Logic of inductive definitions with formal neighbourhoods

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1. Computable functionals of finite types

- Gödel 1958: "Über eine bisher noch nicht benützte Erweiterung des finiten Standpunkts": computable finite type functions.
- Need partial continuous functionals as their intendend domain (Scott 1969). The total ones then appear as a dense subset (Kreisel 1959, Ershov 1972, Berger 1990).
- We define them concretely, based on (a simplified form of) information systems (Scott 1982).

Atomic coherent information systems (acis's)

- Acis: $(A, \frown, \geq)$ such that $\frown$ (consistent) is reflexive and symmetric, $\geq$ (entails) is reflexive and transitive and

$$a \frown b \to b \geq c \to a \frown c.$$

- Formal neighborhood: $U \subseteq A$ finite and consistent. We write $U \geq a$ for $\exists_{b \in U}(b \geq a)$, and $U \geq V$ for $\forall_{a \in V}(U \geq a)$.
- Function space: Let $\mathbf{A} = (A, \frown_A, \geq_A)$ and $\mathbf{B} = (B, \frown_B, \geq_B)$ be acis's. Define $\mathbf{A} \to \mathbf{B} = (C, \frown, \geq)$ by

$$\begin{aligned} C &:= \mathrm{Con}_A \times B, \\ (U, b) \frown (V, c) &:= (U \frown_A V \to b \frown_B c), \\ (U, b) \geq (V, c) &:= (V \geq_A U \wedge b \geq_B c). \end{aligned}$$

$\mathbf{A} \to \mathbf{B}$ is an acis again.

Ideals, Scott topology

- Ideal: $x \subseteq A$ consistent and deductively closed. $|\mathbf{A}|$ is the set of ideals (points, objects) of $\mathbf{A}$.
- $|\mathbf{A}|$ carries a natural topology, with cones $\tilde{U} := \{ z \mid z \supseteq U \}$ generated by the formal neighborhoods U as basis.

Theorem (Scott 1982)

The continuous maps $f : |\mathbf{A}| \to |\mathbf{B}|$ and the ideals $r \in |\mathbf{A} \to \mathbf{B}|$ are in a bijective correspondence.

Definition

An ideal $x \subseteq A$ is computable if it is recursively enumerable as a set of (finite) tokens.

Turning a free algebra into an acis

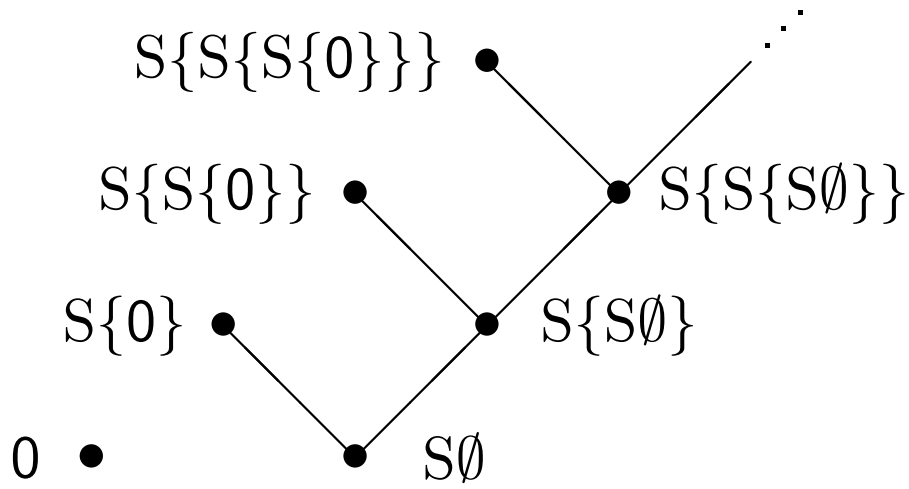
Commonly done by adding $\perp$: "flat cpo".

- Problem 1: Constructors are not injective: $C(\perp, b) = \perp = C(a, \perp)$.
- Problem 2: Constructors do not have disjoint ranges: $C_1(\perp) = \perp = C_2(\perp)$.

Solution: Use as tokens constructor trees of the form $C_i U_1 \ldots U_n$.

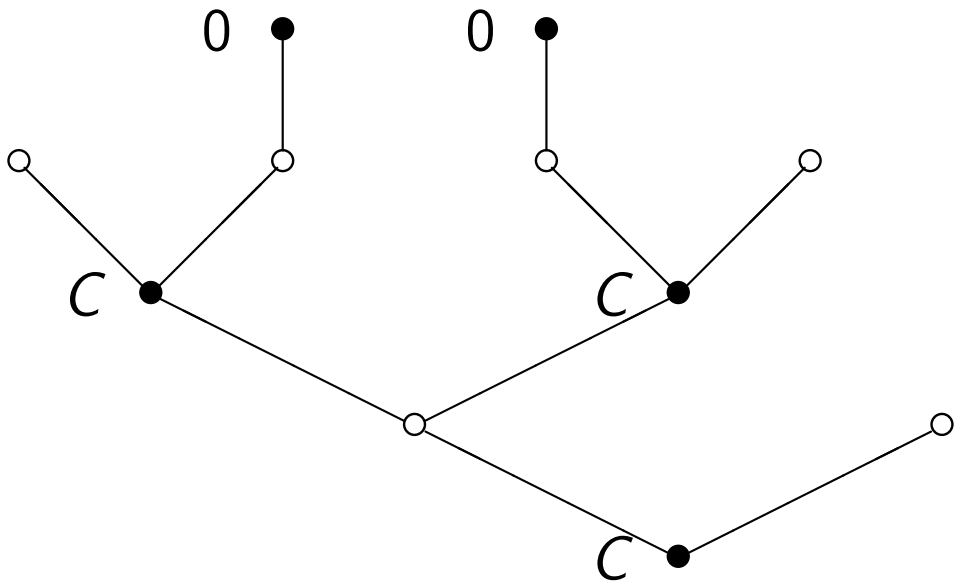
- Essentially the same idea is carried out in an appendix "Nonstandard elements as formal points" by Venanzio Capretta and Giovanni Sambin to the second author's forthcoming book.

Example: tokens and entailment for **N**



Constructor trees

Nodes alternate between atomic and neighborhood nodes. Every atomic node is labelled by a constructor, and has as many (typed) neighborhood successor nodes as the constructor has arguments. Every neighborhood node has finitely many (possibly zero) atomic successor nodes. Example: $C\{C\emptyset\{0\}, C\{0\}\emptyset\}\emptyset$



Total and cototal ideals for finitary algebras

- A total ideal for the acis of a finitary algebra is the deductive closure of a constructor tree all of whose neighborhood nodes have successors (i.e., formed without empty neighborhoods).

Example: every total ideal for the algebra **N** is the deductive closure of a token $\mathrm{S}\{\mathrm{S}\ldots\{\mathrm{S}\{0\}\}\ldots\}$.

- A cototal ideal for the acis of a finitary algebra is determined by a possibly non-wellfounded constructor tree, as the deductive closure of all finite constructor subtrees obtained by removing branches of neighborhood nodes.

Example: a cototal ideal of **N** is the deductive closure of set of all tokens of the form $\mathrm{S}\{\mathrm{S}\ldots\{\mathrm{S}\emptyset\}\ldots\}$.

A common extension T^+ of Gödel's T and Plotkin's PCF

- Terms $M, N ::= x^\rho \mid C \mid D \mid (\lambda_{x^\rho} M^\sigma)^{\rho\to\sigma} \mid (M^{\rho\to\sigma} N^\rho)^\sigma$.
- Constants D defined by computation rules. Examples:
  Recursion $\mathcal{R}^\tau_{\mathbf{N}} \colon \mathbf{N} \to \tau \to (\mathbf{N} \to \tau \to \tau) \to \tau$.

$$\mathcal{R}0xy = x, \quad \mathcal{R}(\mathrm{S}n)xy = yn(\mathcal{R}nxy).$$

  Corecursion $\mathcal{C}^\tau_{\mathbf{N}} \colon \tau \to (\tau \to \mathbf{U} + \tau + \mathbf{N}) \to \mathbf{N}$.

$$\mathcal{C}xy = [\textbf{case } yx \textbf{ of } 0 \mid \lambda_z(\mathrm{S}[\textbf{case } z^{\tau+\mathbf{N}} \textbf{ of } \lambda_u(\mathcal{C}uy) \mid \lambda_n n])].$$

The computation rules consist of finitely many equations

$$D\vec{P}_i(\vec{y}_i) = M_i(\vec{y}_i) \quad i = 1, \ldots, n$$

with constructor patterns $\vec{P}_i$. To ensure consistency, we require that for $i \neq j$ either $\vec{P}_i$ and $\vec{P}_j$ are non-unifiable, or else for the most general unifier ξ of $\vec{P}_i$ and $\vec{P}_j$ we have $M_i\xi = M_j\xi$.

Destructors

Every algebra ι with k constructors each of arity n_i $(i < k)$ has a destructor D_ι of type

$$\iota \to \sum_{i<k} \prod_{j<n_i} \iota.$$

Computation rules:

$$D_\iota(C_i(\vec{x})) = \mathrm{in}_i(\vec{x}).$$

Example: $D_{\mathbf{N}} : \mathbf{N} \to \mathbf{U} + \mathbf{N}$ is defined by the computation rules

$$D_{\mathbf{N}}(\mathrm{S}n) = \mathrm{inr}(n),$$
$$D_{\mathbf{N}}(0) = \mathrm{inl}(\mathbf{u}).$$

Denotational semantics: definition of $(\vec{U}, b) \in [\![\lambda_{\vec{x}} M]\!]$

$$\frac{U_i \geq b}{(\vec{U}, b) \in [\![\lambda_{\vec{x}} x_i]\!]}(V), \qquad \frac{(\vec{U}, V) \subseteq [\![\lambda_{\vec{x}} N]\!] \qquad (\vec{U}, V, c) \in [\![\lambda_{\vec{x}} M]\!]}{(\vec{U}, c) \in [\![\lambda_{\vec{x}} MN]\!]}(A).$$

For every constructor C and defined constant D we have

$$\frac{\vec{V} \geq \vec{W}}{(\vec{U}, \vec{V}, C\vec{W}) \in [\![\lambda_{\vec{x}} C]\!]}(C), \qquad \frac{(\vec{U}, \vec{V}, b) \in [\![\lambda_{\vec{x},\vec{y}} M]\!]}{(\vec{U}, \vec{P}(\vec{V}), b) \in [\![\lambda_{\vec{x}} D]\!]}(D)$$

with one such rule (D) for every computation rule $D\vec{P}(\vec{y}\,) = M$.

Theorem (Adequacy; Plotkin 1977, Martin-Löf 1983)

If $b \in [\![M]\!]$ for a closed term M, then M head-reduces to a token entailing b.

2. Logic of inductive definitions LID

- Based on T^+. Terms with a common reduct are identified.
- Contains inductively and coinductively defined predicates, given by their clauses and (least and greatest) fixed point axioms. Examples: $T, T^\infty, \mathrm{Eq}, \exists$ (cf. Martin-Löf 1971).
- Ex-falso-quodlibet is provable, when one defines falsity by $\mathbf{F} := \mathrm{Eq}_{\mathbf{B}}(\mathrm{ff}, \mathrm{tt})$.

Minimal logic: introduction and elimination rules only, for

- computational $\to^{\mathrm{c}}$, $\forall^{\mathrm{c}}$:

$$\tau(A \to^{\mathrm{c}} B) := (\tau(A) \to \tau(B)), \qquad \tau(\forall^{\mathrm{c}}_{x^\rho} B) := (\rho \to \tau(B)).$$

- non-computational $\to$, $\forall$:

$$\tau(A \to B) := \tau(\forall_{x^\rho} B) := \tau(B).$$

Correct derivations and extracted terms

Restrictions to $\rightarrow^+$ and $\forall^+$: consider

$$\dfrac{\begin{array}{c}[u\colon A]\\ \mid M\\ B\end{array}}{A \rightarrow B}\rightarrow^+ u \qquad \text{or as term} \quad (\lambda_{u^A} M)^{A\rightarrow B}.$$

$(\lambda_{u^A} M)^{A\rightarrow B}$ is correct if M is and $x_u \notin \mathrm{FV}([\![M]\!])$. Similarly: Consider

$$\dfrac{\begin{array}{c}\mid M\\ A\end{array}}{\forall_x A}\forall^+ x \qquad \text{or as term} \quad (\lambda_x M)^{\forall_x A} \qquad \text{(VarC)}.$$

$(\lambda_x M)^{\forall_x A}$ is correct if M is and $x \notin \mathrm{FV}([\![M]\!])$.

Totality for **N**

- Inductively define totality by the clauses

$$T0, \qquad \forall_n(Tn \to^{\mathrm{c}} T(\mathrm{S}n)).$$

  Its (least) fixed point axiom is

$$\forall_n(Tn \to^{\mathrm{c}} A(0) \to^{\mathrm{c}} \forall_n(Tn \to^{\mathrm{c}} A(n) \to^{\mathrm{c}} A(\mathrm{S}n)) \to^{\mathrm{c}} A(n)).$$

- T can be understood as the least set of pairs witness - argument satisfying the clauses. Witness: total ideal in **N**.
- Inductively define $T^{\mathbf{r}}(s, n)$ to express that s witnesses $T(n)$.

Cototality for **N**

- Coinductively define cototality by the clause

$$\forall_{n \in T^\infty}(n = 0 \vee^{\mathrm{c}} \exists^{\mathrm{r}}_m(n = \mathrm{S}m \wedge^{\mathrm{r}} T^\infty m))$$

  Its (greatest) fixed point axiom is

$$\forall_n(A(n) \to^{\mathrm{c}} \forall_n(A(n) \to^{\mathrm{c}} n = 0 \vee^{\mathrm{c}} \exists^{\mathrm{r}}_m(n = \mathrm{S}m \wedge^{\mathrm{r}} (A(m) \vee^{\mathrm{c}} T^\infty m))) \to^{\mathrm{c}} T^\infty n).$$

- T^∞ can be understood as the greatest set of pairs witness - argument satisfying the clause. Witness: cototal ideal in **N**.
- Coinductively define $(T^\infty)^{\mathsf{r}}(s, n)$ to express that s witnesses $T^\infty(n)$.

Example: existential quantifiers $\exists^{\mathrm{c}}$, $\exists^{\mathrm{r}}$, $\exists^{\mathrm{l}}$, $\exists$

The respective clause is

$$\forall_x^{\mathrm{c}}(A \to^{\mathrm{c}} \exists_x^{\mathrm{c}} A),$$
$$\forall_x^{\mathrm{c}}(A \to \exists_x^{\mathrm{l}} A),$$
$$\forall_x(A \to^{\mathrm{c}} \exists_x^{\mathrm{r}} A),$$
$$\forall_x(A \to \exists_x A).$$

and the (least) fixed point axiom (with $x \notin \mathrm{FV}(C)$)

$$\exists_x^{\mathrm{c}} A \to^{\mathrm{c}} \forall_x^{\mathrm{c}}(A \to^{\mathrm{c}} C) \to^{\mathrm{c}} C,$$
$$\exists_x^{\mathrm{l}} A \to^{\mathrm{c}} \forall_x^{\mathrm{c}}(A \to C) \to^{\mathrm{c}} C,$$
$$\exists_x^{\mathrm{r}} A \to^{\mathrm{c}} \forall_x(A \to^{\mathrm{c}} C) \to^{\mathrm{c}} C,$$
$$\exists_x A \to \forall_x(A \to C) \to^{\mathrm{c}} C.$$

Similarly for $\wedge$, $\vee$.

Realizers for the T-axioms

- Abbreviating $\forall_x(Tx \to^c A)$ by $\forall_{x\in T}A$ the clauses and the (least) fixed point axiom for T are

$$T0, \qquad \forall_{n\in T}T(\mathrm{S}n),$$
$$\forall_{n\in T}(A(0) \to^c \forall_{n\in T}(A(n) \to^c A(\mathrm{S}n)) \to^c A(n))$$

- Its types are $\mathbf{N}$ and $\mathbf{N} \to \mathbf{N}$ for the clauses and

$$\mathbf{N} \to \tau \to (\mathbf{N} \to \tau \to \tau) \to \tau$$

  for the fixed point axiom.

- Its extracted terms are the constructors and the recursion operator of $\mathbf{N}$.

Realizing the clause for T^∞

- Recall the clause for T^∞

$$\forall_{n \in T^\infty}(n = 0 \vee^{\mathrm{c}} \exists^{\mathrm{r}}_m(n = \mathrm{S}m \wedge^{\mathrm{r}} T^\infty m))$$

- Its type is

$$\mathbf{N} \to \mathbf{U} + \mathbf{N}$$

since $\tau(T^\infty) := \mathbf{N}$ and $\tau(\forall_x A) := \tau(\exists^{\mathrm{r}}_x A) := \tau(A)$.

- Its extracted term is the destructor $D_{\mathbf{N}}$.

Realizing the fixed point axiom for T^∞

- Recall the (greatest) fixed point axiom for T^∞

  $\forall_n(A(n) \to^{\mathrm{c}}$
  $\quad \forall_n(A(n) \to^{\mathrm{c}} n = 0 \vee^{\mathrm{c}} \exists^{\mathrm{r}}_m(n = \mathrm{S}m \wedge^{\mathrm{r}} (A(m) \vee^{\mathrm{c}} T^\infty m))) \to^{\mathrm{c}}$
  $\quad T^\infty n)$.

- Its type is

$$\mathbf{N} \to (\mathbf{N} \to \mathbf{U} + \tau + \mathbf{N}) \to \mathbf{N},$$

  since $\tau(T^\infty) := \mathbf{N}$ and $\tau(\forall_x A) := \tau(\exists^{\mathrm{r}}_x A) := \tau(A)$.

- Its extracted term is the corecursion operator.

Leibniz equality Eq

- Inductively defined by the clause

$$\forall_x \mathrm{Eq}(x^\rho, x^\rho).$$

- The (least) fixed point axiom is

$$\forall_{x,y}\big(\mathrm{Eq}(x,y) \to \forall_x C(x,x) \to^{\mathrm{c}} C(x,y)\big).$$

- The fixed point axiom with $C(x,y) := A(x) \to^{\mathrm{c}} A(y)$ implies

$$\forall_{x,y}(\mathrm{Eq}(x,y) \to A(x) \to^{\mathrm{c}} A(y)) \quad \text{(compatibility of } \mathrm{Eq}).$$

- Compatibility gives symmetry and transitivity of Eq.

Lemma (Ex-Falso-Quodlibet)

$\mathbf{F} \to A$ *for* $\tau(A) = \varepsilon$, *with falsity defined by* $\mathbf{F} := \mathrm{Eq}(\mathrm{ff}, \mathrm{tt})$.

Proof.

(1) $\mathbf{F} \to \mathrm{Eq}(x^\rho, y^\rho)$, since from $\mathrm{Eq}(\mathrm{ff}, \mathrm{tt})$ by compatibility

$$\mathrm{Eq}\, \underbrace{[\textbf{if } \mathrm{tt} \textbf{ then } x \textbf{ else } y]}_{x}\, \underbrace{[\textbf{if } \mathrm{ff} \textbf{ then } x \textbf{ else } y]}_{y}.$$

Hence $\mathrm{Eq}(x^\rho, y^\rho)$.

(2) Induction on $A \in \mathrm{F}$.

- Case $I\vec{s}$. Let K_i be the nullary clause, with final conclusion $I\vec{t}$. By IH: $\mathbf{F} \to A_i$. Hence $I\vec{t}$. From $\mathbf{F}$ we also obtain $\mathrm{Eq}(s_i, t_i)$, by (1). Hence $I\vec{s}$ by compatibility.
- Case $J\vec{s}$. Use the greatest fixed point axiom for J with $C(\vec{x}) := \mathbf{F}$. Since $k > 0$ and $n_0 = 0$ it suffices to prove $\mathbf{F} \to \exists_{\vec{y}_i} \bigwedge \vec{A}_i$. This holds by IH.
- The cases $A \to B$ and $\forall_x A$ are clear.

(AC), (IP) and (IQ)

These axioms express the intended meaning of computational and non-computational connectives. (AC) is the axiom of choice:

$$\forall^{\mathrm{c}}_{x^\rho}\exists^{\mathrm{c}}_{y^\sigma}A(x,y) \to^{\mathrm{c}} \exists^{\mathrm{c}}_{f^{\rho\to\sigma}}\forall^{\mathrm{c}}_{x^\rho}A(x,f(x)).$$

Independence of premise axiom (IP)

$$(A \to \exists^{\mathrm{c}}_x B) \to^{\mathrm{c}} \exists^{\mathrm{c}}_x(A \to B) \quad (x \notin \mathrm{FV}(A)).$$

Independence of quantifier axiom (IQ)

$$\forall_x\exists^{\mathrm{c}}_y A \to^{\mathrm{c}} \exists^{\mathrm{c}}_y\forall_x A.$$

Similarly for $\exists^{\mathrm{l}}$ and $\exists^{\mathrm{r}}$.

Characterization and soundness theorems

Using (AC), (IP) and (IQ) we can prove

Theorem (Characterization)

Every formula A is computationally equivalent to $\exists^{\mathrm{c}}_x(x\ \mathbf{r}\ A)$ if it has computational content, and to $\varepsilon\ \mathbf{r}\ A$ if not.

Corollary (Ex-falso-quodlibet)

$\mathbf{F} \to A$ *for arbitrary (possibly c.r.) formulas* A.

Proof.

Recall $\mathbf{F} \to \varepsilon^\rho\ \mathbf{r}\ A$, where ε^ρ is the canonical inhabitant of type ρ.
Hence $\exists^{\mathrm{c}}_x(x\ \mathbf{r}\ A)$ and therefore A. □

Theorem (Soundness)

Assume M derives A from assumptions $u_i : C_i$ $(i < n)$. Then we can find a derivation of $[\![M]\!]\ \mathbf{r}\ A$ from assumptions $\bar{u}_i : x_{u_i}\ \mathbf{r}\ C_i$ for u_i computational (i.e., $x_{u_i} \in \mathrm{FV}([\![M]\!])$), and $\bar{u}_i : C_i$ for the other.

3. Implicit computation with streams

- Based on a recent draft of Ulrich Berger "From coinductive proofs to exact real arithmetic".
- The exact computational nature of the relevant notions is left implicit, in the clauses of their (co)inductive definitions. It is only at the level of realizers that computational details come to the surface. This can help for a better understanding, by allowing a more abstract treatment.
- LID might be a proper framework to carry this out.

Coinductive definition of real numbers

Let ξ be the the (abstract) type of reals, and x, y of type ξ. Let Rx abbreviate "x is a real in $[-1,1]$". Coinductive definition of W_0:

$$\forall_x(W_0x \to^{\mathrm{c}} x = 0 \vee^{\mathrm{c}} \exists_{d\in T_{\mathbf{SD}}}\exists^{\mathrm{r}}_y(x = \frac{y+d}{2} \wedge^{\mathrm{r}} W_0y)).$$

Let $\mathbb{I}_{p,k} := [p - 2^{-k}, p + 2^{-k}]$ and $\mathrm{B}_kx := \exists^{\mathrm{l}}_q(x \in \mathbb{I}_{q,k})$. Assume that in the abstract theory we can prove

$$\forall_{x\in R}\forall^{\mathrm{c}}_{p,q\in Q}(p < q \to x \leq q \vee p \leq x),$$
$$\forall_x(Rx \leftrightarrow^{\mathrm{c}} \forall^{\mathrm{c}}_k B_kx).$$

Coinductive definition of real numbers (continued)

Lemma

- $\forall_x(Rx \to^{\mathrm{c}} W_0 x)$.
- $\forall_x(W_0 x \to^{\mathrm{c}} \forall^{\mathrm{c}}_k B_k x)$.

Proof:

- Use the fixed point axiom for W_0 with Rx for $A(x)$.
- Prove $\forall^{\mathrm{c}}_k \forall_x(W_0 x \to^{\mathrm{c}} B_k x)$ by induction on k. In the step case use the clause for W_0 and the IH.

Using the lemma we can derive

$$\forall_{x,y}(W_0 x \to^{\mathrm{c}} W_0 y \to^{\mathrm{c}} W_0(x * y))$$

from $\forall_{x,y}(Rx \to^{\mathrm{c}} Ry \to^{\mathrm{c}} R(x * y))$ in the abstract setting. Its type is that of a stream transformer.

Inductive/coinductive definition of continuous functions

Let $\mathbb{I} := [-1, 1]$ and let f range over (abstract) functions $\mathbb{I} \to \mathbb{I}$. Let Cf abbreviate "f is (uniformly) continuous". Assume that in the abstract theory we can prove

$$\forall^{\mathrm{c}}_{f\colon \mathbb{I}\to\mathbb{I}}(C(f) \leftrightarrow^{\mathrm{c}} \forall^{\mathrm{c}}_k \exists^{\mathrm{c}}_l B_{l,k} f) \quad \text{with } \mathrm{B}_{l,k} f := \forall^{\mathrm{c}}_p \exists^{\mathrm{c}}_q (f[\mathbb{I}_{p,l}] \subseteq \mathbb{I}_{q,k}).$$

For $d \in \mathbf{SD} := \{-1, 0, 1\}$ let $\mathbb{I}_d$ be defined by

$$\mathbb{I}_{-1} := [-1, 0] \qquad \mathbb{I}_0 := [-\frac{1}{2}, \frac{1}{2}] \qquad \mathbb{I}_1 := [0, 1].$$

Define $\mathrm{in}_d, \mathrm{out}_d$ such that $\mathrm{in}_d[\mathbb{I}] = \mathbb{I}_d$ and $\mathrm{out}_d[\mathbb{I}_d] = \mathbb{I}$ by

$$\mathrm{in}_d(x) := \frac{d + x}{2}, \quad \mathrm{out}_d(x) := 2x - d.$$

Both functions are inverse to each other.

Inductive/coinductive def. of continuous functions (ctd.)

- Inductive definition of a predicate R_X depending on a parameter X:

$$\forall^{\mathrm{c}}_f(f[\mathbb{I}] \subseteq \mathbb{I}_d \to X(\mathrm{out}_d \circ f) \to^{\mathrm{c}} R_X f),$$
$$\forall^{\mathrm{c}}_f(\forall^{\mathrm{c}}_{d \in \mathbf{SD}} R_X(f \circ \mathrm{in}_d) \to^{\mathrm{c}} R_X f).$$

- Using R, we give a coinductive definition of W by the clause

$$\forall_f(Wf \to^{\mathrm{c}} \mathrm{Id} f \vee^{\mathrm{c}} R_W f).$$

- Realizers of Wf: cototal/total ideals (i.e., alternating in write and read mode).

Inductive/coinductive def. of continuous functions (ctd.)

Lemma

- $\forall_f(Cf \to^{\mathrm{c}} Wf)$.
- $\forall_f(Wf \to^{\mathrm{c}} \forall^{\mathrm{c}}_k \exists^{\mathrm{c}}_l B_{l,k} f)$.

Proof:

- Use the fixed point axiom for W with Cf for $A(f)$.
- Prove $\forall^{\mathrm{c}}_k \forall_f(Wf \to^{\mathrm{c}} \exists^{\mathrm{c}}_l B_{l,k+1} x)$ by induction on k. In the step use a side induction on R_W and the IH.

Using the lemma one can derive

$$\forall_{f,g}(Wf \to^{\mathrm{c}} Wg \to^{\mathrm{c}} W(f \circ g))$$

from $\forall_{f,g}(Cf \to^{\mathrm{c}} Cg \to^{\mathrm{c}} C(f \circ g))$ in the abstract setting. Its content again is a stream transformer, as in Peter Hancock's talk.

Conclusion

- Partial continuous functionals: Acis's, ideals, free algebras, totality and cototality.
- T^+, a common extension of Gödel's T and Plotkin's PCF: Constants defined by computation rules, denotational semantics, adequacy theorem.
- Logic of inductive definitions LID: based on T^+. $\to, \forall$ as well as $\to^{\mathrm{c}}, \forall^{\mathrm{c}}$. Characterization and soundness theorems.
- Application: Extraction of stream transformers from abstract proofs in real analysis.