proof of Theorem 5.6, there exists a Radon measure λ on $\bar{\Omega}$ such that for every open set $U \subset \Omega$,

$$\lambda(U) \le \mathscr{F}_{\mathrm{loc}}(u,U) \le \lambda(\bar{U})\,.$$

For a given open set $U \subset \Omega$, we shall show additionally that

$$\lambda(U) \ge \mathscr{F}_{\mathrm{loc}}(u,U).$$

Take an increasing sequence of open, bounded, smooth sets $U_j \subset\subset U$, $j \in \mathbb{N}$, such that $\bar{U}_j \subset U_{j+1}$ for all j and $U = \bigcup_{j=1}^{\infty} U_j$. By the definition of $\mathscr{F}_{\mathrm{loc}}$, for each $j \ge 3$ there exists a sequence $(u_{j,k}) \subset W^{1,r}_{\mathrm{loc}}(U_j \setminus \bar{U}_{j-2};\mathbb{R}^N)$ such that

$$u_{j,k} \overset{*}{\rightharpoonup} u \text{ weakly* in } \mathrm{BV}(U_j \setminus \bar{U}_{j-2};\mathbb{R}^N) \text{ as } k \to \infty\,,$$

and

$$\int_{U_j\setminus\bar{U}_{j-2}} f(\nabla u_{j,k})\,\mathrm{d}x \le \mathscr{F}_{\mathrm{loc}}(u, U_j \setminus \bar{U}_{j-2}) + 2^{-j}\,. \tag{5.23}$$

Fix positive integers α_j, which will be determined later, and note that by taking a subsequence (for convenience not relabelled) we may assume $u_{j,k} \to u$ almost everywhere in $U_j \setminus \bar{U}_{j-2}$ as $k \to \infty$, and

$$\|u_{j,k} - u\|_{L^1(U_j\setminus\bar{U}_{j-2})} \le 2^{-j-k}\alpha_j^{-1}\,.$$

Now use Lemma 5.4 to connect $u_{j,k}$ to $u_{j+1,k}$ across $U_j \setminus \bar{U}_{j-1}$. There exist open sets $V^+_{j,k}$, $V^-_{j+1,k}$ such that

$$\begin{cases} V^+_{j,k} \subset U_j \setminus \bar{U}_{j-2} \\ V^-_{j+1,k} \subset U_{j+1} \setminus \bar{U}_{j-1} \\ U_{j+1} \setminus \bar{U}_{j-2} = V^+_{j,k} \cup V^-_{j+1,k} \\ \mathscr{L}^n(V^+_{j,k} \cap V^-_{j+1,k}) \leq C_j 2^{-j-k} \alpha_j^{-1}\,, \end{cases}$$

and there exist functions $(z_{j,k}) \subset W^{1,r}(U_{j+1} \setminus \bar{U}_{j-2}; \mathbb{R}^N)$ such that

$$z_{j,k} = \begin{cases} u_{j,k} & \text{on } (U_j \setminus \bar{U}_{j-2}) \setminus V^-_{j+1,k}\,, \\ u_{j+1,k} & \text{on } (U_{j+1} \setminus \bar{U}_{j-1}) \setminus V^+_{j,k}\,, \end{cases}$$

and

$$\begin{aligned} \int_{V^+_{j,k} \cap V^-_{j+1,k}} f(\nabla z_{j,k})\,\mathrm{d}x &\leq L \int_{V^+_{j,k} \cap V^-_{j+1,k}} \big(1 + |\nabla z_{j,k}|^r\big)\,\mathrm{d}x \\ &\leq LC_j 2^{-j-k}\alpha_j^{-1} + C_j (2^{j+k}\alpha_j)^{r(n-1)-n} \Big(\|u_{j,k}\|_{W^{1,1}(U_j \setminus \bar{U}_{j-1})} \\ &\qquad + \|u_{j+1,k}\|_{W^{1,1}(U_j \setminus \bar{U}_{j-1})} + 2^{j+k}\alpha_j \|u_{j+1,k} - u_{j,k}\|_{L^1(U_j \setminus \bar{U}_{j-1})} \Big)^r \\ &\leq C_j (2^{j+k}\alpha_j)^{r(n-1)-n}\,, \end{aligned}$$

where C_j is a constant depending on j. Hence we may specify our choice of α_j so that $\alpha_j^{r(n-1)-n} C_j \leq 1$. Now define $(z_k) \subset W^{1,r}_{\mathrm{loc}}(\Omega \setminus U_1; \mathbb{R}^N)$ by

$$z_k := \begin{cases} z_{j,k} & \text{on } V^+_{j,k} \cap V^-_{j+1,k}\,, \\ u_{j+1,k} & \text{on } (U_{j+1} \setminus U_{j-1}) \setminus (V^+_{j,k} \cup V^-_{j+2,k})\,. \end{cases}$$

Now fix $m \in \mathbb{N}$, $m \geq 2$. We have

$$\begin{aligned} \int_{U \setminus \bar{U}_m} f(\nabla z_k)\,\mathrm{d}x &\leq \sum_{j=m+1}^{\infty} \int_{U_j \setminus \bar{U}_{j-1}} f(\nabla z_k)\,\mathrm{d}x \\ &\leq \sum_{j=m+1}^{\infty} \bigg(\int_{U_{j+1} \setminus \bar{U}_{j-1}} f(\nabla u_{j+1,k})\,\mathrm{d}x \\ &\qquad + \int_{U_j \setminus \bar{U}_{j-1}} f(\nabla u_{j,k})\,\mathrm{d}x + \int_{V^+_{j,k} \cap V^-_{j+1,k}} f(\nabla z_{j,k})\,\mathrm{d}x \bigg) \\ &\leq \sum_{j=m+1}^{\infty} \big(2\mathscr{F}_{\mathrm{loc}}(u, U_{j+1} \setminus \bar{U}_{j-1}) + 2^{-j+1} + 2^{(j+k)(r(n-1)-n)} \big) \\ &\leq \sum_{j=m+1}^{\infty} \big(2\lambda(U_{j+2} \setminus U_{j-1}) + 2^{-j+1} + 2^{(j+k)(r(n-1)-n)} \big) \end{aligned}$$

$$\begin{aligned}
&\leq 6\lambda(U \setminus U_{m-1}) + 2^{-m+1} + 2^{k(r(n-1)-n)} \sum_{j=m+1}^{\infty} (2^{n-r(n-1)})^{-j} \\
&\leq 6\lambda(U \setminus U_{m-1}) + 2^{-m+1} + 2^{k(r(n-1)-n)} \cdot o(m)\,.
\end{aligned}$$

By the coercivity condition (5.16) and the above, we have

$$\begin{aligned}
\int_{U\setminus\bar{U}_m} |\nabla z_k| &\leq C \int_{\Omega\setminus\bar{U}_m} f(\nabla z_k)\,\mathrm{d}x \\
&\leq C6\lambda(U \setminus U_{m-1}) + C \\
&\leq C6\mathscr{F}_{\mathrm{loc}}(u, U \setminus U_{m-1}) + C
\end{aligned}$$

for all k so, since $\mathscr{F}_{\mathrm{loc}}(u,\Omega) < \infty$, the sequence (z_k) is bounded in $W^{1,1}(U\setminus\bar{U}_m;\mathbb{R}^N)$. Now note that

$$\begin{aligned}
\mathscr{L}^n\Big(\bigcup_{j=1}^{\infty}(V_{j,k}^{+} \cap V_{j+1,k}^{-})\Big) &\leq \sum_{j=1}^{\infty} \mathscr{L}^n(V_{j,k}^{+} \cap V_{j+1,k}^{-}) \\
&\leq \sum_{j=1}^{\infty} C_j 2^{-j-k}\alpha_j^{-1} \\
&\leq \sum_{j=1}^{\infty} 2^{-j-k} \\
&\to 0 \quad \text{as } k \to \infty\,.
\end{aligned}$$

Hence, arguing using Rellich-Kondrachoff as in Lemma 5.7, we have that $z_k \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(U \setminus \bar{U}_m;\mathbb{R}^N)$. Therefore

$$\mathscr{F}_{\mathrm{loc}}(u, U \setminus \bar{U}_m) \leq 6\lambda(U \setminus U_{m-1}) + 2^{-m+1}\,,$$

and so

$$\begin{aligned}
\inf_K\{\mathscr{F}(u, U \setminus K) : K \subset U \text{ is compact }\} &\leq \lim_{m\to\infty} \mathscr{F}_{\mathrm{loc}}(u, U \setminus \bar{U}_m) \\
&\leq \lim_{m\to\infty} \big(6\lambda(U \setminus U_{m-1}) + 2^{-m+1}\big) \\
&= 0\,.
\end{aligned}$$

Thus condition (5.21) of Lemma 5.8 is satisfied, allowing us to conclude that indeed

$$\lambda(U) = \mathscr{F}_{\mathrm{loc}}(u, U)\,.$$

We remove the coercivity assumption (5.16) using the same argument as in the proof of Theorem 5.6. □

Chapter 6

Lower semicontinuity in BV of integrals with superlinear growth

In this chapter we present a proof of the final main new result of this thesis. As in Chapter 5, we are considering the variational integral

$$F(u;\Omega) := \int_\Omega f(\nabla u(x))\,\mathrm{d}x\,, \tag{6.1}$$

where Ω is a bounded, open subset of $\mathbb{R}^n$, $n \geq 2$, and f is an integrand satisfying the following growth condition:

$$0 \leq f(\xi) \leq L(|\xi|^r + 1) \tag{6.2}$$

for a fixed finite $L > 0$ and all $\xi \in \mathbb{R}^{N\times n}$, where $r \in [1, \frac{n}{n-1})$. As in Chapter 4, we also now assume in addition that f is quasiconvex, and as in Chapter 5, we define the Lebesgue-Serrin extension

$$\mathscr{F}_{\mathrm{loc}}(u,\Omega) := \inf_{(u_j)}\left\{ \liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \;\middle|\; \begin{array}{l} (u_j) \subset W^{1,r}_{\mathrm{loc}}(\Omega,\mathbb{R}^N) \\ u_j \overset{*}{\rightharpoonup} u \text{ in } \mathrm{BV}(\Omega,\mathbb{R}^N) \end{array} \right\}. \tag{6.3}$$

Now define the *recession function* f_∞ of f as

$$f_\infty(\xi) := \limsup_{t\to\infty} \frac{f(t\xi)}{t}\,. \tag{6.4}$$

The properties of $\mathscr{F}_{\mathrm{loc}}$ in the case $r = 1$ have been studied extensively by Ambrosio and Dal Maso in [12] (see Introduction). Most notably they prove that for every open set $\Omega \subset \mathbb{R}^n$ and every $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$ we have

$$\mathscr{F}_{\mathrm{loc}}(u,\Omega) = \int_\Omega f(\nabla u(x))\,\mathrm{d}x + \int_\Omega f_\infty\left(\frac{D^s u}{|D^s u|}\right)|D^s u|\,,$$

where, as described in Chapter 2, ∇u is the density of the absolutely continuous part of the measure Du with respect to Lebesgue measure, $D^s u$ is the singular part of Du, and $\frac{D^s u}{|D^s u|}$ is the Radon-Nikoým derivative of the measure $D^s u$ with respect to its variation $|D^s u|$.

In this chapter we obtain a lower bound for $\mathscr{F}_{\mathrm{loc}}$ under superlinear growth conditions, i.e. when $r \in [1, \frac{n}{n-1})$, provided we assume additionally that f_∞ is finite in certain rank-one directions. That is, for a given $u \in \mathrm{BV}(\Omega; \mathbb{R}^N)$,

$$f_\infty(u(y) \otimes \nu) < \infty \quad \text{for } \mathscr{L}^n\text{-a.a. } y \in \Omega \text{ and all } \nu \in \mathbb{R}^n . \tag{6.5}$$

This is a natural assumption, since otherwise $f_\infty(D^s u/|D^s u|)$ may just be infinity for general BV functions. Henceforth, taking a suitable precise representative if necessary, we shall assume without loss of generality that (6.5) holds for all $y \in \Omega$. Note that since f is quasiconvex, f_∞ is rank-one convex (see, for example, [79]), meaning that it is finite also on rank-one matrices of the form $\xi = \eta \otimes \nu$ whenever $\nu \in \mathbb{R}^n$ and $\eta \in \mathrm{span}\{u(y) : y \in \Omega\}$. Observe that the definition of the recession function immediately implies that f has linear growth in any direction where f_∞ is finite. Moreover, since $f_\infty(0) = 0$, we have the linear growth condition

$$f_\infty(\xi) \leq C|\xi| \tag{6.6}$$

for a fixed finite $C > 0$, for all $\xi \in \mathbb{R}^{N\times n}$ such that $\xi = \eta \otimes \nu$, $\eta \in \mathrm{span}\{u(y) : y \in \Omega\}$, $\nu \in \mathbb{R}^n$.

It is also important to note that we are most interested in the case where

$$\mathrm{span}\{u(y) : y \in \Omega\} \neq \mathbb{R}^N ,$$

as the following proposition indicates that it is likely that whenever f has subquadratic growth conditions, then finiteness of f_∞ on the full rank-one cone in fact implies f has at most linear growth in all directions. Although our result is limited to the case $n = N = 2$, we believe that it ought to be possible to generalise this result for higher dimensions, and are currently working on this. Recall that if f is quasiconvex and satisfies the growth condition (6.2) for some exponent r, then it is $W^{1,r}$-quasiconvex.

Proposition 6.1. *Let $n = N = 2$, and B denote the open unit ball in $\mathbb{R}^2$. Let $1 < r < 2$ and $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ be a $W^{1,r}$-quasiconvex function. Suppose f has linear growth on matrices of rank at most one, i.e.*

$$0 \leq f(\xi) \leq L(|\xi| + 1) \tag{6.7}$$

for a fixed finite $L > 0$ and all $\xi \in \mathbb{R}^{2\times 2}$ satisfying rank$(\xi) \leq 1$. Then f has linear growth in all directions, i.e. (6.7) *holds for all $\xi \in \mathbb{R}^{2\times 2}$ (for perhaps a larger constant).*

Proof of Proposition 6.1. Let $\xi \in \mathbb{R}^{2\times 2}$. Now define the map $u_\xi \colon B \to \mathbb{R}^2$ as

$$u_\xi(x) = \frac{\xi x}{|x|}\,.$$

Note that u_ξ maps $B \setminus \{0\}$ to the surface $\xi(\partial B)$, so $\det(\nabla u_\xi(x)) = 0$ for all $x \in B \setminus \{0\}$. Hence rank$(\det(\nabla u_\xi)) \leq 1$ on $B \setminus \{0\}$. Indeed,

$$\begin{aligned}\nabla u_\xi(x_1, x_2) &= \frac{\xi}{(x_1^2 + x_2^2)^{\frac{3}{2}}} \cdot \begin{pmatrix} x_2^2 & -x_1 x_2 \\ -x_1 x_2 & x_1^2 \end{pmatrix} \\ &= \frac{\xi}{(x_1^2 + x_2^2)^{\frac{3}{2}}} \cdot \big((x_2, -x_1)^t \otimes (x_2, -x_1)^t\big)\,.\end{aligned}$$

Hence, by assumption,

$$f(\nabla u_\xi(x)) \leq L(|\nabla u_\xi(x)| + 1) \tag{6.8}$$

for all $x \in B \setminus \{0\}$. It is well-known that $u_\xi \in W^{1,q}(B;\mathbb{R}^2)$ for all $1 \leq q < n = 2$ when ξ is the identity (see for example [20]), and consequently clearly also for any other $\xi \in \mathbb{R}^{2\times 2}$. Moreover, $u_\xi(x) = \xi x$ on ∂B. Therefore, since f is $W^{1,r}$-quasiconvex and $1 < r < 2$, we have

$$\int_B f(\nabla u_\xi)\,\mathrm{d}x \geq \mathscr{L}^2(B) f(\xi)\,. \tag{6.9}$$

Thus, using (6.8) and (6.9), we get

$$\begin{aligned} f(\xi) &\leq (L/|B|) \int_B 1 + |\nabla u_\xi|\,\mathrm{d}x \\ &\leq (L/|B|)\Big(1 + \int_B \big|\xi\big(\nabla(x/|x|)\big)\big|\,\mathrm{d}x\Big) \\ &\leq (L/|B|)\Big(1 + |\xi| \int_B \big|\nabla(x/|x|)\big|\,\mathrm{d}x\Big)\,. \end{aligned}$$

Since $x \mapsto x/|x|$ is in $W^{1,1}(B;\mathbb{R}^2)$, the required result follows with L replaced by $(L/|B|)\int_B \big|\nabla(x/|x|)\big|\,\mathrm{d}x = 2L$. $\square$

The statement of the main theorem is as follows.

Theorem 6.2. *Let Ω be a bounded, open set in $\mathbb{R}^n$ and $u \in BV(\Omega;\mathbb{R}^N)$. Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a quasiconvex function satisfying the growth condition* (6.2) *for $r \in [1, \frac{n}{n-1})$. Let the recession function f_∞ be as defined in* (6.4)*, and suppose it is finite on rank-one matrices of the form $u(y) \otimes \nu$, $y \in \Omega$, $\nu \in \mathbb{R}^n$.*

Suppose (u_j) is a sequence in $W^{1,r}_{loc}(\Omega;\mathbb{R}^N)$ such that

$$u_j \overset{*}{\rightharpoonup} u \ \text{ in } BV(\Omega;\mathbb{R}^N)\,. \tag{6.10}$$

Then

$$\liminf_{j\to\infty} F(u_j;\Omega) \geq \int_\Omega f(\nabla u(x))\,\mathrm{d}x + \int_\Omega f_\infty\left(\frac{D^s u}{|D^s u|}\right)|D^s u|\,, \tag{6.11}$$

and hence the Lebesgue-Serrin extension $\mathscr{F}_{loc}$ as defined in (6.3) *satisfies*

$$\mathscr{F}_{loc}(u,\Omega) \geq \int_\Omega f(\nabla u(x))\,\mathrm{d}x + \int_\Omega f_\infty\left(\frac{D^s u}{|D^s u|}\right)|D^s u|\,. \tag{6.12}$$

The structure of the rest of this chapter is as follows. First we show that the proof of this result involves establishing two inequalities: one on the absolutely continuous part of the measure Du, and one on the singlular part. The first inequality is essentially a direct application of a result by Kristensen in [64]. To prove the inequality on the singular part of Du, we first need to prove further bounds on $\mathscr{F}_{\mathrm{loc}}$. One such lower bound is a straightforward adaptation of that of Ambrosio and Dal Maso [12]. We also obtain an upper bound, with a new technique involving mollification, for functions $u \in \mathrm{SBV}(\Omega;\mathbb{R}^N)$ that are constant almost everywhere, whose jump set is the union of finitely many polyhedra; we then adapt a method of Braides and Coscia [22] to extend this result to general BV functions. Equipped with these additional upper and lower bounds on $\mathscr{F}_{\mathrm{loc}}$, we then establish the remaining inequality by using a non-standard blow-up technique. Throughout the latter half of this chapter, we shall make use of Theorem 5.5 from Chapter 5, which tells us that if $\mathscr{F}_{\mathrm{loc}}(u,\cdot)$ is finite, it is representable by some non-negative, finite Radon measure λ on Ω.

6.1 Preliminaries

Let f be as stated in the assumptions of Theorem 6.2, and likewise let (u_j) be a sequence in $W^{1,r}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$, $u \in \mathrm{BV}(\Omega;\mathbb{R}^N)$, and $u_j \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^N)$. We may assume that

$$\liminf_{j\to\infty} F(u_j;\Omega)$$

is finite, as otherwise there is nothing to prove. Moreover, by taking a subsequence (for convenience not relabelled), we can also assume

$$\lim_{j\to\infty} F(u_j;\Omega) = \liminf_{j\to\infty} F(u_j;\Omega)\,.$$

Thus the sequence $f(Du_j)\mathscr{L}^n$ is bounded in $\mathscr{M}(\bar{\Omega})$, so we that have for some further subsequence (again not relabelled) there exists a measure μ in $\bar{\Omega}$ such that

$$f(Du_j) \overset{*}{\rightharpoonup} \mu \text{ in } \mathscr{M}(\bar{\Omega})\,.$$

Clearly, since f is non-negative, μ must also be a non-negative measure on $\bar{\Omega}$. Now observe that by applying the Radon-Nikodým Theorem twice, first with μ and Lebesgue measure, and then again on the singular part of μ and $|D^s u|$, we may decompose μ as

$$\mu = \frac{d\mu}{d\mathscr{L}^n}\mathscr{L}^n + \frac{d\mu}{|D^s u|}|D^s u| + \mu^*\,,$$

where μ^* is non-negative and singular with respect to both Lebesgue measure and $|D^s u|$. Hence

$$\liminf_{j\to\infty} \int_\Omega f(Du_j)\,\mathrm{d}x \geq \mu(\Omega) = \int_\Omega \frac{d\mu}{d\mathscr{L}^n}\,\mathrm{d}x + \int_\Omega \frac{d\mu}{d|D^s u|}\,|D^s u| + \mu^*(\Omega)\,.$$

Therefore the required lower bound will follow if we can show that

$$\frac{d\mu}{d\mathscr{L}^n}(x) \geq f(\nabla(x)) \text{ for } \mathscr{L}^n\text{-a.a. } x \in \Omega, \tag{6.13}$$

and

$$\frac{d\mu}{d|D^s u|}(x) \geq f_\infty\left(\frac{D^s u}{|D^s u|}(x)\right) \text{ for } |D^s u|\text{-a.a. } x \in \Omega. \tag{6.14}$$

These two inequalities are the subject of the main propositions of this chapter. First, however, we state a lemma, attributable to Kristensen [64], that is particularly important for establishing (6.13), which in turn plays a role in aspects of the proof of (6.14). In the statement of this lemma and subsequently we denote by B_ϱ the open ball in $\mathbb{R}^n$ centred on the origin with radius ϱ, and $B = B_1$.

Lemma 6.3. [64] *Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a quasiconvex function satisfying the growth condition* (6.2) *for some exponent $r \in [1, \frac{n}{n-1})$.*

Let (u_j) be a sequence in $W^{1,r}(B;\mathbb{R}^N)$ and suppose

$$u_j \to 0 \textit{ in } L^1(B;\mathbb{R}^N) \tag{6.15}$$

and

$$\sup_j \int_B |\nabla u_j| \, \mathrm{d}x < +\infty \,. \tag{6.16}$$

Then we have the following inequality:

$$\liminf_{j\to\infty} \int_B f(\nabla u_j) \, \mathrm{d}x \geq \mathscr{L}^n(B) \cdot f(0) \,. \tag{6.17}$$

As we shall see, the proof of Lemma 6.3 is very similar to the proof of Lemma 4.2, but using the following lemma instead of Lemma 4.3.

Lemma 6.4. [64] *Let $1 \leq r < \frac{n}{n-1}$. Then there exists a linear extension operator*

$$\mathbf{E}\colon (W^{1,1})(\partial B; \mathbb{R}^N) \to W^{1,r}(B_2 \setminus \bar{B}; \mathbb{R}^N)$$

with the following properties:

1. *If $g \in C^1(\partial B; \mathbb{R}^N)$ then $\mathbf{E}(g) \in C^\infty(B_2 \setminus \bar{B})$ with $\mathbf{E}(g)|_{\partial B} = g$.*
2. *If $(z_j) \subset C^\infty(\partial B; \mathbb{R}^N)$ and $\lim_{j\to\infty} \int_{\partial B} z_j \cdot \phi \, \mathrm{d}\mathscr{H}^{n-1} = 0$ for all $\phi \in C^\infty(\partial B; \mathbb{R}^N)$, then for any multi-index α, $\partial^\alpha [\mathbf{E} z_j] \to 0$ locally uniformly in $B_2 \setminus \bar{B}$.*
3. *There exist positive constants c_1, c_2, dependent on n, N, r, such that:*
   (a)
   $$\int_{B_2 \setminus B} |\mathbf{E}(g)|^r \leq c_1 \|g\|^r_{L^1(\partial B)}$$
   (b)
   $$\int_{B_2 \setminus B} |\nabla [\mathbf{E} g]|^r \mathscr{L}^n \leq \left(c_2 \int_{\partial B} |\nabla g| \, \mathrm{d}\mathscr{H}^{n-1} \right)^r$$
   for all $g \in C^1(\partial B)$.

For a proof of Lemma 6.4, refer to the proof of Lemma 3.2 in Chapter 3, where by a localisation argument it suffices to consider extending functions defined on $\mathbb{R}^{n-1}$ into the half space $\mathbb{R}^n_+$ consisting of points in $\mathbb{R}^n$ whose n^{th} coordinate is non-negative.

Proof of Lemma 6.3. By approximation we may assume $(u_j) \subset C^1(\bar{B}; \mathbb{R}^N)$. If the left hand side of (6.17) is infinite then there is nothing to prove, so suppose it is finite. Moreover, by extracting a subsequence if necessary, we can assume

$$l_0 := \liminf_{j\to\infty} \int_B f(\nabla u_j) \, \mathrm{d}x = \lim_{j\to\infty} \int_B f(\nabla u_j) \, \mathrm{d}x \,.$$

From (6.15), by the Fubini-Tonelli theorem and the Rellich-Kondrachoff compactness theorem we have

$$\lim_{j\to\infty}\int_0^1\int_{\partial B_\varrho}|u_j|\,\mathrm{d}\mathscr{H}^{n-1}\,\mathrm{d}\varrho=\lim_{j\to\infty}\int_B|u_j|\,\mathrm{d}x=0\,.$$

This implies there exists a subsequence $\{u_j\}_{j\in T}$ such that

$$\lim_{j\to\infty\,,\,j\in T}\int_{\partial B_\varrho}|u_j|\,\mathrm{d}\mathscr{H}^{n-1}=0 \tag{6.18}$$

for almost all $\varrho\in(0,1)$. By Fatou's Lemma and (6.16) we have

$$\int_0^1\liminf_{j\to\infty\,,\,j\in T}\int_{\partial B_\varrho}|\nabla u_j|\,\mathrm{d}\mathscr{H}^{n-1}\,\mathrm{d}\varrho\le\liminf_{j\to\infty\,,\,j\in T}\int_B|\nabla u_j|\,\mathrm{d}x<\infty\,.$$

Thus, for almost all $\varrho\in(0,1)$

$$\liminf_{j\to\infty\,,\,j\in T}\int_{\partial B_\varrho}|\nabla u_j|\,\mathrm{d}\mathscr{H}^{n-1}<\infty\,. \tag{6.19}$$

Now fix $0<\delta<1$. By (6.18) and (6.19) we can choose $\varrho\in(\delta,1)$ such that

$$\lim_{j\to\infty\,,\,j\in T}\int_{\partial B_\varrho}|u_j|\,\mathrm{d}\mathscr{H}^{n-1}=0$$

and

$$\liminf_{j\to\infty\,,\,j\in T}\int_{\partial B_\varrho}|\nabla u_j|\,\mathrm{d}\mathscr{H}^{n-1}<\infty\,.$$

Now take a further subsequence $\{u_j\}_{j\in S}$, where $S\subseteq T$, so that

$$\lim_{j\to\infty\,,\,j\in S}\int_{\partial B_\varrho}|\nabla u_j|\,\mathrm{d}\mathscr{H}^{n-1}=\liminf_{j\to\infty\,,\,j\in T}\int_{\partial B_\varrho}|\nabla u_j|\,\mathrm{d}\mathscr{H}^{n-1}\,.$$

Relabel the sequence (u_j) so that $S=\mathbb{N}$. Now define the sequence $(g_j)\subset W^{1,1}(\partial B;\mathbb{R}^N)$ as:

$$g_j(x):=u_j|_{\partial B_\varrho}(\varrho x)\quad\text{for } x\in\partial B\,.$$

Take a cut-off function $\eta\in C^1(B;\mathbb{R})$ such that $\mathbf{1}_{B_\varrho}\le\eta\le\mathbf{1}_B$, $|\nabla\eta|\le\frac{2}{1-\varrho}$, and define $(v_j)\subset W_0^{1,r}(B;\mathbb{R}^N)$ as:

$$v_j(x):=\begin{cases}\eta(x)\cdot(\mathbf{E}(g_j))(\frac{x}{\varrho}) & \text{if } |x|\ge\varrho\,,\\ u_j(x) & \text{if } |x|<\varrho\,,\end{cases}$$

where $\mathbf{E}$ is the extension operator from Lemma 6.4.

Since the function $t \mapsto t^r$ is convex, $(s+t)^r \leq 2^{r-1}(s^r+t^r)$ for all $s, t \geq 0$. Hence from Lemma 6.4 we have

$$\begin{aligned}
\int_{B\setminus B_\varrho} |\nabla v_j|^r &\leq \int_{B\setminus B_\varrho} \Big(\big| \nabla\eta \cdot \mathbf{E}g_j(\cdot/\varrho) \big| + \big| \eta \cdot \nabla[\mathbf{E}g_j(\cdot/\varrho)] \big| \Big)^r \\
&\leq 2^{r-1} \int_{B\setminus B_\varrho} |\nabla\eta|^r \cdot \big| \mathbf{E}g_j(\cdot/\varrho) \big|^r + 2^{r-1} \int_{B\setminus B_\varrho} |\eta|^r \cdot \big| \nabla[\mathbf{E}g_j(\cdot/\varrho)] \big|^r \\
&\leq C \int_{B\setminus B_\varrho} \big| \mathbf{E}g_j(\cdot/\varrho) \big|^r + C \int_{B\setminus B_\varrho} \big| \nabla[\mathbf{E}g_j(\cdot/\varrho)] \big|^r \qquad (6.20)
\end{aligned}$$

for some constant C. We estimate the two terms in (6.20) using Lemma 6.4 (3) as follows. Firstly, note that we have

$$\begin{aligned}
\int_{B\setminus B_\varrho} \big| [\mathbf{E}g_j(\cdot/\varrho)] \big|^r &\leq c_1 \|g_j\|^r_{L^1(\partial B)} \\
&= c_1 \|u_j\|^r_{L^1(\partial B_\varrho)} \\
&\to 0 \text{ as } j \to \infty\,.
\end{aligned}$$

Now we use (3)(b) to estimate the remaining term:

$$\begin{aligned}
\int_{B\setminus B_\varrho} \big| \nabla[\mathbf{E}g_j(\cdot/\varrho)] \big|^r &\leq \left(c_2 \int_{\partial B} |\nabla g_j| \,\mathrm{d}\mathscr{H}^{n-1} \right)^r \\
&= \left(c_2 \int_{\partial B_\varrho} |\nabla u_j| \,\mathrm{d}\mathscr{H}^{n-1} \right)^r . \qquad (6.21)
\end{aligned}$$

Now note that we may obtain the same inequality (albeit for a different constant) using Lemma 6.4 for any other r' such that $r < r' < \frac{n}{n-1}$. Hence by (6.21) and Lemma 6.4, since

$$\sup_j \int_{\partial B_\varrho} |\nabla u_j| \,\mathrm{d}\mathscr{H}^{n-1} < \infty\,,$$

we can use the De la Vallée Poussin criterion to deduce that the sequence $|\nabla[\mathbf{E}g_j]|^r$ is equi-integrable on $B \setminus B_\varrho$. By Lemma 6.4, since

$$\sup_j \int_{\partial B_\varrho} |u_j| \,\mathrm{d}\mathscr{H}^{n-1} \to 0 \text{ as } j \to \infty\,,$$

$\nabla[\mathbf{E}g_j] \to 0$ locally uniformly on $B \setminus B_\varrho$, and hence so does $|\nabla[\mathbf{E}g_j]|^r$. Thus, by Vitali's Convergence Theorem,

$$\int_{B\setminus B_\varrho} \big| \nabla[\mathbf{E}g_j(\cdot/\varrho)] \big|^r \to 0 \quad \text{as } j \to \infty\,.$$

Combining these estimates in (6.20), we have

$$\limsup_{j\to\infty} \int_{B\setminus B_\varrho} |\nabla v_j|^r \, \mathrm{d}x = 0\,. \tag{6.22}$$

Now we use the quasiconvexity and non-negativity of f to obtain

$$\begin{aligned}\int_B f(\nabla u_j) \geq \int_{B_\varrho} f(\nabla u_j) = \int_B f(\nabla v_j) - \int_{B\setminus B_\varrho} f(\nabla v_j) \\ \geq \mathscr{L}^n(B) f(0) - \int_{B\setminus B_\varrho} f(\nabla v_j) \\ \geq \mathscr{L}^n(B) f(0) - L \int_{B\setminus B_\varrho} \big(1 + |\nabla v_j|^r\big)\,.\end{aligned}$$

Let $j \to \infty$ to get, using (6.22),

$$l_0 \geq \mathscr{L}^n(B) f(0) - L\mathscr{L}^n(B \setminus B_\varrho)\,.$$

Recall $\varrho \in (\delta, 1)$ for fixed $0 < \delta < 1$. Hence we conclude by taking δ arbitrarily close to 1, which completes the proof of the Lemma. □

6.2 Lower bound on the absolutely continuous part

We now state and prove (6.13), which is essentially just the lower semicontinuity result proved by Kristensen in [64]. Note that it does not require any finiteness properties of f_∞ - in fact f_∞ does not feature at all in this context.

Proposition 6.5. *Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a quasiconvex function satisfying the growth condition* (6.2) *for some exponent $1 \leq r < \frac{n}{n-1}$. Let Ω be a bounded, open subset of $\mathbb{R}^n$.*

Let (u_j) be a sequence in $W^{1,r}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $u \in BV(\Omega;\mathbb{R}^N)$. Suppose

$$u_j \overset{*}{\rightharpoonup} u \text{ in } BV(\Omega;\mathbb{R}^N)\,. \tag{6.23}$$

Let μ be a measure in $\bar{\Omega}$ and suppose

$$f(\nabla u_j) \overset{*}{\rightharpoonup} \mu \text{ in } \mathscr{M}(\bar{\Omega})\,.$$

Then for $\mathscr{L}^n$-almost all $x \in \Omega$, we have

$$\frac{d\mu}{d\mathscr{L}^n}(x) \geq f(\nabla(x))\,.$$

The proof of Proposition 6.5 is just a straightforward blow-up argument using Lemma 6.3.

Proof of Proposition 6.5. Since, by (6.23) and the Uniform Boundedness Principle, $|\nabla u_j|\mathscr{L}^n$ is bounded in $\mathscr{M}(\bar{\Omega})$, we have for some subsequence (for convenience not relabelled) that there exists a measure ν in $\bar{\Omega}$ such that

$$|\nabla u_j| \overset{*}{\rightharpoonup} \nu \quad \text{in } \mathscr{M}(\bar{\Omega})\,.$$

Let Ω_0 denote the set of points $x \in \Omega$ such that

1. $$\frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}(x) = \lim_{\varrho\to 0^+} \frac{\mu(\overline{B(x,\varrho)})}{\mathscr{L}^n(B(x,\varrho))} \quad \text{exists and is finite}$$

2. $$\frac{\mathrm{d}\nu}{\mathrm{d}\mathscr{L}^n}(x) = \lim_{\varrho\to 0^+} \frac{\nu(\overline{B(x,\varrho)})}{\mathscr{L}^n(B(x,\varrho))} \quad \text{exists and is finite}$$

3. $$\lim_{\varrho\to 0^+} \frac{1}{\varrho} \fint_{B(x,\varrho)} |u(y) - u(x) - [\nabla u(x)](x-y)|\,\mathrm{d}y = 0$$

where ∇u is the Radon-Nikodým derivative of Du with respect to Lebesgue measure. By standard results (see for example [84], [98]), $\mathscr{L}^n(\Omega \setminus \Omega_0) = 0$. Fix $x_0 \in \Omega_0$, and note that since the set

$$\{\varrho \in (0, \text{dist}(x, \partial\Omega)) : (\mu+\nu)(\partial B(x,\varrho) > 0\}$$

is at most countable we may find a sequence $r_k \searrow 0$ such that $(\mu+\nu)(\partial B(x, r_k)) = 0$ for all k. Now define

$$v_{j,k}(y) := \frac{u_j(x_0 + r_k y) - u(x_0) - [\nabla u(x_0)](r_k y)}{r_k}\,, \quad y \in B\,. \tag{6.24}$$

Then by the above assumptions we have

$$\lim_{k\to\infty}\lim_{j\to\infty} \int_B |v_{j,k}(y)|\,\mathrm{d}y = 0\,,$$

$$\begin{aligned}\lim_{k\to\infty}\lim_{j\to\infty} \fint_B |\nabla v_{j,k}(y) + \nabla u(x_0)|\,\mathrm{d}y &= \lim_{k\to\infty}\lim_{j\to\infty} \fint_B |\nabla u_j(x_0 + r_k y)|\,\mathrm{d}y \\ &= \lim_{k\to\infty} \frac{1}{|B(x_0, r_k)|} \int_{B(x_0,r_k)} |\nabla u(y)|\,\mathrm{d}y\end{aligned}$$

$$= \frac{\mathrm{d}\nu}{\mathrm{d}\mathscr{L}^n}(x_0)\,,$$

and similarly

$$\lim_{k\to\infty}\lim_{j\to\infty} \fint_B f(\nabla v_{j,k}(y) + \nabla u(x_0))\,\mathrm{d}y = \frac{\mathrm{d}\mu}{\mathrm{d}\mathscr{L}^n}(x_0)\,.$$

Hence for each k we can find $j_k \in \mathbb{N}$ such that the all the convergence above occurs for $v_{j_k,k}$ as k tends to infinity. Thus, if we define $z_k := v_{j_k,k}$, then $(z_k) \subset W^{1,r}(B;\mathbb{R}^N)$ satisfies conditions (6.15) and (6.16) of Lemma 6.3. Applying this lemma (to the function $\bar{f}(\xi) = f(\nabla u(x_0) + \xi)$, say), we obtain

$$\liminf_{k\to\infty} \fint_B f(\nabla z_k + \nabla u(x_0))\,\mathrm{d}y \geq f(\nabla u(x_0))\,,$$

i.e.

$$\frac{d\mu}{d\mathscr{L}^n}(x_0) \geq f(\nabla(x_0))\,,$$

as required. □

Now we remark that the following result follows immediately from this proposition by integrating $\frac{d\mu}{dx}$ with respect to Lebesgue measure over Ω. It gives a first lower bound for the Lebesgue-Serrin extension, which Theorem 6.2 improves upon, provided additional assumptions on f are satisfied.

Corollary 6.6. *Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a quasiconvex function satisfying the growth condition* (6.2) *for some exponent $1 \leq r < \frac{n}{n-1}$. Let Ω be a bounded, open subset of $\mathbb{R}^n$.*

Let (u_j) be a sequence in $W^{1,r}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ and $u \in BV(\Omega;\mathbb{R}^N)$. Suppose

$$u_j \overset{*}{\rightharpoonup} u \text{ in } BV(\Omega;\mathbb{R}^N)\,.$$

Then

$$\liminf_{j\to\infty} \int_\Omega f(\nabla u_j)\,\mathrm{d}x \geq \int_\Omega f(\nabla u)\,\mathrm{d}x\,,$$

and hence, for $\mathscr{F}_{loc}$ as defined in (6.3)*,*

$$\mathscr{F}_{loc}(u,\Omega) \geq \int_\Omega f(\nabla u)\,\mathrm{d}x\,.$$

6.3 Bounds on the Lebesgue-Serrin Extension

In this section we establish some additional properties of $\mathscr{F}_{\text{loc}}$ as defined in (6.3), which are essential for proving the lower bound (6.14). In particular, we find upper and lower bounds of $\mathscr{F}_{\text{loc}}(u,\Omega)$ for BV functions u satisfying specific properties. For basic properties of $\mathscr{F}_{\text{loc}}$, we refer to Chapter 5.

The first principal result here provides us with an upper bound for $\mathscr{F}_{\text{loc}}(u,\Omega)$ for specific types of functions u in SBV, namely those that are constant almost everywhere (and hence have absolutely continuous part zero), whose jump set is the union of finitely many polyhedra.

Lemma 6.7. *Let Ω be a bounded, open subset of $\mathbb{R}^n$ with Lipschitz boundary. Suppose $u \in SBV(\Omega;\mathbb{R}^N)$ is such that*

$$|\nabla u(x)| = 0$$

for $\mathscr{L}^n$-almost all $x \in \Omega$, and that the set J_u of approximate jump points of u is the union of finitely many polyhedra. Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous function satisfying the growth condition (6.2) *for some exponent $1 \leq r < \frac{n}{n-1}$. Let the recession function f_∞ be as defined in* (6.4)*, and suppose it is finite on rank-one matrices of the form $u(y)\otimes\nu$, $y \in \Omega$, $\nu \in \mathbb{R}^n$. Let $\mathscr{F}_{loc}$ be as defined in* (6.3)*. Then*

$$\mathscr{F}_{loc}(u,\Omega) \leq C\big(\mathscr{L}^n(\Omega) + |D^s u|(\Omega)\big) \tag{6.25}$$

for some constant $C > 0$ dependent on f.

Proof of Lemma 6.7. We argue by mollification. Let $(\phi_\epsilon)_{\epsilon>0}$ be family of mollifiers, i.e. $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$, where ϕ is a symmetric convolution kernel in $\mathbb{R}^n$ (so it satisfies $\phi \in C_c^\infty(B(0,1))$, $\phi \geq 0$, $\int \phi = 1$, $\phi(x) = \phi(-x)$, and $\operatorname{supp}(\phi) \subset\subset B(0,1)$). We wish to mollify on all of Ω: since it has a Lipschitz boundary, we can extend u onto all of $\mathbb{R}^n$ so that

$$|Du|(\mathbb{R}^n) \leq C|Du|(\Omega)\,,$$

and u still satisfies $\nabla u = 0$, J_u is the union of finitely many polyhedra, and $\operatorname{span}\{u(y) : y \in \mathbb{R}^n\} = \operatorname{span}\{u(y) : y \in \Omega\}$. Now define, for $\epsilon > 0$, $u_\epsilon(x) := (u * \phi_\epsilon)$, $x \in \Omega$. Recall from Proposition 2.3 that we have

$$\nabla u_\epsilon(x) = (Du * \phi_\epsilon)(x) = \epsilon^{-n}\int_{B(x,\epsilon)} \phi\Big(\frac{y}{\epsilon}\Big)\,\mathrm{d}Du(y)\,.$$

Let $x \in \Omega$ and consider $B(x,\epsilon)$: if $B(x,\epsilon) \cap J_u = \emptyset$, then $Du = \nabla u = 0$ on $B(x,\epsilon)$, and so

$$f(\nabla u_\epsilon(x)) = f(0)\,. \tag{6.26}$$

If $B(x,\epsilon) \cap J_u \neq \emptyset$, and the intersection of this ball and the jump set is just part of the face of a single polyhedron, then, on this ball, we have

$$Du = D^s u = a \otimes \nu \mathscr{H}^{n-1} \lfloor J_u\,,$$

where a is just the difference of (constant) values of u on either side of the face, and ν is a unit normal to this face in the appropriate direction. Hence we have

$$\begin{aligned}|\nabla u_\epsilon(x)| &= \left|\epsilon^{-n} \int_{B(x,\epsilon)\cap J_u} \phi\Big(\frac{y}{\epsilon}\Big)(a \otimes \nu)\,\mathrm{d}\mathscr{H}^{n-1}(y)\right| \\ &\leq \epsilon^{-n} \mathscr{H}^{n-1}(J_u \cap B(x,\epsilon))|a \otimes \nu|\end{aligned}$$

Recall that by the given finiteness condition on f_∞, it follows that f satisfies the linear growth condition

$$0 \leq f(\xi) \leq C(1 + |\xi|)$$

for all matrices ξ of the form $\eta \otimes \nu$, where $\nu \in \mathbb{R}^n$ and $\eta \in \text{span}\{u(y) : y \in \Omega\}$. Since $a \otimes \nu$ is of this form, we get

$$\begin{aligned}f(\nabla u_\epsilon) &\leq C\big(1 + \epsilon^{-n}\mathscr{H}^{n-1}(J_u \cap B(x,\epsilon))|a \otimes \nu|\big) \\ &= C\big(1 + \epsilon^{-n}|D^s u|(J_u \cap B(x,\epsilon))\big)\end{aligned} \tag{6.27}$$

Note that this inequality holds even if $B(x,\epsilon) \cap J_u = \emptyset$. Lastly, suppose $B(x,\epsilon) \cap J_u \neq \emptyset$, and the intersection of this ball and the jump set contains more than just a face - i.e. it contains a corner of a polyhedron and/or multiple (albeit finitely many) polyhedra. Then we have, on this ball,

$$Du = D^s u = \Big(\sum_{i=1}^m \xi_i\Big)\mathscr{H}^{n-1}\lfloor J_u$$

for some $m \in \mathbb{N}$, where ξ_i, similarly to above, are rank-one matrices of the form $a_i \otimes \nu_i$ corresponding to jumps a_i along the unit normal vector ν_i of some face of some polyhedron in this intersection. Note that $\mathscr{H}^{n-1}(B(x,\epsilon) \cap J_u)$ is of order ϵ^{n-1}, so

$$|\nabla u_\epsilon(x)| = \left|\epsilon^{-n} \int_{B(x,\epsilon)\cap J_u} \phi\Big(\frac{y}{\epsilon}\Big)\,\mathrm{d}D^s u(y)\right|$$

$$
\begin{aligned}
&\leq \epsilon^{-n}\Big(\sum_{i=1}^{m}|\xi_i|\Big)\mathscr{H}^{n-1}(B(x,\epsilon)\cap J_u) \\
&\leq C\epsilon^{-1}\sum_{i=1}^{m}|\xi_i|\,.
\end{aligned}
$$

Now use the growth condition (6.2) on f to get

$$
\begin{aligned}
f(\nabla u_\epsilon(x)) &\leq C\bigg(1+\epsilon^{-r}\sum_{i=1}^{m}|\xi_i|^r\bigg) \\
&\leq C(u)(1+\epsilon^{-r})\,, \qquad (6.28)
\end{aligned}
$$

where $C(u)$ is a constant depending on u. Similarly to before, this inequality holds even if $B(x,\epsilon)\cap J_u=\emptyset$, or if this intersection only contains just a face.

Now let $\mathcal{B}$ be a maximal collection of disjoint balls of radius $\epsilon/5$ in Ω. That is, $\mathcal{B}$ is a (finite) disjoint collection of balls, and for any other ball $B'\subset\Omega$ of radius $\epsilon/5$,

$$
B'\cap\bigcup_{B\in\mathcal{B}}B\neq\emptyset\,.
$$

For $B\in\mathcal{B}$, let RB denote the ball with the same centre, but of radius $\frac{R}{5}\epsilon$. Then (see, for example, [73])

$$
\Omega\subset\bigcup_{B\in\mathcal{B}}5B\,.
$$

For each $B\in\mathcal{B}$, we now consider cases as above. If $10B\cap J_u=\emptyset$, then for each $x\in 5B$, $B(x,\epsilon)\cap J_u=\emptyset$. Thus we have, from (6.26),

$$
\int_{5B} f(\nabla u_\epsilon)\,\mathrm{d}x = |5B|f(0)\,. \qquad (6.29)
$$

If $10B\cap J_u\neq\emptyset$, and the intersection of this ball and the jump set is just the part of a face of a single polyhedron, then for each $x\in 5B$, $B(x,\epsilon)$ is contained in $10B$, and so either $B(x,\epsilon)\cap J_u$ is just part of a face or is empty. Hence, using (6.27),

$$
\begin{aligned}
\int_{5B} f(\nabla u_\epsilon)\,\mathrm{d}x &\leq C|5B|\big(1+\epsilon^{-n}|D^s u|(J_u\cap B(x,\epsilon))\big) \\
&\leq C\big((\epsilon^n+|D^s u|(J_u\cap 10B)\big)\,. \qquad (6.30)
\end{aligned}
$$

Finally, if $10B\cap J_u\neq\emptyset$, and the intersection of this ball and the jump set contains more than just a face, then for each $x\in 5B$, $B(x,\epsilon)\cap J_u$ may be empty, or just part of a face, or more than just a face. Thus we use (6.28) to get

$$
\int_{5B} f(\nabla u_\epsilon)\,\mathrm{d}x \leq |5B|C(u)\big(1+\epsilon^{-r}\big)
$$

$$\leq C(u)\epsilon^{n-r}\,. \tag{6.31}$$

Now let $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ be the balls in $\mathcal{B}$ where (6.29), (6.30) and (6.31) hold respectively. Then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ and

$$\sum_{B\in\mathcal{B}_1} \int_{5B} f(\nabla u_\epsilon)\,\mathrm{d}x \leq C\mathscr{L}^n(\Omega)f(0)\,.$$

Note that $\mathcal{B}_2$, since it only contains balls B such that $10B$ contains the polyhedral jump set of u (which has Hausdorff dimension $n-1$), contains less than $C\epsilon^{1-n}$-many balls, where this constant depends on the jump set J_u. Hence

$$\begin{aligned}\sum_{B\in\mathcal{B}_2} \int_{5B} f(\nabla u_\epsilon)\,\mathrm{d}x &\leq C\sum_{B\in\mathcal{B}_2} \epsilon^n + |D^s u|(J_u \cap 10B)\\ &\leq C(u)\epsilon + C|D^s u|(J_u\cap\Omega)\,.\end{aligned}$$

Lastly, we observe that $\mathcal{B}_3$, since it only contains balls B such that $10B$ contains parts of the jump set that are not faces (which has Hausdorff dimension at most $n-2$), has cardinality of order ϵ^{2-n}. Therefore

$$\begin{aligned}\sum_{B\in\mathcal{B}_3} \int_{5B} f(\nabla u_\epsilon)\,\mathrm{d}x &\leq C(u)\sum_{B\in\mathcal{B}_3} \epsilon^{n-r}\\ &\leq C(u)\epsilon^{2-r}\,.\end{aligned}$$

Now take a sequence (ϵ_j) such that $\epsilon_j \searrow 0$. Then $u_{\epsilon_j} \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathbb{R}^N)$, and

$$\begin{aligned}\mathscr{F}_{\mathrm{loc}}(u,\Omega) &\leq \liminf_{j\to\infty} \int_\Omega f(\nabla u_{\epsilon_j})\,\mathrm{d}x\\ &\leq \liminf_{j\to\infty} \sum_{B\in\mathcal{B}} \int_{5B} f(\nabla u_{\epsilon_j})\,\mathrm{d}x\\ &\leq \liminf_{j\to\infty} C\big(\mathscr{L}^n(\Omega) + |D^s u|(J_u\cap\Omega)\big) + C(u)\big(\epsilon_j + \epsilon_j^{2-r}\big)\\ &= C\big(\mathscr{L}^n(\Omega) + |D^s u|(J_u\cap\Omega)\big)\,.\end{aligned}$$

This completes the proof. □

Remark. Localising the proof of this result, we also obtain the upper bound

$$\mathscr{F}_{\mathrm{loc}}(u,U) \leq C(\mathscr{L}^n(U) + |D^s u|(U)) \tag{6.32}$$

for any open subset $U \subset \Omega$.

It is interesting to add that if $u \in \mathrm{SBV}(\Omega;\mathbb{R}^N)$ satisfies the conditions of this lemma, then the result tells us that $\mathscr{F}_{\mathrm{loc}}(u,\Omega) < \infty$. Hence, by Theorem 5.5, $\mathscr{F}_{\mathrm{loc}}(u,\cdot)$ is representable by some non-negative, finite Radon measure λ on Ω. Moreover, we have

$$\lambda \ll \mathscr{L}^n + \mathscr{H}^{n-1}\lfloor J_u \,. \tag{6.33}$$

This allows us to refine the upper bound (6.25), as the following result shows. It makes use of the following corollary of Besicovitch's Covering Theorem, which we state first. For a proof refer to, for example, [46].

Theorem 6.8. *Let μ be a Borel measure on $\mathbb{R}^n$ and $\mathcal{B}$ be any collection of nondegenerate closed balls. Let A denote the centres of the balls in $\mathcal{B}$. Assume $\mu(A) < \infty$ and $\inf\{\varrho : B(a,\varrho) \in \mathcal{B}\} = 0$ for each $a \in A$. Let $U \subset \mathbb{R}^n$ be an open set. Then there exists a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{B}$ such that*

$$\bigcup_{B\in\mathcal{G}} B \subset U$$

and

$$\mu\Bigg((A\cap U) \setminus \bigcup_{B\in\mathcal{G}} B\Bigg) = 0\,.$$

Corollary 6.9. *Let Ω be a bounded, open subset of $\mathbb{R}^n$ with Lipschitz boundary. Suppose $u \in SBV(\Omega;\mathbb{R}^N)$ is such that*

$$|\nabla u(x)| = 0$$

for $\mathscr{L}^n$-almost all $x \in \Omega$, and that the set J_u of approximate jump points of u is the union of finitely many polyhedra. Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a continuous function satisfying the growth condition (6.2) *for some exponent $1 \le r < \frac{n}{n-1}$. Let the recession function f_∞ be as defined in* (6.4)*, and suppose it is finite on rank-one matrices of the form $u(y)\otimes\nu$, $y \in \Omega$, $\nu \in \mathbb{R}^n$. Let $\mathscr{F}_{loc}$ be as defined in* (6.3)*. Then*

$$\mathscr{F}_{loc}(u,\Omega) \le \mathscr{L}^n(\Omega) f(0) + \int_\Omega f_\infty\left(\frac{D^s u}{|D^s u|}\right) |D^s u|\,. \tag{6.34}$$

Proof of Corollary 6.9. Using Lemma 6.7 and Theorem 5.5, let λ be a non-negative finite Radon measure on Ω representing $\mathscr{F}_{\mathrm{loc}}$, i.e.

$$\lambda(U) = \mathscr{F}_{\mathrm{loc}}(u,U)$$

for all open sets $U \subset \Omega$. Now let $\varrho > 0$ and let $\mathcal{B}^{(\varrho)}$ be a collection of closed balls of radius at most ϱ that is a fine cover of Ω. Consider any individual ball $B \in \mathcal{B}^{(\varrho)}$. Now take any open ball $B' \supset B$ of radius less than 2ϱ. If $B' \cap J_u = \emptyset$, then $Du = \nabla u = 0$ on B', and $u = a$ for some constant $a \in \mathbb{R}^N$. Hence by the definition of $\mathscr{F}_{\mathrm{loc}}$, noting that if $u_j = a$ for all j, then $u_j \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(B';\mathbb{R}^N)$,

$$\lambda(B') = \mathscr{F}_{\mathrm{loc}}(u, B') \leq \int_{B'} f(0)\,\mathrm{d}x = \mathscr{L}^n(B')f(0)\,. \tag{6.35}$$

Now suppose $B' \cap J_u \neq \emptyset$ and J_u has only a single polyhedron intersecting with B'. Then the jump set cuts the ball into two parts B'_a and B'_b, with (since $\nabla u = 0$)

$$\begin{cases} u(y) = a & \text{on } B'_a\,, \\ u(y) = b & \text{on } B'_b\,, \end{cases}$$

for some a, $b \in \mathbb{R}^N$. Moreover, since any point on a polyhedron is characterised by the intersection of finitely many $n-1$-dimensional hyperplanes, there exists a vector ν, say, such that

$$\begin{cases} B'_a = B' \cap (J_u + t\nu) & \text{for } t < 0\,, \\ B'_b = B' \cap (J_u + t\nu) & \text{for } t > 0\,. \end{cases}$$

Now let $0 < \delta < \varrho$ and define $\chi\colon (-\varrho, \varrho) \to \mathbb{R}^N$ by

$$\chi(t) := \begin{cases} a & \text{if } t \leq -\delta\,, \\ \left(\dfrac{b-a}{2\delta}\right)(x-\delta) + b & \text{if } t \in (-\delta, \delta)\,, \\ b & \text{if } t \geq \delta\,. \end{cases}$$

Now define a function $u_\delta \in C(B';\mathbb{R}^N)$ as follows: note that for each $y \in B'$, there exists a unique $t \in (-\varrho, \varrho)$ with $y \in J_u + t\nu$, and let

$$u_\delta(y) := \chi(t)\,.$$

We therefore have

$$\nabla u_\delta(y) = \begin{cases} 0 & \text{if } y \in J_u + t\nu \ \text{ for } t \notin (-\delta, \delta)\,, \\ \dfrac{(b-a)\otimes\nu}{2\delta} & \text{if } y \in J_u + t\nu \ \text{ for } t \in (-\delta, \delta)\,. \end{cases}$$

Now use the co-area formula to get

$$\begin{aligned} \int_{B'} f(\nabla u_\delta)\,\mathrm{d}x &\leq \mathscr{L}^n(B')f(0) + C\int_{-\delta}^{\delta}\int_{(J_u\cap B')+t\nu} f\left(\frac{(b-a)\otimes\nu}{2\delta}\right)\mathrm{d}\mathscr{H}^{n-1}\,\mathrm{d}t \\ &= \mathscr{L}^n(B')f(0) + C\cdot 2\delta f\left(\frac{(b-a)\otimes\nu}{2\delta}\right) \times \mathscr{H}^{n-1}(J_u \cap B') \end{aligned}$$

$$\to \mathscr{L}^n(B')f(0) + Cf_\infty((b-a)\otimes\nu)\mathscr{H}^{n-1}(J_u \cap B')\,,$$

as $\delta \to 0$. Note that the final term here is finite, since $b - a \in \operatorname{span}\{u(y) : y \in \Omega\}$. Now take any decreasing sequence δ_j converging to zero, and define u_j as u_{δ_j}. It is easily verified that u_j converges almost everywhere to u in B', and that the gradients ∇u_j are bounded in $L^1(B';\mathbb{R}^N)$. Hence, taking a subsequence if necessary, we have $u_j \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(B';\mathbb{R}^N)$. Thus

$$\lambda(B') = \mathscr{F}_{\text{loc}}(u, B') \leq \mathscr{L}^n(B')f(0) + Cf_\infty((b-a)\otimes\nu)\mathscr{H}^{n-1}(J_u \cap B')\,. \quad (6.36)$$

Now note that by taking $B' \searrow B$ we obtain the inequalities (6.35) and (6.36) for the closed ball B.

We also have that $\mathscr{H}^{n-1}$-almost all points $x \in J_u$ are on a face of the polyhedron. Hence for balls $B \in \mathcal{B}^{(\varrho)}$ that only intersect with a face, we have that J_u is characterised by an $n-1$-dimensional hyperplane passing through B. Therefore in this case we may take ν to be the normal vector of this plane, and a, b are just the one-sided traces $u^-(y)$, $u^+(y)$ on either side of the jump, for any $y \in B \cap J_u$. Moreover, we can take the constant C in (6.36), obtained from the co-area formula used above, to be one. Therefore for such a ball B, using the one-homogeneity of f_∞, we have

$$\begin{aligned}
\lambda(B) &\leq \mathscr{L}^n(B)f(0) + \int_{J_u\cap B} f_\infty((u^+(y) - u^-(y))\otimes\nu_u(y))\,\mathrm{d}\mathscr{H}^{n-1}(y) \\
&= \mathscr{L}^n(B)f(0) + \int_{J_u\cap B} f_\infty\left(\frac{u^+(y)-u^-(y)}{|u^+(y)-u^-(y)|}\otimes\nu_u(y)\right)|u^+(y)-u^-(y)|\,\mathrm{d}\mathscr{H}^{n-1}(y) \\
&= \mathscr{L}^n(B)f(0) + \int_{J_u\cap B} f_\infty\left(\frac{D^s u}{|D^s u|}(y)\right)|D^s u|(y)\,. \qquad (6.37)
\end{aligned}$$

Now we apply Theorem 6.8 with $\mu = \mathscr{L}^n + \mathscr{H}^{n-1}\lfloor J_u + \lambda$ and $U = \Omega$. Moreover, we can assume centres of the balls in $\mathcal{B}$, which is a fine partition, is all of Ω. There exists a countable collection of balls $\mathcal{G} \subset \mathcal{B}^{(\varrho)}$ such that

$$\bigcup_{B\in\mathcal{G}} B \subset U$$

and

$$(\mathscr{L}^n + \mathscr{H}^{n-1}\lfloor J_u + \lambda)\Big(\Omega \setminus \bigcup_{B\in\mathcal{G}} B\Big) = 0\,,$$

Let $\mathcal{G}_1$ denote the set of balls where (6.35) holds and $\mathcal{G}_2$ denote the set of balls where (6.37) holds. Then $\mathcal{G} \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)$ is the set of balls B where $B \cap J_u$ is nonempty and not just an $n-1$-dimensional hyperplane. We have already remarked that that $\mathscr{H}^{n-1}$-almost all points $x \in J_u$ are locally characterised by a hyperplane, so given $\epsilon > 0$, in light of (6.33) we may chose ϱ small enough so that

$$\sum_{B \in \mathcal{G} \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)} \lambda(B) \leq \epsilon \,.$$

Thus

$$\begin{aligned}
\lambda(\Omega) &= \sum_{B \in \mathcal{G}} \lambda(B) \\
&= \sum_{B \in \mathcal{G}_1} \lambda(B) + \sum_{B \in \mathcal{G}_2} \lambda(B) + \epsilon \\
&\leq \sum_{B \in \mathcal{G}_1} \mathscr{L}^n(B) f(0) + \sum_{B \in \mathcal{G}_2} \left(\mathscr{L}^n(B) f(0) + \int_{J_u \cap B} f_\infty\left(\frac{D^s u}{|D^s u|} \right) |D^s u| \right) + \epsilon \\
&= \mathscr{L}^n(\Omega) f(0) + \int_{J_u} f_\infty\left(\frac{D^s u}{|D^s u|} \right) |D^s u| + \epsilon \,,
\end{aligned}$$

from where the required result follows. □

These two results enable us to obtain the following upper bound for $\mathscr{F}_{\text{loc}}(u, \Omega)$, adapting a result of Braides and Coscia [22], which applies to general functions u in $\mathrm{BV}(\Omega; \mathbb{R}^N)$.

Lemma 6.10. *Let Ω be a bounded, open subset of $\mathbb{R}^n$ with Lipschitz boundary, and $u \in BV(\Omega; \mathbb{R}^N)$. Let $f \colon \mathbb{R}^{N \times n} \to \mathbb{R}$ be a continuous function satisfying the growth condition* (6.2) *for some exponent $1 \leq r < \frac{n}{n-1}$, and also the coercivity condition*

$$f(\xi) \geq c_0 |\xi|$$

for some constant $c_0 > 0$, for all $\xi \in \mathbb{R}^{N \times n}$. Let the recession function f_∞ be as defined in (6.4)*, and suppose it is finite on rank-one matrices of the form $u(y) \otimes \nu$, $y \in \Omega$, $\nu \in \mathbb{R}^n$. Then*

$$\mathscr{F}_{loc}(u, \Omega) \leq C(\mathscr{L}^n(\Omega) + |Du|(\Omega)) \,, \tag{6.38}$$

where $C > 0$ is a fixed constant depending on n, N and f.

Proof of Lemma 6.10. First assume that $u \in (C^1 \cap \mathrm{BV})(\Omega; \mathbb{R}^N)$. Write u in terms of its components, i.e. $u = (u^{(1)}, \dots, u^{(N)})$. Now note that we are interested specifically in the case where the dimension of span$\{u(y) : y \in \Omega\}$ is less than N: otherwise, as noted at the beginning of this chapter, we believe that if f additionally satisfies subquadratic growth conditions, then it has at most linear growth in all directions, and so the result would be more straightforward. Hence we may assume for simplicity that there exists $m < N$ such that $u^{(i)} = 0$ for $i > m$ and span$\{u(y) : y \in \Omega\} =$ span$\{\epsilon_1, \ldots, \epsilon_m\}$, where $\{\epsilon_1 \ldots, \epsilon_N\}$ is the canonical basis for $\mathbb{R}^N$. Otherwise we may use a change of variables. Note that the proof we give here works even if we were to assume span$\{u(y) : y \in \Omega\}$ has dimension N and $m = N$.

Take any $i \in \{1, \ldots, m\}$ and fix $k \in \mathbb{N}$. By the co-area formula, we have

$$|Du^{(i)}|(\Omega) = \sum_{j\in\mathbb{Z}} \int_{j/k}^{(j+1)/k} \mathscr{H}^{n-1}(\partial^*\{u^{(i)} > t\} \cap \Omega)\, \mathrm{d}t\,.$$

Hence, by the mean value theorem, for every $j \in \mathbb{Z}$ there exists $s_j^{i,k} \in (j/k, (j+1)/k)$ such that

$$\frac{1}{k}\mathscr{H}^{n-1}(\partial^*\{u > s_j^{i,k}\} \cap \Omega) \leq \int_{j/k}^{(j+1)/k} \mathscr{H}^{n-1}(\partial^*\{u^{(i)} > t\} \cap \Omega)\, \mathrm{d}t\,,$$

so that

$$\sum_{j\in\mathbb{Z}} \frac{1}{k}\mathscr{H}^{n-1}(\partial^*\{u > s_j^{i,k}\} \cap \Omega) \leq |Du^{(i)}|(\Omega)\,.$$

Now take, for every $j \in \mathbb{Z}$, a polyhedron $P_j^{i,k}$ such that

$$\left\{u^{(i)} > \frac{j+1}{k}\right\} \subset P_j^{i,k} \subset \left\{u^{(i)} > \frac{j}{k}\right\},$$

and

$$\mathscr{H}^{n-1}(\partial P_j^{i,k} \cap \Omega) \leq \mathscr{H}^{n-1}(\partial^*\{u > s_j^{i,k}\} \cap \Omega) + \frac{1}{k}2^{-|j|}\,.$$

Do this for all $i = 1, \ldots m$. Now define $u_k \in \mathrm{SBV}(\Omega; \mathbb{R}^N)$ by setting

$$w_k^{(i)}(y) := \frac{j}{k} \quad \text{on } P_{j-1}^{i,k} \setminus P_j^{i,k}\,,$$

and then letting $u_k^{(i)} := \mathrm{mid}\{-k, w_k^{(i)}, k\}$. Clearly we have $\nabla u_k^{(i)}(x) = 0$ for $\mathscr{L}^n$-almost all $x \in \Omega$, and there exists $j(i,k) \in \mathbb{N}$ such that

$$J_{u_k^{(i)}} \cap \Omega = \bigcup_{-j(i,k)\leq j\leq j(i,k)} \partial P_j^{i,k}\,,$$

and

$$Du_k^{(i)} = D^s u_k^{(i)} = \sum_{j=-j(i,k)}^{j(i,k)} \frac{1}{k} \nu_j^{i,k} \mathscr{H}^{n-1} \lfloor_{\partial P_j^{i,k}} ,$$

where $\nu_j^{i,k}$ is defined by

$$D\mathbf{1}_{P_j^{i,k}} = \nu_j^{i,k} \mathscr{H}^{n-1} \lfloor_{\partial P_j^{i,k}} .$$

Hence

$$Du_k = \sum_{i=1}^{m} \sum_{j=-j(i,k)}^{j(i,k)} \frac{1}{k} \epsilon_i \otimes \nu_j^{i,k} \mathscr{H}^{n-1} \lfloor_{\partial P_j^{i,k}} .$$

Since the jump set J_{u_k} is the union of finitely many polyhedra, and by assumption f_∞ is finite on matrices of the form $\epsilon_i \otimes \nu$ where $1 \leq i \leq m$ and $\nu \in \mathbb{R}^n$, we use Corollary 6.9 to get, for $1 \leq i \leq m$,

$$\begin{aligned}
\mathscr{F}_{\text{loc}}(u_k) &\leq \mathscr{L}^n(\Omega) f(0) + \int_{J_{u_k}} f_\infty\left(\frac{D^s u_k}{|D^s u_k|}\right) |D^s u_k| \\
&= \mathscr{L}^n(\Omega) f(0) + \sum_{i=1}^{m} \sum_{j=-j(i,k)}^{j(i,k)} \int_{P_j^{i,k} \cap \Omega} f_\infty\left(\frac{1}{k} \epsilon_i \otimes \nu_j^{i,k}\right) \mathrm{d}\mathscr{H}^{n-1} \\
&\leq \mathscr{L}^n(\Omega) f(0) + \frac{C}{k} \sum_{i=1}^{m} \sum_{j=-j(i,k)}^{j(i,k)} \mathscr{H}^{n-1}(\partial P_j^{i,k} \cap \Omega) \\
&\leq \mathscr{L}^n(\Omega) f(0) + \frac{C}{k} \sum_{i=1}^{m} \sum_{j=-j(i,k)}^{j(i,k)} \mathscr{H}^{n-1}(\partial^* \{u > s_j^{i,k}\} \cap \Omega) + \frac{1}{k} 2^{-|j|} \\
&\leq \mathscr{L}^n(\Omega) f(0) + C \sum_{i=1}^{m} |Du^{(i)}|(\Omega) + \frac{1}{k} \\
&\leq \mathscr{L}^n(\Omega) f(0) + C |Du|(\Omega) + \frac{1}{k} .
\end{aligned}$$

Note that the sequence (w_k) converges strongly to u in $L^\infty(\Omega; \mathbb{R}^N)$, so the truncated sequence (u_k) converges strongly to u in $L^1(\Omega; \mathbb{R}^N)$. Moreover, the measures $|Du_k|$ are bounded. Hence also $u_k \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega; \mathbb{R}^N)$, and using the lower semicontinuity of $\mathscr{F}_{\text{loc}}$ (see Proposition 5.2) we have

$$\mathscr{F}_{\text{loc}}(u, \Omega) \leq \liminf_{k \to \infty} \mathscr{F}_{\text{loc}}(u_k \Omega) \leq \mathscr{L}^n(\Omega) f(0) + C |Du|(\Omega) .$$

The result has been proved for $u \in (C^1 \cap \mathrm{BV})(\Omega; \mathbb{R}^N)$. For general $u \in \mathrm{BV}(\Omega; \mathbb{R}^N)$, it suffices to recall that by convolution and using a partition of unity (see, for example, [98]), there exists a sequence $(v_k) \subset (C^\infty \cap \mathrm{BV})(\Omega; \mathbb{R}^N)$ such that $v_k \overset{*}{\rightharpoonup} u$ in

BV$(\Omega;\mathbb{R}^N)$. Moreover, clearly

$$\operatorname{span}\{v_k(y) : y \in \Omega\} = \operatorname{span}\{u(y) : y \in \Omega\}\,,$$

so using the result for (v_k) and again the lower semicontinuity of $\mathscr{F}_{\mathrm{loc}}$, we get

$$\begin{aligned}\mathscr{F}_{\mathrm{loc}}(u,\Omega) &\leq \liminf_{k\to\infty} \mathscr{F}_{\mathrm{loc}}(v_k,\Omega)\\ &\leq \mathscr{L}^n(\Omega)f(0) + C\liminf_{k\to\infty} |Dv_k|(\Omega)\\ &= \mathscr{L}^n(\Omega)f(0) + C|Du|(\Omega)\,.\end{aligned}$$

This completes the proof. □

Remark. Localising the proof of this result, we also obtain the upper bound

$$\mathscr{F}_{\mathrm{loc}}(u,U) \leq C(\mathscr{L}^n(U) + |Du|(U)) \tag{6.39}$$

for any open subset $U \subset \Omega$. Related work concerning SBV and polyhedral approximation may be found in [8, 11, 13, 16].

Theorem 2.10 plays a key part in our proof of the inequality (6.14). This is because it shows us that the blow-up of a BV function on the singular part of the derivative is essentially a function of one variable; this allows us to apply the following lemma, similar to one of Ambrosio and Dal Maso [12], which gives us a useful lower bound for the Lebesgue-Serrin extension of such functions.

Lemma 6.11. *Let $Q \subset \mathbb{R}^n$ be a unit n-cube whose sides are either orthogonal or parallel to a unit vector $\nu \in \mathbb{R}^n$, let η be a unit vector in $\mathbb{R}^N$, and let $v \in BV(Q;\mathbb{R}^N)$ be a function representable as*

$$v(y) = \psi(\langle y,\nu\rangle)\eta$$

for a some non-decreasing function $\psi\colon (a,b) \to \mathbb{R}$. Suppose $u \in BV(Q;\mathbb{R}^N)$ satisfies supp$(v-u) \subset\subset Q$.

Let $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ be a quasiconvex function satisfying the growth condition (6.2) *for some exponent $1 \leq r < \frac{n}{n-1}$, and also the coercivity condition*

$$f(\xi) \geq c_0|\xi|$$

for some constant $c_0 > 0$, for all $\xi \in \mathbb{R}^{N\times n}$. Let the recession function f_∞ be as defined in (6.4)*, and suppose it is finite on rank-one matrices of the form $u(y)\otimes\nu$, $y \in \Omega$, $\nu \in \mathbb{R}^n$. Let $\mathscr{F}_{loc}$ be as defined in* (6.3)*. Then*

$$\mathscr{F}_{loc}(u,Q) \geq f(Du(Q))\,.$$

Proof of Lemma 6.11. We may assume without loss of generality that $\nu = e_1$ and $Q = (0,1)^n$. Let $\psi\colon (0,1) \to \mathbb{R}$ be a non-decreasing function such that $v(y) = \psi(y_1)\eta$, and let α denote the increment of ψ in $(0,1)$, i.e.

$$\alpha = \lim_{t\to 1^-} \psi(t) - \lim_{t\to 0^+} \psi(t) = |D\psi|(0,1) = |Dv|(Q) < +\infty\,.$$

Now define $w \in \mathrm{BV}_{\mathrm{loc}}((0,+\infty)^n;\mathbb{R}^N)$ by

$$w(y) := u(y-[y]) + \alpha[y_1]\eta\,,$$

where, for every $t \in \mathbb{R}$, $[t]$ denotes the integer part of t, and for $y = (y_1,\dots,y_n) \in \mathbb{R}^n$, $[y]$ is defined to be $([y_1],\dots,[y_n])$. Now define, for $y \in Q$,

$$u_k(y) := \frac{w(ky)}{k} \in \mathrm{BV}(Q;\mathbb{R}^N)\,.$$

Note that

$$u_k(y) = \frac{u(ky-[ky])}{k} + \alpha\frac{[ky_1]}{k}\eta\,,$$

$[ky_1]/k$ converges to y_1 as $k \to \infty$, and

$$\int_Q \left|\frac{u(ky-[ky])}{k}\right| \mathrm{d}y = \frac{1}{k^{n+1}}\int_{(0,k)^n} |u(y-[y])|\,\mathrm{d}y = \frac{1}{k}\int_Q |u(y)|\,\mathrm{d}y \to 0.$$

Therefore u_k converges to the affine function $u_0(y) = \alpha y_1\eta$ in $L^1(Q;\mathbb{R}^N)$. Now let $Q_1 \dots Q_{k^n}$ be the standard decomposition of Q into k^n congruent cubes of side length $1/k$. Since, by construction, Dw does not have any jumps on any hyperplane of the form $y_j = h$ where h is an integer and $1 \le j \le n$, it follows that

$$|Du_k|(Q \cap \partial(Q_i)) = 0 \quad \text{for all } 1 \le i \le k^n\,. \tag{6.40}$$

This implies

$$Du_k(Q) = Dw(Q) = Du(Q)\,,$$

so (u_k) is bounded in $\mathrm{BV}(Q;\mathbb{R}^N)$. Hence by Proposition 2.4 in fact the sequence converges weakly* in BV to u_0. By Proposition 5.1, we get

$$\mathscr{F}_{\mathrm{loc}}(u_k,(0,1/k)^n) = (1/k)^n\mathscr{F}_{\mathrm{loc}}(u,Q)\,,\quad \mathscr{F}_{\mathrm{loc}}(u_k,(0,1/k)^n) = \mathscr{F}_{\mathrm{loc}}(u_k,Q_i) \tag{6.41}$$

for all $1 \le i \le k^n$. Now, using the fact that $\mathscr{F}_{\mathrm{loc}}$ has a measure representation, (6.39) and (6.40), it follows that

$$\mathscr{F}_{\mathrm{loc}}(u_k,Q) = \sum_{i=1}^{k^n} \mathscr{F}_{\mathrm{loc}}(u_k,Q_i)\,.$$

This implies, together with (6.41), that $\mathscr{F}_{\mathrm{loc}}(u_k, Q) = \mathscr{F}_{\mathrm{loc}}(u, Q)$. By Corollary 6.6 and the lower semicontinuity of $\mathscr{F}_{\mathrm{loc}}$, we get

$$\mathscr{F}_{\mathrm{loc}}(u, Q) = \lim_{k \to \infty} \mathscr{F}_{\mathrm{loc}}(u_k, Q) \geq \mathscr{F}_{\mathrm{loc}}(u_0, Q) \geq f(\alpha\eta \otimes e_1)\,.$$

Noting that $Du(Q) = Dv(Q) = \alpha\eta \otimes e_1$, the proof is complete. □

6.4 Lower bound on the singular part

We are now in a position to be able to prove the inequality (6.14) which, combined with the proof of (6.13) established in Proposition 6.5, allows us to conclude our proof of Theorem 6.2. In order to use the results of the previous section, we first need to assume that the integrand f is coercive, before then showing how this assumption can be removed.

Proposition 6.12. *Let Ω be a bounded, open subset of $\mathbb{R}^n$, and let (u_j) be a sequence in $W^{1,r}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ and $u \in BV(\Omega; \mathbb{R}^N)$. Let $f \colon \mathbb{R}^{N \times n} \to \mathbb{R}$ be a quasiconvex function satisfying the growth condition* (6.2) *for some exponent $1 \leq r < \frac{n}{n-1}$, that also satisfies the coercivity condition*

$$f(\xi) \geq c_0 |\xi|$$

for some constant $c_0 > 0$, for all $\xi \in \mathbb{R}^{N \times n}$. Let the recession function f_∞ be as defined in (6.4)*, and suppose it is finite on rank-one matrices of the form $u(y) \otimes \nu$, $y \in \Omega$, $\nu \in \mathbb{R}^n$.*

Suppose

$$u_j \overset{*}{\rightharpoonup} u \text{ in } BV(\Omega; \mathbb{R}^N)\,. \tag{6.42}$$

Let μ be a measure in $\bar{\Omega}$ and suppose

$$f(Du_j) \overset{*}{\rightharpoonup} \mu \text{ in } \mathscr{M}(\bar{\Omega})\,.$$

Then for $|D^s u|$-almost all $x \in \Omega$, we have

$$\frac{d\mu}{d|D^s u|}(x) \geq f_\infty\left(\frac{D^s u}{|D^s u|}(x)\right).$$

Proof of Proposition 6.12. By Theorem 2.10, letting $\xi \colon \Omega \to \mathbb{R}^{N \times n}$ denote the density of Du with respect to $|Du|$, we have, for $|D^s u|$-almost all $x_0 \in \Omega$, $|\xi(x_0)| = 1$, $\mathrm{rank}(\xi(x_0)) = 1$, and

$$\lim_{\varrho \to 0^+} \frac{Du(Q(x_0, \varrho))}{|Du|(Q(x_0, \varrho))} = \xi(x_0)\,, \quad \lim_{\varrho \to 0^+} \frac{Du(Q(x_0, \varrho))}{\varrho^n} = +\infty\,,$$

where $Q(x_0, \varrho)$ is any cube centred at x_0 with side-length ϱ. Fix $x_0 \in \operatorname{supp}(|D^s u|)$ with these properties, and write $\xi(x_0) = \eta \otimes \nu$ where $\eta \in \mathbb{R}^n$, $\nu \in \mathbb{R}^N$, $|\eta| = |\nu| = 1$. Without loss of generality, suppose $\nu = e_1$. Let $Q = Q(0,1) = (-\frac{1}{2}, \frac{1}{2})^n$ be the unit cube in $\mathbb{R}^n$, so Q has faces either orthogonal or parallel to e_1. Also, subsequently, we shall let $Q(x_0, \varrho)$ specifically denote the cube $x_0 + \varrho Q$. Let $(r_k) \subset (0, \operatorname{dist}(x_0, \Omega))$ be a sequence decreasing to 0. Now define the functions $(v_{j,k}) \subset W^{1,r}(Q; \mathbb{R}^N)$ by

$$v_{j,k}(y) := \frac{r_k^n}{|Du|(Q(x_0, r_k))} \left(\frac{u_j(x_0 + r_k y)}{r_k} - m_k \right), \tag{6.43}$$

and $(v_k) \subset \mathrm{BV}(Q; \mathbb{R}^N)$ by

$$v_k(y) := \frac{r_k^n}{|Du|(Q(x_0, r_k))} \left(\frac{u(x_0 + r_k y)}{r_k} - m_k \right), \tag{6.44}$$

where

$$m_k := \int_Q \frac{u(x_0 + r_k y)}{r_k} \, \mathrm{d}\mathscr{L}^n .$$

Then, by (6.42), $v_{j,k} \overset{*}{\rightharpoonup} v_k$ in $\mathrm{BV}(Q; \Omega)$ as $j \to \infty$ for each k. By Theorem 2.10, we can chose our sequence (r_k) so that v_k converges weakly* in $\mathrm{BV}(Q; \mathbb{R}^N)$ to a function $v \in \mathrm{BV}(Q; \mathbb{R}^N)$ which can be represented as

$$v(y) = \psi(y_1)\eta$$

for a suitable non-decreasing function $\psi \colon (a,b) \to \mathbb{R}$. Moreover, for a given $\sigma \in (0,1)$, we have

$$1 \geq |Dv|(Q) \geq |Dv|(\sigma \bar{Q}) \geq \sigma^n , \quad \lim_{k \to \infty} |Dv_k|(Q) \geq \sigma^n . \tag{6.45}$$

By Fubini, there exists $s \in (\sigma, 1)$ such that

$$\lim_{k \to \infty} \int_{\partial(sQ)} |v - v_k| \, \mathrm{d}\mathscr{H}^{n-1} = 0 . \tag{6.46}$$

Now define $(w_k) \subset \mathrm{BV}(Q; \mathbb{R}^N)$ by

$$w_k := \begin{cases} v_k & \text{on } sQ , \\ v & \text{on } Q \setminus sQ . \end{cases}$$

Now define the sequence (t_k) converging to $+\infty$ by

$$t_k := \frac{|Du| Q(x_0, r_k)}{r_k^n} .$$

By Lemma 6.11 we have

$$\mathscr{F}_{\mathrm{loc}}(w_k, Q) \geq f(Dv(Q)),$$

so clearly, as t_k behaves like a constant for fixed k,

$$t_k^{-1}\mathscr{F}_{\mathrm{loc}}(t_k w_k, Q) \geq t_k^{-1} f(t_k Dv(Q)). \tag{6.47}$$

Moreover, by the measure representation of $\mathscr{F}_{\mathrm{loc}}$ in Theorem 5.5, we have

$$t_k^{-1}\mathscr{F}_{\mathrm{loc}}(t_k w_k, Q) \leq t_k^{-1}\mathscr{F}_{\mathrm{loc}}(t_k v_k, sQ) + t_k^{-1}\mathscr{F}_{\mathrm{loc}}(t_k w_k, Q \setminus \sigma\bar{Q}). \tag{6.48}$$

We now obtain various estimates for the terms in (6.48). First note that we have

$$t_k^{-1}\mathscr{F}_{\mathrm{loc}}(t_k v_k, sQ) \leq t_k^{-1}\mathscr{F}_{\mathrm{loc}}(t_k v_k, Q) \leq \liminf_{j\to\infty} t_k^{-1}\int_Q f(t_k \nabla v_{j,k})\,\mathrm{d}x.$$

However

$$\begin{aligned} t_k^{-1}\int_Q f(t_k\nabla v_{j,k})\,\mathrm{d}x &= \frac{r_k^n}{|Du|Q(x_0,r_k)}\int_Q f(\nabla u_j(x_0 + r_k y)\,\mathrm{d}y \\ &= \frac{1}{|Du|Q(x_0,r_k)}\int_{Q(x_0,r_k)} f(\nabla u_j(y))\,\mathrm{d}y \\ &\xrightarrow{j\to\infty} \frac{\mu(Q(x_0,r_k))}{|Du|Q(x_0,r_k)}, \end{aligned}$$

and so

$$\begin{aligned} t_k^{-1}\mathscr{F}_{\mathrm{loc}}(t_k v_k, sQ) &\leq \frac{\mu(Q(x_0,r_k))}{|Du|Q(x_0,r_k)} \\ &\to \frac{d\mu}{d|Du|}(x_0) \quad \text{as } k\to\infty. \end{aligned} \tag{6.49}$$

Observe that $w_k(y) \in \mathrm{span}\{u(z) : z \in Q\}$ for all $y \in Q$. Hence we may use the upper bound (6.39) in Lemma 6.10 to obtain

$$\begin{aligned} t_k^{-1}\mathscr{F}_{\mathrm{loc}}(t_k w_k, Q\setminus\sigma\bar{Q}) &\leq t_k^{-1}C(\mathscr{L}^n(Q\setminus\sigma\bar{Q}) + |t_k Dw_k|(Q\setminus\sigma\bar{Q}) \\ &= C\big(t_k^{-1}(1-\sigma^n) + |Dw_k|(Q\setminus\sigma\bar{Q})\big). \end{aligned}$$

Note that

$$|Dw_k|(Q\setminus\sigma\bar{Q}) \leq |Dv|(Q\setminus\sigma\bar{Q}) + |Dv_k|(Q\setminus\sigma\bar{Q}) + \int_{\partial(sQ)} |v - v_k|\,\mathrm{d}\mathscr{H}^{n-1},$$

and hence, using (6.45) and (6.46), we have

$$\limsup_{k\to\infty} |Dw_k|(Q\setminus\sigma\bar{Q}) \leq 2(1-\sigma^n)\,.$$

This means

$$\limsup_{k\to\infty} t_k^{-1}\mathscr{F}_{\mathrm{loc}}(t_k w_k, Q\setminus\sigma\bar{Q}) \leq C(1-\sigma^n)\,. \tag{6.50}$$

Lastly, recall the definition of the recession function in (6.4): since f is quasiconvex and hence rank-one convex, we have

$$\limsup_{t\to\infty}\frac{f(t\xi)}{t} = \lim_{t\to\infty}\frac{f(t\xi)}{t}$$

whenever $\operatorname{rank}(\xi)\leq 1$. Therefore, noting that for $x_0\in\operatorname{supp}(|D^s u|)$ we have $Dv(Q) = \eta\otimes e_1 = \xi(x_0)$, we obtain

$$\lim_{k\to\infty} t_k^{-1} f(t_k Dv(Q)) = f_\infty(Dv(Q))\,. \tag{6.51}$$

Also, for such x_0,

$$\frac{d\mu}{d|Du|}(x_0) = \frac{d\mu}{d|D^s u|}(x_0)$$

Now let k tend to infinity in (6.48), and use (6.49), (6.50) and (6.51) to get

$$f_\infty(\xi(x_0)) \leq \frac{d\mu}{d|D^s u|}(x_0) + C(1-\sigma^n)\,.$$

We conclude the proof by letting $\sigma\nearrow 1$. $\square$

We now remove the coercivity condition on f to prove Theorem 6.2.

Proof of Theorem 6.2. By Propositions 6.5 and 6.12, we have established the inequalities (6.13) and (6.14) for when f is coercive, allowing us establish the Theorem in this case. Otherwise, define $f^\epsilon\colon\mathbb{R}^{N\times n}\to\mathbb{R}$ as

$$f^\epsilon(\xi) := f(\xi) + \epsilon|\xi|\,,$$

for all $\xi\in\mathbb{R}^{N\times n}$, for some $\epsilon>0$. Let (u_j) be a sequence in $W^{1,r}(\Omega;\mathbb{R}^N)$ such that

$$u_j \overset{*}{\rightharpoonup} u \ \text{ in } \mathrm{BV}(\Omega;\mathbb{R}^N)\,.$$

Then we have

$$\liminf_{j\to\infty} F(u_j;\Omega) \geq \int_\Omega f^\epsilon(\nabla u(x))\,\mathrm{d}x + \int_\Omega f^\epsilon_\infty\left(\frac{D^s u}{|D^s u|}\right)|D^s u|\,.$$

Now note that

$$\int_\Omega f^\epsilon(\nabla u(x))\,\mathrm{d}x = \int_\Omega f(\nabla u(x))\,\mathrm{d}x + \epsilon \int_\Omega |\nabla u(x)|\,\mathrm{d}x$$

It is also clear that the recession function f^ϵ_∞ satisfies

$$f^\epsilon_\infty(\xi) = f_\infty(\xi) + \epsilon|\xi|\,,$$

so

$$\int_\Omega f^\epsilon_\infty\left(\frac{D^s u}{|D^s u|}\right)|D^s u| = \int_\Omega f_\infty\left(\frac{D^s u}{|D^s u|}\right)|D^s u| + \epsilon|D^s u|(\Omega)\,.$$

Therefore we have

$$\liminf_{j\to\infty} F(u_j;\Omega) \geq \int_\Omega f(\nabla u(x))\,\mathrm{d}x + \int_\Omega f_\infty\left(\frac{D^s u}{|D^s u|}\right)|D^s u| + \epsilon|Du|(\Omega)\,,$$

and conclude by letting ϵ tend to 0. $\square$

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