Substitutions in typed object terms

Some basic material

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- Consider very basic object terms only: typed lambda calculus.
- Call lambda terms "object terms" as opposed to formulas, comprehension terms and proof terms, which we will consider later.
- In particular don't consider inner structure of base types (e.g. as free algebras). No recursion operator for term formation.

1 Definition (Types)

Types are built from type variables by arrow type formation $\sigma \to \tau$.
Symbols used for type variables and types: σ, τ.

2 Definition (Object terms)

For each type our language includes countably many object variables of this type. Object terms and their free variables are defined by:

1. x^σ, $\mathrm{FV_o}(x) := \{x\}$,
2. $(\lambda_{x^\sigma} r^\tau)^{\sigma\to\tau}$, $\mathrm{FV_o}(\lambda_x r) := \mathrm{FV_o}(r)\backslash\{x\}$,
3. $(r^{\sigma\to\tau} s^\sigma)^\tau$, $\mathrm{FV_o}(rs) := \mathrm{FV_o}(r) \cup \mathrm{FV_o}(s)$.

Let $\overline{r}$ denote the type of r.
Symbols for object variables: x, y, z. For object terms: r, s, t.

3 Definition (Substitution in types)

Substitution in types is inductively defined by:

1. $\sigma\vartheta := \begin{cases} \vartheta(\sigma) & \text{if } \sigma \in \mathrm{dom}(\vartheta), \\ \sigma & \text{otherwise,} \end{cases}$
2. $(\sigma \to \tau)\vartheta := (\sigma\vartheta) \to (\tau\vartheta)$.

4 Definition (Admissible substitutions)

A Substitution ϑ is called admissible for an object variable x if $\overline{x\vartheta} = \overline{x}\vartheta$, where

$$x\vartheta := \begin{cases} \vartheta(x) & \text{if } x \in \text{dom}(\vartheta), \\ x & \text{otherwise.} \end{cases}$$

Furthermore ϑ is called admissible for an object term r if ϑ is admissible for all $x \in \text{FV}_\text{o}(r)$.

This definition is as suggested by Prof. Wilfried Buchholz.

5 Definition

Let ϑ be admissible for r. We call s an application of ϑ to r if one of the following holds:

1. $r = x$ for an object variable x and $s = x\vartheta$.
2. $r = \lambda_x r'$, $s = \lambda_y s'$ and the following holds: (i) y is an object variable fulfilling $\overline{y} = \overline{x}\vartheta$ and $y \notin \bigcup_{z \in \text{FV}_\text{o}(r)} \text{FV}_\text{o}(z\vartheta)$ and (ii) s' is an application of ϑ^y_x to r'.
3. $r = r_1 r_2$, $s = s_1 s_2$ and s_i is an application of ϑ to r_i for $i = 1, 2$.

6 Theorem

Let ϑ be admissible for r. Then the following holds:

(a) There is an object term s such that s is an application of ϑ to r.

(b) If $r\vartheta$ is any *application of ϑ to r then we have $\overline{r\vartheta} = \overline{r}\vartheta$ and $FV_\text{o}(r\vartheta) = \bigcup_{x \in FV_\text{o}(r)} FV_\text{o}(x\vartheta)$.*

Proof: We show the claims by simultaneous induction on r.
Case 1. $r = x$. By definition of a substitution $x\vartheta$ is an object term. Thus $s := x\vartheta$ satisfies (a). Furthermore any application of ϑ to x is equal to $x\vartheta$. As ϑ is admissible for x we have $\overline{x\vartheta} = \overline{x}\vartheta$. Since $\text{FV}_\text{o}(r) = \{x\}$ we also have $\text{FV}_\text{o}(x\vartheta) = \bigcup_{x \in \text{FV}_\text{o}(r)} \text{FV}_\text{o}(x\vartheta)$.
Case 2. $r = \lambda_x r'$. Choose a object variable y such that $\overline{y} = \overline{x}\vartheta$ and $y \notin \bigcup_{z \in \text{FV}_\text{o}(r)} \text{FV}_\text{o}(z\vartheta)$. Then ϑ^y_x is admissible for r' (requires proof!). By induction hypothesis there is a term s' which is an application of ϑ^y_x to r'. Then $s := \lambda_y s'$ is as required. Furthermore any application $r\vartheta$ is of the form $\lambda_y(r'\vartheta^y_x)$ where $\overline{y} = \overline{x}\vartheta$, $y \notin \bigcup_{z \in \text{FV}_\text{o}(r)} \text{FV}_\text{o}(z\vartheta)$ and where $r'\vartheta^y_x$ is an application of ϑ^y_x to r'. Using the induction hypothesis we have $\overline{r\vartheta} = \overline{y} \to \overline{r'\vartheta^y_x} = \overline{x}\vartheta \to \overline{r}'\vartheta^y_x$. Since ϑ and ϑ^y_x coincide on all type variables we have $\overline{r}'\vartheta^y_x = \overline{r}'\vartheta$ (requires proof!). Using this, we have $\overline{r\vartheta} = \overline{x}\vartheta \to \overline{r}'\vartheta = \overline{r}\vartheta$.Using the induction hypotheseis and $\text{FV}_\text{o}(r) = \text{FV}_\text{o}(r') \backslash \{x\}$ yields $\text{FV}_\text{o}(r\vartheta) = \text{FV}_\text{o}(r'\vartheta^y_x) \backslash \{y\} = [\bigcup_{z \in \text{FV}_\text{o}(r')} \text{FV}_\text{o}(z\vartheta^y_x)] \backslash \{y\} = [\bigcup_{z \in \text{FV}_\text{o}(r)} \text{FV}_\text{o}(z\vartheta)] \backslash \{y\}$ and the claim follows with the variable condition $y \notin \bigcup_{z \in \text{FV}_\text{o}(r)} \text{FV}_\text{o}(z\vartheta)$.
Case 3. $r^\tau = r_1^{\sigma \to \tau} r_2^\sigma$. For $i = 1, 2$ we have $\text{FV}_\text{o}(r_i) \subseteq \text{FV}_\text{o}(r)$, thus ϑ is admissible for r_1 and r_2. By the induction hypothesis there exist object terms s_1, s_2 which are

applications of ϑ to r_1, r_2. Also by induction hypothesis we have $\overline{s_1} = \overline{r_1}\vartheta = (\sigma \to \tau)\vartheta = \sigma\vartheta \to \tau\vartheta$ and $\overline{s_2} = \overline{r_2}\vartheta = \sigma\vartheta$. Thus $s := s_1 s_2$ is an object term satisfying (a). The further claims follow easily with the induction hypothesis. □

Plan for the following:

- Define Alpha-equality on object terms (definition by Robert Stärk, as implemented in Minlog).
- $r =_\alpha s$ is supposed to mean that r and s are equal modulo renaming of bound variables.
- Let ϑ be admissible for r. Show that $r =_\alpha s$ implies $r\vartheta =_\alpha s\vartheta$ where $r\vartheta$ and $s\vartheta$ are arbitrary applications.

7 Definition (Alpha-equality of object terms)

Let r, s be object terms and let $((x_1\, y_1), \ldots, (x_n\, y_n))$ be a list of pairs of object variables. We say that r is equal-via-$((x_1\, y_1), \ldots, (x_n\, y_n))$ to s, if one of the following cases holds:

1. $r = x$, $s = y$ for object variables x, y and either
   - (i) $x = y$, x not one of the x_i and y is not one of the y_i, or
   - (ii) there is a $j \in \{1, \ldots, n\}$ such that $x = x_j$, $y = y_j$ and $x \neq x_k$, $y \neq y_k$ for all $k \in \{j+1, \ldots, n\}$.
2. $r = \lambda_x r'$, $s = \lambda_y s'$, $\overline{x} = \overline{y}$ and r' is equal-via-$((x_1\, y_1), \ldots, (x_n\, y_n), (x\, y))$ to s'.
3. $r = r_1 r_2$, $s = s_1 s_2$ and r_1 / r_2 is equal-via-$((x_1\, y_1), \ldots, (x_n\, y_n))$ to s_1 / s_2.

Finally we say that r is alpha-equal to s, written $r =_\alpha s$, if r is equal-via-() to s.

8 Theorem

The relation $=_\alpha$ between object terms is an equivalence relation.

Proof: Requires some work, about three quarters of a A4 page. □

9 Theorem

Let r be an object term an let ϑ be a substitution admissible for r. If r_1 and r_2 are two applications of ϑ to r then we have $r_1 =_\alpha r_2$.

Proof: This is the case $n = 0$ in the following lemma. □

10 Lemma

Let ϑ be a substitution and r be an object term. Let $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, $z_1, \ldots, z_n$ be object variables such that for all $i \in \{1, \ldots, n\}$ the following holds:

(i) $\overline{x_i}\vartheta = \overline{y_i} = \overline{z_i}$.

(ii) *For all* $z \in FV_o(r)$*: If* $y_i \in FV_o(z\vartheta^{y_1 \dots y_{i-1}}_{x_1 \dots x_{i-1}})$ *then* $z \in \{x_i, \dots, x_n\}$.

(iii) *For all* $z \in FV_o(r)$*: If* $z_i \in FV_o(z\vartheta^{z_1 \dots z_{i-1}}_{x_1 \dots x_{i-1}})$ *then* $z \in \{x_i, \dots, x_n\}$.

Suppose that $\vartheta^{y_1 \dots y_n}_{x_1 \dots x_n}$ *is admissible for* r *(hence so is* $\vartheta^{z_1 \dots z_n}_{x_1 \dots x_n}$*). Let* r_1 */* r_2 *be two applications of* $\vartheta^{y_1 \dots y_n}_{x_1 \dots x_n}$ */* $\vartheta^{z_1 \dots z_n}_{x_1 \dots x_n}$ *to* r*. Then* r_1 *is equal-via-*$((y_1\, z_1), \dots, (y_n\, z_n))$ *to* r_2*.*

Proof: Induction on r.
Case 1. $r = x$ for an object variable x. *Subcase 1.1.* x is not one of the x_i. Then we have $r_1 = x\vartheta = r_2$. We want to show that $x\vartheta$ is equal-via-$((y_1\, z_1), \dots, (y_n\, z_n))$ to $x\vartheta$. This follows if we can show that $y_i, z_i \notin \mathrm{FV_o}(x\vartheta)$ for all $i \in \{1, \dots, n\}$ (requires proof!). So show the latter: Suppose there exists $i \in \{1, \dots, n\}$ such that $y_i \in \mathrm{FV_o}(x\vartheta)$. Since $x \notin \{x_1, \dots, x_n\}$ in this subcase we have $x\vartheta^{y_1 \dots y_{i-1}}_{x_1 \dots x_{i-1}} = x\vartheta$. Because of $x \in \mathrm{FV_o}(r)$ it follows from (ii) that $x \in \{x_i, \dots, x_n\}$ must hold. But the latter contradicts the assumption that $x \notin \{x_1, \dots, x_n\}$. Analogously one uses (iii) to show that $z_i \notin \mathrm{FV_o}(x\vartheta)$ for all $i \in \{1, \dots, n\}$. *Subcase 1.2.* $x = x_j$ and $x \neq x_k$ for $k > j$. Then $x\vartheta^{y_1 \dots y_n}_{x_1 \dots x_n} = y_j$ and $x\vartheta^{z_1 \dots z_n}_{x_1 \dots x_n} = z_j$. We need to show that for all $k \in \{j+1, \dots, n\}$ we have $y_k \neq y_j$ and $z_k \neq z_j$. This can be done using (ii) and (iii) very similarly to subcase 1.1.
Case 2. $r = \lambda_x r'$. Then $r_1 = \lambda_{y_{n+1}} r' \vartheta^{y_1 \dots y_n y_{n+1}}_{x_1 \dots x_n x}$. Here y_{n+1} is an object variable satisfying $\overline{y_{n+1}} = \overline{x}\vartheta$ and $y_{n+1} \notin \bigcup_{z \in \mathrm{FV_o}(r)} \mathrm{FV_o}(z\vartheta^{y_1 \dots y_n}_{x_1 \dots x_n})$. Furthermore $r'\vartheta^{y_1 \dots y_n y_{n+1}}_{x_1 \dots x_n x}$ is an application of $\vartheta^{y_1 \dots y_n y_{n+1}}_{x_1 \dots x_n x}$ to r'. Analogously $r_2 = \lambda_{z_{n+1}} r' \vartheta^{z_1 \dots z_n z_{n+1}}_{x_1 \dots x_n x}$. We would like to apply the induction hypothesis to $r'\vartheta^{y_1 \dots y_n y_{n+1}}_{x_1 \dots x_n x}$ and $r'\vartheta^{z_1 \dots z_n z_{n+1}}_{x_1 \dots x_n x}$ and therefore need to show that all conditions are satisfied. Let $i \in \{1, \dots, n+1\}$ be arbitrary. Obviously (i) is satisfied. Now consider (ii): Let $z \in \mathrm{FV_o}(r') \subseteq \mathrm{FV_o}(r) \cup \{x\}$. *Subcase 2.1.* If $z \in \mathrm{FV_o}(r)$ then the implication in (ii) is satisfied by assumption (for $i \in \{1, \dots, n\}$) and because of the variable condition $y_{n+1} \notin \bigcup_{z \in \mathrm{FV_o}(r)} \mathrm{FV_o}(z\vartheta^{y_1 \dots y_n}_{x_1 \dots x_n})$ (for $i = n+1$). *Subcase 2.2.* If $z = x$ then the implication in (ii) is satisfied because its consequent $z \in \{x_i, \dots, x_n, x\}$ is satisfied. Analogously one shows that (iii) is satisfied. Furthermore $\vartheta^{y_1 \dots y_n}_{x_1 \dots x_n}$ is admissible for r', as in the proof of 6. Thus by induction hypothesis $r'\vartheta^{y_1 \dots y_n y_{n+1}}_{x_1 \dots x_n x}$ is equal-via-$((y_1\, z_1), \dots, (y_{n+1}\, z_{n+1}))$ to $r'\vartheta^{z_1 \dots z_n z_{n+1}}_{x_1 \dots x_n x}$. Using $\overline{y_{n+1}} = \overline{z_{n+1}}$ we may conclude that r_1 is equal-via-$((y_1\, z_1), \dots, (y_n\, z_n))$ to r_2, as required.
Case 3. Easily shown using the induction hypothesis. Note that for $r = s_1 s_2$ we have $\mathrm{FV_o}(s_1), \mathrm{FV_o}(s_2) \subseteq \mathrm{FV_o}(r)$. Thus (ii) and (iii) hold for s_1 and s_2. □

11 Theorem

Let $r =_\alpha s$ *be object terms and let* ϑ *be admissible for* r*. Then* ϑ *is admissible for* s *and we have* $r\vartheta =_\alpha s\vartheta$ *(for arbitrary applications).*

Proof: Much easier than the previous one. □

The theory of substitution yields an elegant characterization of alpha-equality:

12 Definition

A relation $\mathcal{R}$ on object terms is called *compatible with substitution* if the following holds: Let ϑ be admissible for an object term r and let r_1, r_2 be two applications of ϑ to r. Then $r_1 \mathcal{R} r_2$.

13 Theorem

Alpha-equality is the smalles relation on object terms which is compatible with substitution, that is: If $\mathcal{R}$ is a relation on object terms compatible with substitution and if r_1, r_2 are object terms then $r_1 =_\alpha r_2$ implies $r_1 \mathcal{R} r_2$.