CHAPTER 4

Computational content of proofs

We have already mentioned that (co)inductive predicates can be declared as either computationally relevant (c.r.) or non-computational (n.c.). But what is the computational content in the c.r. case? We first address this question for (co)inductive predicates, and then extend it to arbitrary formulas. Next we study in what sense a proof of a c.r. formula A provides us with concrete computational content. This can be seen as a "witness" for the validity of A, or (in the sense of Kolmogorov (1932)) a "solution" to problem A.

Finally we take a step back and reflect on what we have done. We formally define what it means for a term to "realize" the c.r. formula A. We extract from a proof M of A a term $\mathrm{et}(M)$ and (again formally) prove that it is a realizer of A. In this proof we need "invariance axioms" stating that every c.r. formula not involving realizability is invariant under realizability, formally $A \leftrightarrow \exists_z(z\ \mathbf{r}\ A)$, where $z\ \mathbf{r}\ A$ means "z realizes A".

4.1. Realizers

Assume that we have a global assignment giving for every c.r. predicate variable X of arity $\vec{\rho}$ an n.c. predicate variable $X^{\mathbf{r}}$ of arity $(\vec{\rho}, \xi)$ where ξ is the type variable associated with X. We will also introduce $I^{\mathbf{r}}/{}^{\mathrm{co}}I^{\mathbf{r}}$ for (co)inductive predicates $I/{}^{\mathrm{co}}I$. A formula or predicate C is called $\mathbf{r}$-free if it does not contain any of these $X^{\mathbf{r}}$, $I^{\mathbf{r}}$ or ${}^{\mathrm{co}}I^{\mathbf{r}}$. A derivation M is called $\mathbf{r}$-free if it contains $\mathbf{r}$-free formulas only.

Definition ($C^{\mathbf{r}}$ for $\mathbf{r}$-free predicates and formulas C). For every $\mathbf{r}$-free predicate or formula C we define a predicate or formula $C^{\mathbf{r}}$. For n.c. C let $C^{\mathbf{r}} := C$. In case C is c.r. $C^{\mathbf{r}}$ is an n.c. predicate of arity $(\vec{\sigma}, \tau(C))$ with $\vec{\sigma}$ the arity of C. We often write $z\ \mathbf{r}\ C$ for $C^{\mathbf{r}}z$ in case C is a c.r. formula. For c.r. *predicates* X let $X^{\mathbf{r}}$ be the n.c. predicate variable provided, and

$$\{\, \vec{x} \mid A \,\}^{\mathbf{r}} := \{\, \vec{x}, z \mid z\ \mathbf{r}\ A \,\}.$$

Now consider a c.r. (co)inductive predicate

$$I/{}^{\mathrm{co}}I := (\mu/\nu)_X((K_i(X))_{i<k}$$

with associated algebra form $\iota_I = \mu_\xi(\kappa_i(\xi))_{i<k}$ where $\kappa_i(\xi) := \tau(K_i(X))$. The i-th constructor of ι_I is $\mathrm{C}_i\colon \kappa_i(\iota_I)$. Let s be a variable of type $\tau(I)$ and ϑ the substitution $\xi \mapsto \tau(I)$, $X^{\mathbf{r}} \mapsto \{\, \vec{x}, s \mid Y\vec{x}s \,\}$. We define n.c. predicates $I^{\mathbf{r}}$ and ${}^{\mathrm{co}}I^{\mathbf{r}}$ by

$$I^{\mathbf{r}}/{}^{\mathrm{co}}I^{\mathbf{r}} := (\mu/\nu)_Y((C_i\ \mathbf{r}\ K_i(X))\vartheta)_{i<k}.$$

The substitution ϑ is necessary since the arity of Y (and hence of $I^{\mathbf{r}}/{}^{\mathrm{co}}I^{\mathbf{r}}$) must be $(\vec{\rho}, \tau(I))$ and not $(\vec{\rho}, \xi)$. For c.r. *formulas* let

$$z\ \mathbf{r}\ P\vec{t} := P^{\mathbf{r}}\vec{t}z,$$

$$z\ \mathbf{r}\ (A \to B) := \begin{cases} \forall_w(w\ \mathbf{r}\ A \to zw\ \mathbf{r}\ B) & \text{if } A \text{ is c.r.} \\ A \to z\ \mathbf{r}\ B & \text{if } A \text{ is n.c.} \end{cases}$$

$$z\ \mathbf{r}\ \forall_x A := \forall_x(z\ \mathbf{r}\ A).$$

Example. As an easy example for the construction of $I^{\mathbf{r}}$ consider the predicate Even, defined by $\mu_X(K_0(X), K_1(X))$ with $K_0(X) := (0 \in X)$ and $K_1(X) := \forall_n(n \in X \to S(Sn) \in X)$. The associated algebra form is $\mu_\xi(\kappa_0(\xi), \kappa_1(\xi))$ with $\kappa_0(\xi) := \xi$ and $\kappa_1(\xi) := \xi \to \xi$, i.e., the algebra $\mathbb{N}$ with constructors $\mathrm{C}_0 := 0$ and $\mathrm{C}_1 := S$. Let ϑ be the substitution $\xi \mapsto \mathbb{N}$, $X^{\mathbf{r}} \mapsto \{\, n, m \mid Ynm \,\}$. Since $S\ \mathbf{r}\ K_1(X)$ is $\forall_{n,m}(X^{\mathbf{r}}nm \to X^{\mathbf{r}}(S(Sn), Sm))$ we obtain

$$I^{\mathbf{r}} := \mu_Y(Y00, \forall_{n,m}(Ynm \to Y(S(Sn), Sm)).$$

We express Kolmogorov's view of formulas as problems by means of *invariance axioms*:

Axiom (Invariance under realizability). *For* $\mathbf{r}$*-free c.r. formulas* A *we require as axioms*

$$\mathrm{InvAll}_A\colon \forall_z(z\ \mathbf{r}\ A \to A). \tag{14}$$

$$\mathrm{InvEx}_A\colon A \to \exists_z(z\ \mathbf{r}\ A). \tag{15}$$

Realizers of totality and cototality predicates will be of special interest for us. Notice that the types $\tau(T_\iota)$ and $\tau({}^{\mathrm{co}}T_\iota)$ are both ι. Moreover we have

Lemma 4.1.1 (Realizers of totality). *For closed base types* ι *the following are equivalent.*

(a) $T_\iota^r xy$,
(b) $x \sim_\iota^{\mathrm{nc}} y$,
(c) $x \in T_\iota^{\mathrm{nc}} \wedge x \equiv y$.

Proof. (a) $\leftrightarrow$ (b). Both $T_\iota^{\mathbf{r}} xy$ and $x \sim_\iota^{\mathrm{nc}} y$ satify the same clauses. Use the respective elimination axiom in each of the two directions.

(b) $\leftrightarrow$ (c). Use Lemma 3.3.1. □

LEMMA 4.1.2 (Realizers of cototality). *For closed base types ι the following are equivalent.*

(a) ${}^{\mathrm{co}}T_\iota^{\boldsymbol{r}} xy$,
(b) $x \approx_\iota^{\mathrm{nc}} y$,
(c) $x \in {}^{\mathrm{co}}T_\iota^{\mathrm{nc}} \wedge x \equiv y$.

PROOF. As an example we give the proof for $\mathbb{N}$. Since we have n.c. goals only, decorations are omitted.

(a) $\to$ (b). We use the greatest-fixed-point axiom for $\approx_{\mathbb{N}}$:

$$\forall_{n,m}(Xnm \to (n \equiv 0 \wedge m \equiv 0) \vee \exists_{n',m'}((n' \approx_{\mathbb{N}} m' \vee Xn'm') \wedge n \equiv Sn' \wedge m \equiv Sm')) \to X \subseteq \approx_{\mathbb{N}}$$

and apply it with ${}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}}$ for X. It suffices to prove the premise. Assume ${}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}} nm$; the goal is

$$C := {}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}} 00 \vee \exists_{n',m'}((n' \approx_{\mathbb{N}} m' \vee {}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}} n'm') \wedge n \equiv Sn' \wedge m \equiv Sm').$$

By the closure axiom $({}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}})^-$ we have

$$(n \equiv 0 \wedge m \equiv 0) \vee \exists_{n',m'}({}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}} n'm' \wedge n \equiv Sn' \wedge m \equiv Sm').$$

We argue by cases (i.e., use $\vee^-$).

Case 1. $n \equiv 0 \wedge m \equiv 0$. Go for the l.h.s. of the disjunction C and show ${}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}} 00$. But this follows from the greatest-fixed-point axiom for ${}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}}$ with competitor predicate $\{\, n, m \mid n \equiv 0 \wedge m \equiv 0 \,\}$.

Case 2. $\exists_{n',m'}({}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}} n'm' \wedge n \equiv Sn' \wedge m \equiv Sm')$. Go for the r.h.s. of C.

(b) $\to$ (a). Recall ${}^{\mathrm{co}}T_{\mathbb{N}} := \nu_X(0 \in X,\ \forall_{n \in X}(Sn \in X))$, hence by definition

$${}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}} := \nu_{X^{\mathbf{r}}}(X^{\mathbf{r}} 00,\ \forall_{n,m}(X^{\mathbf{r}} nm \to X^{\mathbf{r}}(Sn)(Sm))).$$

We need to show $m \approx_{\mathbb{N}} n \to {}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}} mn$. To this end we use the greatest-fixed-point axiom for ${}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}}$:

$$\forall_{n,m}(Xnm \to X00 \vee \exists_{n',m'}(n', m' \in ({}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}} \cup X) \wedge n \equiv Sn' \wedge m \equiv Sm')) \to X \subseteq {}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}}$$

and apply it with $\approx_{\mathbb{N}}$ for X. It suffices to prove the premise. Assume $n \approx_{\mathbb{N}} m$; the goal is

$$C := (0 \approx_{\mathbb{N}} 0) \vee \exists_{n',m'}((n', m' \in ({}^{\mathrm{co}}T_{\mathbb{N}}^{\mathbf{r}} \cup \approx_{\mathbb{N}}) \wedge n \equiv Sn' \wedge m \equiv Sm')).$$

By the closure axiom $(\approx_{\mathbb{N}})^-$ we have

$$n \approx_{\mathbb{N}} m \to (n \equiv 0 \wedge m \equiv 0) \vee \exists_{n',m'}(n' \approx_{\mathbb{N}} m' \wedge n \equiv Sn' \wedge m \equiv Sm').$$

We argue by cases (i.e., use $\vee^-$).

Case 1. $n \equiv 0 \wedge m \equiv 0$. Go for the l.h.s. of the disjunction C and show $0 \approx_{\mathbb{N}} 0$. But this follows from $n \approx_{\mathbb{N}} m$.

Case 2. $\exists_{n',m'}(n' \approx_{\mathbb{N}} m' \wedge n \equiv Sn' \wedge m \equiv Sm')$. Go for the r.h.s. of C.

(b) $\leftrightarrow$ (c). Use the Bisimilarity axiom and Lemma 3.3.1. $\square$

Next we study what our general definition says about realizers for the c.r. inductively defined decorated connectives.

LEMMA 4.1.3 (Realizers for $\exists$). *$z\ \mathbf{r}\ \exists_x A \leftrightarrow \exists_x(z\ \mathbf{r}\ A)$ for A c.r.*

PROOF. Recall $\mathrm{Ex}_Y := \mu_X(\forall_x(x \in Y \to X))$. Then

$$\mathrm{Ex}^{\mathbf{r}}_{Y^{\mathbf{r}}} := \mu_{X^{\mathbf{r}}}(\forall_{x,z}(Y^{\mathbf{r}}xz \to X^{\mathbf{r}}z)).$$

Now substituting $Y^{\mathbf{r}}$ by $\{\, x, z \mid z\ \mathbf{r}\ A \,\}$ in the introduction axiom gives

$$(\mathrm{Ex}^{\mathbf{r}}_{\{x,z|z\mathbf{r}A\}})^+_0 \colon \forall_{x,z}(z\ \mathbf{r}\ A \to z\ \mathbf{r}\ \exists_x A)$$

Conversely, the elimination axiom $(\mathrm{Ex}^{\mathbf{r}}_{Y^{\mathbf{r}}})^-$ is

$$\forall_z(z \in \mathrm{Ex}^{\mathbf{r}}_{Y^{\mathbf{r}}} \to \forall_{x,z}(Y^{\mathbf{r}}xz \to z \in X) \to z \in X).$$

which is equivalent to

$$\forall_z(z \in \mathrm{Ex}^{\mathbf{r}}_{Y^{\mathbf{r}}} \to \forall_z(\exists_x Y^{\mathbf{r}}xz \to z \in X) \to z \in X).$$

Substituting X by $\{\, z \mid \exists_x(Y^{\mathbf{r}}xz) \,\}$ makes the middle part provable. Thus with $\{\, x, z \mid z\ \mathbf{r}\ A \,\}$ for $Y^{\mathbf{r}}$ we obtain $\forall_z(z\ \mathbf{r}\ \exists_x A \to \exists_x(z\ \mathbf{r}\ A))$ from $(\mathrm{Ex}^{\mathbf{r}}_{\{x,z|z\mathbf{r}A\}})^-$. $\square$

Similarly we have

LEMMA 4.1.4 (Realizers for $\wedge$). *$z\ \mathbf{r}\ (A \wedge B)$ is equivalent to*

$z \equiv \langle \mathrm{lft}(z), \mathrm{rht}(z)\rangle \wedge (\mathrm{lft}(z)\ \mathbf{r}\ A) \wedge (\mathrm{rht}(z)\ \mathbf{r}\ B)$	*for A c.r. and B c.r.*
$(z\ \mathbf{r}\ A) \wedge B$	*for A c.r. and B n.c.*
$A \wedge (z\ \mathbf{r}\ B)$	*for A n.c. and B c.r.*

PROOF. *Case A, B* c.r. Recall $\mathrm{AndD}_{X^c,Y^c} := \mu_{Z^c}(X^c \to Y^c \to Z^c)$. Then

$$\mathrm{AndD}^{\mathbf{r}}_{X^{\mathbf{r}},Y^{\mathbf{r}}} := \mu_{Z^{\mathbf{r}}}(\forall_x(x \in X^{\mathbf{r}} \to \forall_y(y \in Y^{\mathbf{r}} \to \langle x, y\rangle \in Z^{\mathbf{r}}))).$$

Now substituting $X^{\mathbf{r}}$ by $\{\, x \mid x\ \mathbf{r}\ A \,\}$ and $Y^{\mathbf{r}}$ by $\{\, y \mid y\ \mathbf{r}\ B \,\}$ in the introduction axiom gives

$$(\mathrm{AndD}^{\mathbf{r}}_{\{x|x\mathbf{r}A\},\{y|y\mathbf{r}B\}})^+_0 \colon \forall_x((x\ \mathbf{r}\ A) \to \forall_y(y\ \mathbf{r}\ B \to \langle x, y\rangle\ \mathbf{r}\ (A \wedge B))).$$

This suffices for "$\leftarrow$". Conversely, the elimination axiom $(\mathrm{AndD}^{\mathbf{r}}_{X^{\mathbf{r}},Y^{\mathbf{r}}})^-$ is

$$\forall_x(x \in X^{\mathbf{r}} \to \forall_y(y \in Y^{\mathbf{r}} \to \langle x, y\rangle \in Z)) \to \mathrm{AndD}^{\mathbf{r}}_{X^{\mathbf{r}},Y^{\mathbf{r}}} \subseteq Z.$$

Substitute Z by $\{\, z \mid z \equiv \langle \mathrm{lft}(z), \mathrm{rht}(z)\rangle \wedge (\mathrm{lft}(z)\ \mathbf{r}\ A) \wedge (\mathrm{rht}(z)\ \mathbf{r}\ B) \,\}$. Then with $\{\, x \mid x\ \mathbf{r}\ A \,\}$ for $X^{\mathbf{r}}$ and $\{\, y \mid y\ \mathbf{r}\ B \,\}$ for $Y^{\mathbf{r}}$ the premise is provable and we obtain

$$\forall_z(z\ \mathbf{r}\ (A \wedge B) \to z \equiv \langle \mathrm{lft}(z), \mathrm{rht}(z)\rangle \wedge (\mathrm{lft}(z)\ \mathbf{r}\ A) \wedge (\mathrm{rht}(z)\ \mathbf{r}\ B)).$$

Case A c.r. and B n.c. Recall $\mathrm{AndL}_{X^c,Y^{\mathrm{nc}}} := \mu_{Z^c}(X^c \to Y^{\mathrm{nc}} \to Z^c)$. Then

$$\mathrm{AndL}^{\mathbf{r}}_{X^{\mathbf{r}},Y^{\mathrm{nc}}} := \mu_{Z^{\mathbf{r}}}(\forall_z(z\ \mathbf{r}\ X \to Y^{\mathrm{nc}} \to z \in Z^{\mathbf{r}})).$$

Now substituting $X^{\mathbf{r}}$ by $\{\, z \mid z\ \mathbf{r}\ A \,\}$ and Y^{nc} by B in the introduction axiom gives

$$(\mathrm{AndL}^{\mathbf{r}}_{\{z|z\mathbf{r}A\},B})^+_0 \colon \forall_z((z\ \mathbf{r}\ A) \to B \to z\ \mathbf{r}\ (A \wedge B)).$$

This suffices for "$\leftarrow$". Conversely, the elimination axiom $(\mathrm{AndL}^{\mathbf{r}}_{X,Y^{\mathrm{nc}}})^-$ is

$$\forall_z(z\ \mathbf{r}\ X \to Y^{\mathrm{nc}} \to z \in Z) \to \mathrm{AndL}^{\mathbf{r}}_{X,Y^{\mathrm{nc}}} \subseteq Z.$$

Substitute Z by $\{\, z \mid (z\ \mathbf{r}\ A) \wedge B \,\}$. Then with $\{\, z \mid z\ \mathbf{r}\ A \,\}$ for X and B for Y^{nc} the premise is provable and we obtain

$$\forall_z(z\ \mathbf{r}\ (A \wedge B) \to (z\ \mathbf{r}\ A) \wedge B). \qquad \square$$

Recall that for the sum type $\rho+\sigma$ we had the constructors $(\mathrm{InL}_{\rho\sigma})^{\rho\to\rho+\sigma}$ and $(\mathrm{InR}_{\rho\sigma})^{\sigma\to\rho+\sigma}$. In the special situation that one of the two parameter types is the unit type $\mathbb{U}$ it is common to view the sum type $\mathbb{U}+\sigma$ as a unary algebra form, with constructors DummyL of type $\mathbb{U}+\sigma$ and Inr of type $\sigma \to \mathbb{U}+\sigma$. Similarly $\rho+\mathbb{U}$ is viewed as a unary algebra, with constructors Inl of type $\rho \to \rho+\mathbb{U}$ and DummyR of type $\rho+\mathbb{U}$.

LEMMA 4.1.5 (Realizers for $\vee$). $z\ \mathbf{r}\ (A \vee B)$ *is equivalent to*

$\exists_x(x\ \mathbf{r}\ A \wedge z \equiv \mathrm{InL}(x)) \vee^{\mathrm{nc}} \exists_y(y\ \mathbf{r}\ B \wedge z \equiv \mathrm{InR}(y))$ *for* A, B *c.r.*

$\exists_x(x\ \mathbf{r}\ A \wedge z \equiv \mathrm{Inl}(x)) \vee^{\mathrm{nc}} (B \wedge z \equiv \mathrm{DummyR})$ *for* A *c.r. and* B *n.c.*

$(A \wedge z \equiv \mathrm{DummyL}) \vee^{\mathrm{nc}} \exists_y(y\ \mathbf{r}\ B \wedge z \equiv \mathrm{Inr}(y))$ *for* A *n.c. and* B *c.r.*

$(A \wedge z \equiv \mathtt{tt}) \vee^{\mathrm{nc}} (B \wedge z \equiv \mathtt{ff})$ *for* A, B *n.c.*

PROOF. As an example we consider the case A n.c. and B c.r. Recall $\mathrm{OrR}_{X^{\mathrm{nc}},Y^c} := \mu_Z(X^{\mathrm{nc}} \to Z, Y^c \to Z)$. Then

$$\mathrm{OrR}^{\mathbf{r}}_{X^{\mathrm{nc}},Y^{\mathbf{r}}} := \mu_{Z^{\mathbf{r}}}(X^{\mathrm{nc}} \to \mathrm{DummyL} \in Z^{\mathbf{r}}, \forall_y(y\ \mathbf{r}\ Y \to \mathrm{Inr}(y) \in Z^{\mathbf{r}})).$$

Now substituting X^{nc} by A and $Y^{\mathbf{r}}$ by $\{\, y \mid y\ \mathbf{r}\ B \,\}$ in the introduction axioms gives

$$(\mathrm{OrR}^{\mathbf{r}}_{A,\{y|y\mathbf{r}B\}})^+_0 \colon A \to \mathrm{DummyL}\ \mathbf{r}\ (A \vee B),$$
$$(\mathrm{OrR}^{\mathbf{r}}_{A,\{y|y\mathbf{r}B\}})^+_1 \colon \forall_y(y\ \mathbf{r}\ B \to \mathrm{Inr}(y)\ \mathbf{r}\ (A \vee B)).$$

This suffices for "$\leftarrow$": if $A \wedge z \equiv \mathrm{DummyL}$, then from $(\mathrm{OrR}^{\mathbf{r}}_{A,\{y|y\mathbf{r}B\}})^+_0$ we obtain $z\ \mathbf{r}\ (A \vee B)$, and if we have y with $y\ \mathbf{r}\ B$ and $z \equiv \mathrm{Inr}(y)$, then from $(\mathrm{OrR}^{\mathbf{r}}_{A,\{y|y\mathbf{r}B\}})^+_1$ we again obtain $z\ \mathbf{r}\ (A \vee B)$.

Conversely, the elimination axiom $(\mathrm{OrR}^{\mathbf{r}}_{X^{\mathrm{nc}},Y^{\mathbf{r}}})^-$ is

$$(X^{\mathrm{nc}} \to \mathrm{DummyL} \in Z) \to \forall_y(y\ \mathbf{r}\ Y \to \mathrm{Inr}(y) \in Z) \to \mathrm{OrR}^{\mathbf{r}}_{X^{\mathrm{nc}},Y^{\mathbf{r}}} \subseteq Z.$$

Substitute Z by $\{\, z \mid (A \wedge z \equiv \mathrm{DummyL}) \vee^{\mathrm{nc}} \exists_y(y\ \mathbf{r}\ B \wedge z \equiv \mathrm{Inr}(y)) \,\}$. Then with A for X^{nc} and $\{\, y \mid y\ \mathbf{r}\ B \,\}$ for $Y^{\mathbf{r}}$ the two premises become provable and we obtain

$$\forall_z(z\ \mathbf{r}\ (A \vee B) \to (A \wedge z \equiv \mathrm{DummyL}) \vee^{\mathrm{nc}} \exists_y(y\ \mathbf{r}\ B \wedge z \equiv \mathrm{Inr}(y))). \quad \square$$

4.2. Extracted terms, soundness

Let M be a proof in TCF of a c.r. formula A. Assume M is an $\mathbf{r}$-free proof, i.e., M contains no realizability predicates $I^{\mathbf{r}}$ or ${}^{\mathrm{co}}I^{\mathbf{r}}$. We define its *extracted term* $\mathrm{et}(M)$, of type $\tau(A)$, with the aim to express M's computational content. It will be a term built up from variables, constructors, recursion operators, destructors and corecursion operators by λ-abstraction and application.

Definition (Extracted term). For an $\mathbf{r}$-free proof M of a c.r. formula A we define its extracted term $\mathrm{et}(M)$ by

$$\begin{aligned}
\mathrm{et}(u^A) &:= z_u^{\tau(A)} \quad (z_u^{\tau(A)} \text{ uniquely associated to } u^A),\\
\mathrm{et}((\lambda_{u^A} M^B)^{A\to B}) &:= \begin{cases} \lambda_{z_u}\mathrm{et}(M) & \text{if } A \text{ is c.r.}\\ \mathrm{et}(M) & \text{if } A \text{ is n.c.}\end{cases}\\
\mathrm{et}((M^{A\to B} N^A)^B) &:= \begin{cases} \mathrm{et}(M)\mathrm{et}(N) & \text{if } A \text{ is c.r.}\\ \mathrm{et}(M) & \text{if } A \text{ is n.c.}\end{cases}\\
\mathrm{et}((\lambda_x M^A)^{\forall_x A}) &:= \mathrm{et}(M),\\
\mathrm{et}((M^{\forall_x A(x)} t)^{A(t)}) &:= \mathrm{et}(M).
\end{aligned}$$

It remains to define extracted terms for the axioms. Consider a (c.r.) inductively defined predicate I. For its introduction and elimination axioms define $\mathrm{et}(I_i^+) := \mathrm{C}_i$ and $\mathrm{et}(I^-) := \mathcal{R}$, where both the constructor C_i and the recursion operator $\mathcal{R}$ refer to the algebra ι_I associated with I. For the closure and greatest-fixed-point axioms of ${}^{\mathrm{co}}I$ define $\mathrm{et}({}^{\mathrm{co}}I^-) := \mathrm{D}$ and $\mathrm{et}({}^{\mathrm{co}}I_i^+) := {}^{\mathrm{co}}\mathcal{R}$, where both the destructor D and the corecursion operator ${}^{\mathrm{co}}\mathcal{R}$ refer to the cotype ${}^{\mathrm{co}}\iota_I$ where ι_I is the algebra associated with I. For the elimination axiom $(I^{\mathrm{nc}})^-$ of a one-clause-nc inductive predicate with a c.r. competitor predicate the extracted term is the identity.

From the Soundness Theorem 4.2.6 below it will follow that the term extracted from a closed $\mathbf{r}$-free proof of a c.r. formula A realizes A. As a preparation we first attend the axioms. Let I be an inductive predicate and ι_I its associated algebra. One can show that the extracted term of $I^{\pm}$, ${}^{\mathrm{co}}I^{\pm}$ realizes the respective axiom.

For the first two claims we only consider the inductive predicate $\sim_{\mathbb{L}}$ with $\mathbb{L}$ the algebra of lists of signed digits.

LEMMA 4.2.1. *The constructors of* $\mathbb{L}$ *realize the clauses of* $\sim_{\mathbb{L}}$.

PROOF. We only consider the second constructor ::. We must show that :: realizes the following formula C equivalent to $(\sim_{\mathbb{L}})^+_1$:

$$\forall_{s_1,s_2}(s_1 \sim_{\mathbb{D}} s_2 \to \forall_{\ell_1,\ell_2}(\ell_1 \sim_{\mathbb{L}} \ell_2 \to s_1 :: \ell_1 \sim_{\mathbb{L}} s_2 :: \ell_2))$$

i.e., $:: \mathbf{r}\ C$. Pick s_1, s_2. The goal then is

$$:: \mathbf{r}\ (s_1 \sim_{\mathbb{D}} s_2 \to \forall_{\ell_1,\ell_2}(\ell_1 \sim_{\mathbb{L}} \ell_2 \to s_1 :: \ell_1 \sim_{\mathbb{L}} s_2 :: \ell_2)).$$

Pick s with $\sim^{\mathbf{r}}_{\mathbb{D}}(s_1, s_2, s)$. The goal then is

$$:: s\ \mathbf{r}\ \forall_{\ell_1,\ell_2}(\ell_1 \sim_{\mathbb{L}} \ell_2 \to s_1 :: \ell_1 \sim_{\mathbb{L}} s_2 :: \ell_2).$$

Pick ℓ_1, ℓ_2, ℓ with $\sim^{\mathbf{r}}_{\mathbb{L}}(\ell_1, \ell_2, \ell)$. The goal then is

$$(s :: \ell)\ \mathbf{r}\ (s_1 :: \ell_1 \sim_{\mathbb{L}} s_2 :: \ell_2), \quad \text{i.e.,}$$
$$\sim^{\mathbf{r}}_{\mathbb{L}}\ (s_1 :: \ell_1, s_2 :: \ell_2, s :: \ell).$$

But this follows from what we have by the second clause of $\sim^{\mathbf{r}}_{\mathbb{L}}$:

$$\forall_{s_1,s_2,s,\ell_1,\ell_2,\ell}(\sim^{\mathbf{r}}_{\mathbb{D}}(s_1, s_2, s) \to \sim^{\mathbf{r}}_{\mathbb{L}}(\ell_1, \ell_2, \ell) \to \sim^{\mathbf{r}}_{\mathbb{L}}(s_1 :: \ell_1, s_2 :: \ell_2, s :: \ell)). \quad \square$$

LEMMA 4.2.2. *The recursion operator* $\mathcal{R}^{\alpha}_{\mathbb{L}}$ *realizes the least-fixed-point axiom* $\sim^{-}_{\mathbb{L}}$.

PROOF. We equivalently rewrite $\sim^{-}_{\mathbb{L}}$ as $C :=$

$$\begin{aligned}
&\forall_{\ell_1,\ell_2}(\ell_1 \sim_{\mathbb{L}} \ell_2 \to \\
&\quad X[][] \to \\
&\quad \forall_{s_1,s_2,\ell_1,\ell_2}(s_1 \sim_{\mathbb{D}} s_2 \to \ell_1 \sim_{\mathbb{L}} \ell_2 \to X\ell_1\ell_2 \to X(s_1 :: \ell_1, s_2 :: \ell_2)) \to \\
&\quad X\ell_1\ell_2)
\end{aligned}$$

to make its type the same as the one for $\mathcal{R}^{\alpha}_{\mathbb{L}}$:

$$\mathbb{L} \to \alpha \to (\mathbb{D} \to \mathbb{L} \to \alpha \to \alpha) \to \alpha.$$

We must show $\mathcal{R}^{\alpha}_{\mathbb{L}}\ \mathbf{r}\ C$. Pick ℓ_1, ℓ_2. The goal then is

$$\begin{aligned}
&\mathcal{R}^{\alpha}_{\mathbb{L}}\ \mathbf{r}\ (\ell_1 \sim_{\mathbb{L}} \ell_2 \to \\
&\quad X[][] \to \\
&\quad \forall_{s_1,s_2,\ell_1,\ell_2}(s_1 \sim_{\mathbb{D}} s_2 \to \ell_1 \sim_{\mathbb{L}} \ell_2 \to X\ell_1\ell_2 \to X(s_1 :: \ell_1, s_2 :: \ell_2)) \to \\
&\quad X\ell_1\ell_2).
\end{aligned}$$

Pick ℓ with $\sim^{\mathbf{r}}_{\mathbb{L}}(\ell_1, \ell_2, \ell)$. Then the goal is

$$\begin{aligned}
&\mathcal{R}^{\alpha}_{\mathbb{L}}\ell\ \mathbf{r}\ (X[][] \to \\
&\quad \forall_{s_1,s_2,\ell_1,\ell_2}(s_1 \sim_{\mathbb{D}} s_2 \to \ell_1 \sim_{\mathbb{L}} \ell_2 \to X\ell_1\ell_2 \to X(s_1 :: \ell_1, s_2 :: \ell_2)) \to \\
&\quad X\ell_1\ell_2).
\end{aligned}$$

Pick x with $X^{\mathbf{r}}[][]x$. Then the goal is

$$\mathcal{R}^{\alpha}_{\mathbb{L}}\ell x\,\mathbf{r}\,(\forall_{s_1,s_2,\ell_1,\ell_2}(s_1 \sim_{\mathbb{D}} s_2 \to \ell_1 \sim_{\mathbb{L}} \ell_2 \to X\ell_1\ell_2 \to X(s_1 :: \ell_1, s_2 :: \ell_2)) \to X\ell_1\ell_2).$$

Pick f with

$$f\,\mathbf{r}\,\forall_{s_1,s_2,\ell_1,\ell_2}(s_1 \sim_{\mathbb{D}} s_2 \to \ell_1 \sim_{\mathbb{L}} \ell_2 \to X\ell_1\ell_2 \to X(s_1 :: \ell_1, s_2 :: \ell_2)),$$

which implies

$$\forall_{s_1,s_2,s,\ell_1,\ell_2,\ell,y}(\sim_{\mathbb{D}}(s_1,s_2,s) \to \sim_{\mathbb{L}}(\ell_1,\ell_2,\ell) \to X^{\mathbf{r}}\ell_1\ell_2 y \to X^{\mathbf{r}}(s_1 :: \ell_1, s_2 :: \ell_2, fs\ell y)).$$

Our goal is

$$X^{\mathbf{r}}(\ell_1,\ell_2,\mathcal{R}^{\alpha}_{\mathbb{L}}\ell x f) =: Q\ell_1\ell_2\ell.$$

To this end we use the elimination axiom for $\sim^{\mathbf{r}}_{\mathbb{L}}$:

$$\begin{aligned}\forall_{\ell_1,\ell_2,\ell}(&\sim^{\mathbf{r}}_{\mathbb{L}}(\ell_1,\ell_2,\ell) \to\\ &Q[][][] \to\\ &\forall_{s_1,s_2,s,\ell_1,\ell_2,\ell}(\sim^{\mathbf{r}}_{\mathbb{D}}(s_1,s_2,s) \to \sim^{\mathbf{r}}_{\mathbb{L}}(\ell_1,\ell_2,\ell) \to Q\ell_1\ell_2\ell \to\\ &\qquad Q(s_1 :: \ell_1, s_2 :: \ell_2, s :: \ell)) \to\\ &Q\ell_1\ell_2\ell).\end{aligned}$$

It suffices to prove the premises $Q[][][]$ and $\forall_{s_1,s_2,s,\ell_1,\ell_2,\ell}(\sim^{\mathbf{r}}_{\mathbb{D}}(s_1,s_2,s) \to \sim^{\mathbf{r}}_{\mathbb{L}}(\ell_1,\ell_2,\ell) \to Q\ell_1\ell_2\ell \to Q(s_1 :: \ell_1, s_2 :: \ell_2, s :: \ell))$. By a computation rule for $\mathcal{R}^{\alpha}_{\mathbb{L}}$ the former is $X^{\mathbf{r}}[][]x$, which we have. For the latter assume s_1, s_2, s, ℓ_1, ℓ_2, ℓ and its premises. We show $Q(s_1 :: \ell_1, s_2 :: \ell_2, s :: \ell)$, i.e.,

$$X^{\mathbf{r}}(s_1 :: \ell_1, s_2 :: \ell_2, \mathcal{R}^{\alpha}_{\mathbb{L}}(s :: \ell)xf).$$

By the computation rules for $\mathcal{R}^{\alpha}_{\mathbb{L}}$ this is the same as

$$X^{\mathbf{r}}(s_1 :: \ell_1, s_2 :: \ell_2, fs\ell(\mathcal{R}^{\alpha}_{\mathbb{L}}\ell xf)).$$

But with $y := \mathcal{R}^{\alpha}_{\mathbb{L}}\ell xf$ this follows from what we have. □

For the final two claims we only consider the coinductive predicate $\approx_{\mathbb{S}}$.

Lemma 4.2.3. *The destructor* $\mathrm{D}_{\mathbb{S}}$ *realizes the closure axiom* $\approx^{-}_{\mathbb{S}}$.

Proof. Recall $\approx^{-}_{\mathbb{S}}$:

$$\begin{aligned}\forall_{u_1,u_2}(&u_1 \approx_{\mathbb{S}} u_2 \to\\ &\exists_{s_1,s_2,u_1',u_2'}(s_1 \sim_{\mathbb{D}} s_2 \wedge u_1' \approx_{\mathbb{S}} u_2' \wedge u_1 \equiv s_1 :: u_1' \wedge u_2 \equiv s_2 :: u_2'))\end{aligned}$$

with cotype ${}^{\mathrm{co}}\mathbb{S} \to \mathbb{D} \times {}^{\mathrm{co}}\mathbb{S}$. The goal is $\mathrm{D}_{\mathbb{S}}\,\mathbf{r} \approx^{-}_{\mathbb{S}}$, which unfolds into

$$\begin{aligned}\forall_{u_1,u_2,u}(&\approx^{\mathbf{r}}_{\mathbb{S}}(u_1,u_2,u) \to \mathrm{D}_{\mathbb{S}}u\,\mathbf{r}\,\exists_{s_1,s_2,u_1',u_2'}(s_1 \sim_{\mathbb{D}} s_2 \wedge u_1' \approx_{\mathbb{S}} u_2' \wedge\\ &\qquad u_1 \equiv s_1 :: u_1' \wedge u_2 \equiv s_2 :: u_2')).\end{aligned}$$

Assume $\approx^{\mathbf{r}}_{\mathbb{S}}(u_1, u_2, u)$. We need to prove

$$\exists_{s_1,s_2,u_1',u_2'}(\mathrm{D}_{\mathbb{S}}u\ \mathbf{r}\ (s_1 \sim_{\mathbb{D}} s_2 \land u_1' \approx_{\mathbb{S}} u_2') \land u_1 \equiv s_1 :: u_1' \land u_2 \equiv s_2 :: u_2').$$

By $(\approx^{\mathbf{r}}_{\mathbb{S}})^-$ from $\approx^{\mathbf{r}}_{\mathbb{S}}(u_1, u_2, u)$ we obtain s_1, s_2, s, u_1', u_2', u' such that

$$\sim^{\mathbf{r}}_{\mathbb{D}}(s_1, s_2, s) \land \approx^{\mathbf{r}}_{\mathbb{S}}(u_1', u_2', u') \land u_1 \equiv s_1 :: u_1' \land u_2 \equiv s_2 :: u_2') \land u \equiv s :: u'.$$

Take s_1, s_2, u_1', u_2'. It remains to show $\mathrm{D}_{\mathbb{S}}u\ \mathbf{r}\ (s_1 \sim_{\mathbb{D}} s_2 \land u_1' \approx_{\mathbb{S}} u_2')$. By the computation rule of $\mathrm{D}_{\mathbb{S}}$ we know $\mathrm{D}_{\mathbb{S}}u \equiv \mathrm{D}_{\mathbb{S}}(s :: u') \equiv \langle s, u' \rangle$. Hence we must prove $\sim^{\mathbf{r}}_{\mathbb{D}}(s_1, s_2, s)$ and $\approx^{\mathbf{r}}_{\mathbb{S}}(u_1, u_2, u)$, which we both have. □

Lemma 4.2.4. *The corecursion operator* ${}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}$ *realizes the greatest-fixed-point axiom* $\approx^{+}_{\mathbb{S}}$.

Proof. We equivalently rewrite $\approx^{+}_{\mathbb{S}}$ as $C :=$

$$\begin{aligned}
&\forall_{u_1,u_2}(Xu_1u_2 \to \\
&\qquad \forall_{u_1,u_2}(Xu_1u_2 \to \exists_{s_1,s_2,u_1',u_2'}(s_1 \sim_{\mathbb{D}} s_2 \land (u_1' \approx_{\mathbb{S}} u_2' \lor Xu_1'u_2') \land \\
&\qquad\qquad u_1 \equiv s_1 :: u_1' \land u_2 \equiv s_2 :: u_2')) \to \\
&\qquad u_1 \approx_{\mathbb{S}} u_2)
\end{aligned}$$

to make its cotype the same as the one for ${}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}$:

$$\alpha \to (\alpha \to \mathbb{D} \times ({}^{\mathrm{co}}\mathbb{S} + \alpha)) \to {}^{\mathrm{co}}\mathbb{S}.$$

We must show that ${}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}$ realizes C, or more formally ${}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}\ \mathbf{r}\ C$. The goal then is

$$\begin{aligned}
&{}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}\ \mathbf{r}\ (Xu_1u_2 \to \\
&\qquad \forall_{u_1,u_2}(Xu_1u_2 \to \exists_{s_1,s_2,u_1',u_2'}(s_1 \sim_{\mathbb{D}} s_2 \land (u_1' \approx_{\mathbb{S}} u_2' \lor Xu_1'u_2') \land \\
&\qquad\qquad u_1 \equiv s_1 :: u_1' \land u_2 \equiv s_2 :: u_2')) \to \\
&\qquad u_1 \approx_{\mathbb{S}} u_2).
\end{aligned}$$

Pick u with $X^{\mathbf{r}}u_1u_2u$. Then the goal is

$$\begin{aligned}
&{}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}u\ \mathbf{r}\ \forall_{u_1,u_2}(Xu_1u_2 \to \exists_{s_1,s_2,u_1',u_2'}(s_1 \sim_{\mathbb{D}} s_2 \land (u_1' \approx_{\mathbb{S}} u_2' \lor Xu_1'u_2') \land \\
&\qquad\qquad u_1 \equiv s_1 :: u_1' \land u_2 \equiv s_2 :: u_2')) \to \\
&\qquad u_1 \approx_{\mathbb{S}} u_2).
\end{aligned}$$

Pick f such that

$$\begin{aligned}
f\ \mathbf{r}\ \forall_{u_1,u_2}(Xu_1u_2 \to \exists_{s_1,s_2,u_1',u_2'}(s_1 \sim_{\mathbb{D}} s_2 \land (u_1' \approx_{\mathbb{S}} u_2' \lor Xu_1'u_2') \land \\
u_1 \equiv s_1 :: u_1' \land u_2 \equiv s_2 :: u_2'))
\end{aligned}$$

i.e.,

$$\forall_{u_1,u_2,u}(X^{\mathbf{r}}u_1u_2u \to fu\ \mathbf{r}\ \exists_{s_1,s_2,u_1',u_2'}(s_1 \sim_{\mathbb{D}} s_2 \land (u_1' \approx_{\mathbb{S}} u_2' \lor Xu_1'u_2') \land u_1 \equiv s_1 :: u_1' \land u_2 \equiv s_2 :: u_2')).$$

Our goal is

$$\approx^{\mathbf{r}}_{\mathbb{S}}(u_1, u_2, {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}uf).$$

To this end we use the greatest-fixed-point axiom for $\approx^{\mathbf{r}}_{\mathbb{S}}$ in the form

$$\begin{aligned}&\forall_{u_1,u_2,u}(Qu_1u_2u \to\\ &\quad \forall_{u_1,u_2,u}(Qu_1u_2u \to\\ &\qquad \exists_{s_1,s_2,s,u_1',u_2',u'}(\sim^{\mathbf{r}}_{\mathbb{D}}(s_1,s_2,s) \land (\approx^{\mathbf{r}}_{\mathbb{S}}(u_1',u_2',u') \lor Qu_1u_2u') \land\\ &\qquad\qquad u_1 \equiv s_1{::}u_1' \land u_2 \equiv s_2{::}u_2') \land u \equiv s{::}u')) \to\\ &\quad \approx^{\mathbf{r}}_{\mathbb{S}}(u_1,u_2,u))\end{aligned}$$

with

$$\exists_{z'}(X^{\mathbf{r}}u_1u_2z' \land u \equiv {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}z'f) =: Qu_1u_2u.$$

It suffices to prove the closure property of Q. Let u_1, u_2, u and also u' be given such that

$$X^{\mathbf{r}}u_1u_2u' \land u \equiv {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}u'f.$$

We need to show

$$\begin{aligned}&\exists_{s_1,s_2,s,u_1',u_2',u'}(\\ &\sim^{\mathbf{r}}_{\mathbb{D}}(s_1,s_2,s) \land (\approx^{\mathbf{r}}_{\mathbb{S}}(u_1',u_2',u') \lor \exists_{u'}(X^{\mathbf{r}}u_1u_2u' \land u \equiv {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}u'f)) \land\\ &u_1 \equiv s_1 :: u_1' \land u_2 \equiv s_2 :: u_2' \land u \equiv s :: u').\end{aligned} \tag{16}$$

First note that $\sim^{\mathbf{r}}_{\mathbb{D}}(s_1,s_2,s)$ is equivalent to $s_1 \equiv s_2 \equiv s$. Since $X^{\mathbf{r}}u_1u_2u'$ we know

$$fu'\ \mathbf{r}\ \exists_{s_1,s_2,u_1',u_2'}(s_1 \sim_{\mathbb{D}} s_2 \land (u_1' \approx_{\mathbb{S}} u_2' \lor Xu_1'u_2') \land u_1 \equiv s_1 :: u_1' \land u_2 \equiv s_2 :: u_2').$$

Then $fu' \equiv \langle s, w\rangle$ with $\sim^{\mathbf{r}}_{\mathbb{D}}(s_1,s_2,s)$ and $w\ \mathbf{r}\ (u_1' \approx_{\mathbb{S}} u_2' \lor Xu_1'u_2')$, for some s_1, s_2, u_1', u_2' such that $u_1 \equiv s_1 :: u_1'$ and $u_2 \equiv s_2 :: u_2'$. Hence

$$\exists_{u'}(\approx^{\mathbf{r}}_{\mathbb{S}}(u_1',u_2',u') \land w \equiv \mathrm{InL}(u')) \lor \exists_{u''}(X^{\mathbf{r}}u_1'u_2'u'' \land w \equiv \mathrm{InR}(u'')).$$

We distinguish cases on this disjunction. Recall

$${}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}uf \equiv \begin{cases} s :: u & \text{if } fu \equiv \langle s, \mathrm{InL}(u)\rangle,\\ s :: {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}u'f & \text{if } fu \equiv \langle s, \mathrm{InR}(u')\rangle. \end{cases}$$

Case L. $\approx^{\mathbf{r}}_{\mathbb{S}}(u_1',u_2',u') \land w \equiv \mathrm{InL}(u')$ for some u'. Then (16) holds, since $u \equiv {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}u'f \equiv s :: u'$.

Case R. $X^{\mathbf{r}}u_1'u_2'u'' \land w \equiv \mathrm{InR}(u'')$ for some u''. Then again (16) holds with $u' := {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}u''f$, since $u \equiv {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}u'f \equiv s :: {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}u''f \equiv s :: u'$. □

LEMMA 4.2.5 (Identities realize I^- for one-clause-nc I). *For one-clause-nc inductive predicates the elimination axiom with a c.r. competitor predicate is realized by the identity.*

PROOF. For $\vec{A}$ n.c. we have

$$(\lambda_z z)\ \mathbf{r}\ \forall_{\vec{x}}(I\vec{x} \to \forall_{\vec{y}}(\vec{A} \to X\vec{t}\,) \to X\vec{x}\,)$$
$$\forall_{\vec{x}}(I\vec{x} \to (\lambda_z z)\ \mathbf{r}\ (\forall_{\vec{y}}(\vec{A} \to X\vec{t}\,) \to X\vec{x}\,))$$
$$\forall_{\vec{x}}(I\vec{x} \to \forall_z(z\ \mathbf{r}\ \forall_{\vec{y}}(\vec{A} \to X\vec{t}\,) \to z\ \mathbf{r}\ X\vec{x}\,))$$
$$\forall_{\vec{x}}(I\vec{x} \to \forall_z(\forall_{\vec{y}}(\vec{A} \to z\ \mathbf{r}\ X\vec{t}\,) \to z\ \mathbf{r}\ X\vec{x}\,))$$

which is an instance of the same elimination axiom. □

THEOREM 4.2.6 (Soundness). *Let M be an $\mathbf{r}$-free derivation of a formula A from assumptions $u_i\colon C_i$ $(i < n)$. Then we can derive*

$$\begin{cases} \mathrm{et}(M)\ \mathbf{r}\ A & \text{if } A \text{ is c.r.} \\ A & \text{if } A \text{ is n.c.} \end{cases}$$

from assumptions

$$\begin{cases} z_{u_i}\ \mathbf{r}\ C_i & \text{if } C_i \text{ is c.r.} \\ C_i & \text{if } C_i \text{ is n.c.} \end{cases}$$

PROOF. *Case* $u\colon A$. *Subcase* A c.r. Then $\mathrm{et}(u) = z_u$. *Subcase* A n.c. Immediate.

Case $c\colon A$. *Subcase* A c.r. The axioms have been treated above. *Subcase* A n.c. Immediate.

Case $(\lambda_{u^A} M^B)^{A\to B}$ with B c.r. We must derive $\mathrm{et}(\lambda_u M)\ \mathbf{r}\ (A \to B)$. To this end we distinguish subcases. *Subcase* A c.r. Then the goal

$$\forall_z(z\ \mathbf{r}\ A \to \mathrm{et}(M)(z)\ \mathbf{r}\ B)$$

follows from the induction hypothesis by $\to^+$ and $\forall^+$.

Subcase A n.c. Then the goal is

$$A \to \mathrm{et}(\lambda_u M)\ \mathbf{r}\ B.$$

Recall that $\mathrm{et}(\lambda_u M) = \mathrm{et}(M)$. By induction hypothesis we have a derivation of $\mathrm{et}(M)\ \mathbf{r}\ B$ from A, which is what we want.

Case $(\lambda_{u^A} M^B)^{A\to B}$ with B n.c. We need a derivation of $A \to B$.

Subcase A c.r. By induction hypothesis we have a derivation of B from $z\ \mathbf{r}\ A$. Using the invariance axiom $A \to \exists_z(z\ \mathbf{r}\ A)$ we obtain the required

derivation of B from A as follows.

$$\dfrac{\dfrac{A \to \exists_z(z\ \mathbf{r}\ A) \qquad A}{\exists_z(z\ \mathbf{r}\ A)} \qquad \begin{array}{c}[z\ \mathbf{r}\ A]\\ \mid \text{IH}\\ B\end{array}}{B}\ \exists^-$$

Subcase A n.c. By induction hypothesis we have a derivation of B from A, which is what we want.

Case $(M^{A\to B}N^A)^B$ with B c.r. We need a derivation of $\mathrm{et}(MN)\ \mathbf{r}\ B$. To this end we distinguish subcases. *Subcase* A c.r. Then $\mathrm{et}(MN) = \mathrm{et}(M)\mathrm{et}(N)$. By induction hypothesis we have derivations of $\mathrm{et}(M)\ \mathbf{r}\ (A \to B)$ and hence of

$$\forall_z(z\ \mathbf{r}\ A\ \to\ \mathrm{et}(M)z\ \mathbf{r}\ B)$$

and of $\mathrm{et}(N)\ \mathbf{r}\ A$. This gives the claim. *Subcase* A n.c. Then $\mathrm{et}(MN) = \mathrm{et}(M)$. By induction hypothesis we have derivations of $\mathrm{et}(M)\ \mathbf{r}\ (A \to B)$ and hence of

$$A\ \to\ \mathrm{et}(M)\ \mathbf{r}\ B$$

and of A. Applying the former to the latter gives $\mathrm{et}(M)\ \mathbf{r}\ B$.

Case $(M^{A\to B}N^A)^B$ with B n.c. The goal is to find a derivation of B. *Subcase* A c.r. By induction hypothesis we have derivations of $A \to B$ and of $\mathrm{et}(N)\ \mathbf{r}\ A$. Now using the invariance axiom $\forall_z(z\ \mathbf{r}\ A \to A)$ we obtain the required derivation of B by $\to^-$ from the derivation of $A \to B$ and

$$\dfrac{\dfrac{\forall_z(z\ \mathbf{r}\ A \to A) \qquad \mathrm{et}(N)}{\mathrm{et}(N)\ \mathbf{r}\ A \to A} \qquad \begin{array}{c}\mid \text{IH}\\ \mathrm{et}(N)\ \mathbf{r}\ A\end{array}}{A}$$

Subcase A n.c. By induction hypothesis we have derivations of $A \to B$ and of A, hence also a derivation of B.

Case $(\lambda_x M^A)^{\forall_x A}$ with $\forall_x A$ c.r. We need a derivation of $\mathrm{et}(\lambda_x M)\ \mathbf{r}\ \forall_x A$. By definition $\mathrm{et}(\lambda_x M) = \mathrm{et}(M)$. Hence we must derive

$$\mathrm{et}(M)\ \mathbf{r}\ \forall_x A, \quad \text{which is} \quad \forall_x(\mathrm{et}(M)\ \mathbf{r}\ A).$$

This follows from the induction hypothesis.

Case $(\lambda_x M^A)^{\forall_x A}$ with $\forall_x A$ n.c. By induction hypothesis we have a derivation of A. Apply $\forall^+$.

Case $(M^{\forall_x A(x)}t)^{A(t)}$ with $A(t)$ c.r. We must derive $\mathrm{et}(Mt)\ \mathbf{r}\ A(t)$. By definition $\mathrm{et}(Mt) = \mathrm{et}(M)$, and by induction hypothesis we can derive

$$\mathrm{et}(M)\ \mathbf{r}\ \forall_x A(x), \quad \text{which is} \quad \forall_x(\mathrm{et}(M)\ \mathbf{r}\ A(x)).$$

Case $(M^{\forall_x A(x)}t)^{A(t)}$ with $A(t)$ n.c. By induction hypothesis we have a derivation of $\forall_x A(x)$. Apply $\forall^-$. □

4.3. Extensionality of extracted terms

Let I be an inductive predicate and ι_I its associated algebra. One can show that

- every constructor of ι_I is extensional w.r.t. its clause I_i^+,
- $\mathcal{R}^\alpha_{\iota_I}$ is extensional w.r.t. the least-fixed-point axiom I^-,
- the destructor of ι_I is extensional w.r.t. the closure axiom ${}^{\mathrm{co}}I^-$, and
- ${}^{\mathrm{co}}\mathcal{R}^\alpha_{\iota_I}$ is extensional w.r.t. the greatest-fixed-point axiom ${}^{\mathrm{co}}I^+$.

Since the term $\mathrm{et}(M)$ extracted from a closed proof M of a c.r. formula A is built from these constants by abstraction and application, by Lemma 3.3.7 on extensionality of terms we can conclude that $\mathrm{et}(M)$ is extensional w.r.t. the cotype of A.

We prove the claim above for a special case only, the algebras of lists and streams of "signed digits". Such objects are of interest for the representation of (dyadic) rational numbers and of real numbers.

Let $\sim_{\mathbb{D}}$ be the similarity relation for the three-element algebra $\mathbb{D}$ of signed digits 1, 0, -1 (written $\bar{1}$), defined by the three clauses $s \sim_{\mathbb{D}} s$ for s a signed digit. We will work with lists $\mathbb{L}(\mathbb{D})$ of signed digits and streams $\mathbb{S}(\mathbb{D})$ of signed digits, abbreviated $\mathbb{L}$ and $\mathbb{S}$. The similarity relation $\sim_{\mathbb{L}}$ has clauses

$$(\sim_{\mathbb{L}})^+_0 \colon [] \sim_{\mathbb{L}} [],$$
$$(\sim_{\mathbb{L}})^+_1 \colon \forall_{s_1,s_2,\ell_1,\ell_2}(s_1 \sim_{\mathbb{D}} s_2 \to \ell_1 \sim_{\mathbb{L}} \ell_2 \to s_1 :: \ell_1 \sim_{\mathbb{L}} s_2 :: \ell_2)$$

and the elimination axiom $\sim^-_{\mathbb{L}}$:

$$X[][] \to \forall_{s_1,s_2,\ell_1,\ell_2}(s_1 \sim_{\mathbb{D}} s_2 \to \ell_1 \sim_{\mathbb{L}} \ell_2 \to X\ell_1\ell_2 \to X(s_1 :: \ell_1, s_2 :: \ell_2)) \to \sim_{\mathbb{L}} \subseteq X.$$

For the first two claims we only consider the inductive predicate $\sim_{\mathbb{L}}$.

Lemma 4.3.1. *The constructors of $\mathbb{L}$ are extensional w.r.t. the clauses of $\sim_{\mathbb{L}}$.*

Proof. We only consider the second constructor C. The goal is to show that C is extensional w.r.t. the cotype $\mathbb{D} \to \mathbb{L} \to \mathbb{L}$ of $\sim_{\mathbb{L}}$'s second clause, which by definition of $\doteq$ means

$$\forall_{s_1,s_2,\ell_1,\ell_2}(s_1 \sim_{\mathbb{D}} s_2 \to \ell_1 \sim_{\mathbb{L}} \ell_2 \to s_1 :: \ell_1 \sim_{\mathbb{L}} s_2 :: \ell_2).$$

But this is the second clause of $\sim_{\mathbb{L}}$. □

Lemma 4.3.2. *$\mathcal{R}^\alpha_{\mathbb{L}}$ is extensional w.r.t. the least-fixed-point axiom $\sim^-_{\mathbb{L}}$.*

PROOF. We equivalently rewrite $\sim^-_{\mathbb{L}}$ as $C :=$

$$\begin{aligned}&\forall_{\ell_1,\ell_2}(\ell_1 \sim_{\mathbb{L}} \ell_2 \to\\ &\quad X[][] \to\\ &\quad \forall_{s_1,s_2,\ell_1,\ell_2}(s_1 \sim_{\mathbb{D}} s_2 \to \ell_1 \sim_{\mathbb{L}} \ell_2 \to X\ell_1\ell_2 \to X(s_1 :: \ell_1, s_2 :: \ell_2)) \to\\ &\quad X\ell_1\ell_2)\end{aligned}$$

to make its cotype the same as the one for $\mathcal{R}^\alpha_{\mathbb{L}}$:

$$\mathbb{L} \to \alpha \to (\mathbb{D} \to \mathbb{L} \to \alpha \to \alpha) \to \alpha.$$

We must show $\mathcal{R}^\alpha_{\mathbb{L}} \doteq_C \mathcal{R}^\alpha_{\mathbb{L}}$ with $\alpha := \varphi(X)$. By definition of $\doteq_C$ this is equivalent to

$$\begin{aligned}\forall_{x_1,x_2,f_1,f_2,\ell_1,\ell_2}(x_1 \doteq_\alpha x_2 \to f_1 \doteq_{\mathbb{D}\to\mathbb{L}\to\alpha\to\alpha} f_2 \to \ell_1 \sim_{\mathbb{L}} \ell_2 \to\\ \mathcal{R}^\alpha_{\mathbb{L}}\ell_1 x_1 f_1 \doteq_\alpha \mathcal{R}^\alpha_{\mathbb{L}}\ell_2 x_2 f_2).\end{aligned}$$

Assume $x_1 \doteq_\alpha x_2$ and $f_1 \doteq_{\mathbb{D}\to\mathbb{L}\to\alpha\to\alpha} f_2$. Use the least-fixed-point axiom $\sim^-_{\mathbb{L}}$ (in its original form) with competitor predicate

$$X := \{\, \ell_1, \ell_2 \mid \mathcal{R}^\alpha_{\mathbb{L}}\ell_1 x_1 f_1 \doteq_\alpha \mathcal{R}^\alpha_{\mathbb{L}}\ell_2 x_2 f_2 \,\}.$$

Case []. By the computation rules for $\mathcal{R}^\alpha_{\mathbb{L}}$ the claim $X[][]$ follows from $x_1 \doteq_\alpha x_2$.

Case ::. Assume $s_1 \sim_{\mathbb{D}} s_2$ and $f_1 \doteq_{\mathbb{D}\to\mathbb{L}\to\alpha\to\alpha} f_2$. Let $y_1 := \mathcal{R}^\alpha_{\mathbb{L}}\ell_1 x_1 f_1$ and $y_2 := \mathcal{R}^\alpha_{\mathbb{L}}\ell_2 x_2 f_2$. Then $y_1 \doteq_\alpha y_2$ by assumption. The goal $f_1 s_1 \ell_1 y_1 \doteq_\alpha f_2\ell_2 x_2 y_2$ follows by definition of $\doteq$ from $f_1 \doteq f_2$, $s_1 \sim s_2$, $\ell_1 \sim \ell_2$ and $y_1 \doteq_\alpha y_2$. □

The bisimilarity relation $\approx_{\mathbb{S}}$ is defined by the closure axiom

$$\begin{aligned}&\approx^-_{\mathbb{S}} : \forall_{u_1,u_2}(u_1 \approx_{\mathbb{S}} u_2 \to\\ &\qquad \exists_{s_1,s_2,u_1',u_2'}(s_1 \sim_{\mathbb{D}} s_2 \land u_1' \approx_{\mathbb{S}} u_2' \land u_1 \equiv s_1 :: u_1' \land u_2 \equiv s_2 :: u_2'))\end{aligned}$$

and the greatest-fixed-point axiom $\approx^+_{\mathbb{S}}$:

$$\begin{aligned}&\forall_{u_1,u_2}(Xu_1u_2 \to \exists_{s_1,s_2,u_1',u_2'}(s_1 \sim_{\mathbb{D}} s_2 \land (u_1' \approx_{\mathbb{S}} u_2' \lor Xu_1'u_2') \land\\ &\qquad\qquad u_1 \equiv s_1 :: u_1' \land u_2 \equiv s_2 :: u_2')) \to\\ &X \subseteq \approx_{\mathbb{S}}.\end{aligned}$$

For the final two claims we only consider the coinductive predicate $\approx_{\mathbb{S}}$.

LEMMA 4.3.3. *The destructor* $\mathrm{D}_{\mathbb{S}}$ *is extensional w.r.t. the closure axiom* $\approx^-_{\mathbb{S}}$.

PROOF. The closure axiom $\approx^-_{\mathbb{S}}$ has cotype ${}^{\mathrm{co}}\mathbb{S} \to \mathbb{D} \times {}^{\mathrm{co}}\mathbb{S}$. The goal is $\mathrm{D}_{\mathbb{S}} \doteq_{({}^{\mathrm{co}}\mathbb{S}\to\mathbb{D}\times{}^{\mathrm{co}}\mathbb{S})} \mathrm{D}_{\mathbb{S}}$, which unfolds into

$$\forall_{u_1,u_2}(u_1 \approx_{\mathbb{S}} u_2 \to \mathrm{D}_{\mathbb{S}}u_1 \sim_{\mathbb{D}\times{}^{\mathrm{co}}\mathbb{S}} \mathrm{D}_{\mathbb{S}}u_2).$$

Assume $u_1 \approx_{\mathbb{S}} u_2$. By $\approx^-_{\mathbb{S}}$ we obtain s_1, s_2, u_1', u_2' with

$$s_1 \sim_{\mathbb{D}} s_2 \land u_1' \approx_{\mathbb{S}} u_2' \land u_1 \equiv s_1 :: u_1' \land u_2 \equiv s_2 :: u_2'.$$

By the computation rule for $\mathrm{D}_{\mathbb{S}}$ we have $\mathrm{D}_{\mathbb{S}} u_i \equiv \langle s_i, u_i' \rangle$. By the clause for $\sim_{\mathbb{D}\times{}^{\mathrm{co}}\mathbb{S}}$ this implies the claim $\mathrm{D}_{\mathbb{S}} u_1 \sim_{\mathbb{D}\times{}^{\mathrm{co}}\mathbb{S}} \mathrm{D}_{\mathbb{S}} u_2$. □

Lemma 4.3.4. ${}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}$ *is extensional w.r.t. the greatest-fixed-point axiom* $\approx^+_{\mathbb{S}}$.

Proof. We equivalently rewrite $\approx^+_{\mathbb{S}}$ as

$$\begin{aligned}
&\forall_{u_1,u_2}(Xu_1u_2 \to \\
&\qquad \forall_{u_1,u_2}(Xu_1u_2 \to \exists_{s_1,s_2,u_1',u_2'}(s_1 \sim_{\mathbb{D}} s_2 \land (u_1' \approx_{\mathbb{S}} u_2' \lor Xu_1'u_2') \land \\
&\qquad\qquad\qquad\qquad u_1 \equiv s_1 :: u_1' \land u_2 \equiv s_2 :: u_2')) \to \\
&\qquad u_1 \approx_{\mathbb{S}} u_2)
\end{aligned}$$

to make its cotype the same as the one for ${}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}$:

$$\alpha \to (\alpha \to \mathbb{D} \times ({}^{\mathrm{co}}\mathbb{S} + \alpha)) \to {}^{\mathrm{co}}\mathbb{S}.$$

Call this cotype ψ. The goal is ${}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}} \doteq_{\psi} {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}}$, which unfolds into

$$\forall_{x_1,x_2}(x_1 \doteq_{\alpha} x_2 \to \forall_{f_1,f_2}(f_1 \doteq_{\alpha\to\mathbb{D}\times({}^{\mathrm{co}}\mathbb{S}+\alpha)} f_2 \to {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}} x_1 f_1 \approx_{\mathbb{S}} {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}} x_2 f_2)).$$

Assume $x_1 \doteq_{\alpha} x_2$ and $f_1 \doteq_{\alpha\to\mathbb{D}\times({}^{\mathrm{co}}\mathbb{S}+\alpha)} f_2$. Let $u_1 := {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}} x_1 f_1$ and $u_2 := {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}} x_2 f_2$. To prove the goal $u_1 \approx_{\mathbb{S}} u_2$ we use coinduction, or more precisely $\approx^+_{\mathbb{S}}$ with competitor predicate

$$X := \{\, u_1, u_2 \mid \exists_{y_1,y_2}(u_1 \equiv {}^{\mathrm{co}}\mathcal{R} y_1 f_1 \land u_2 \equiv {}^{\mathrm{co}}\mathcal{R} y_2 f_2 \land y_1 \doteq_{\alpha} y_2) \,\}.$$

This means that we have to show

$$\begin{aligned}
&\exists_{s_1,s_2,u_1',u_2'}(\\
&s_1 \sim_{\mathbb{D}} s_2 \land (u_1' \approx_{\mathbb{S}} u_2' \lor \exists_{y_1,y_2}(u_1' \equiv {}^{\mathrm{co}}\mathcal{R} y_1 f_1 \land u_2' \equiv {}^{\mathrm{co}}\mathcal{R} y_2 f_2 \land y_1 \doteq_{\alpha} y_2)) \land \\
&u_1 \equiv s_1 :: u_1' \land u_2 \equiv s_2 :: u_2').
\end{aligned}$$

From $x_1 \doteq_{\alpha} x_2$ and $f_1 \doteq_{\alpha\to\mathbb{D}\times({}^{\mathrm{co}}\mathbb{S}+\alpha)} f_2$ we obtain $f_1 x_1 \sim_{\mathbb{D}\times({}^{\mathrm{co}}\mathbb{S}+\alpha)} f_2 x_2$. By definition of $\sim_{\times}$ this implies the existence of s_1, s_2, a_1, a_2 with

$$f_1 x_1 \equiv \langle s_1, a_1 \rangle \land f_2 x_2 \equiv \langle s_2, a_2 \rangle \land s_1 \sim_{\mathbb{D}} s_2 \land a_1 \sim_{({}^{\mathrm{co}}\mathbb{S}+\alpha)} a_2,$$

and by definition of $\sim_+$ from $a_1 \sim_{({}^{\mathrm{co}}\mathbb{S}+\alpha)} a_2$ we obtain the disjunction

$$\begin{aligned}
&(a_1 \equiv \mathrm{InL}(u_1') \land a_2 \equiv \mathrm{InL}(u_2') \land u_1' \approx_{\mathbb{S}} u_2') \lor \\
&(a_1 \equiv \mathrm{InR}(x_1') \land a_2 \equiv \mathrm{InR}(x_2') \land x_1' \doteq_{\alpha} x_2').
\end{aligned}$$

We argue by cases on this disjunction. Recall

$${}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}} x f \equiv \begin{cases} s :: u & \text{if } fx \equiv \langle s, \mathrm{InL}(u) \rangle, \\ s :: {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}} x' f & \text{if } fx \equiv \langle s, \mathrm{InR}(x') \rangle. \end{cases}$$

Case L. Then we have s_1, s_2, u_1', u_2' with $s_1 \sim_{\mathbb{D}} s_2$ and $u_1' \approx_{\mathbb{S}} u_2'$ such that $f_i x_i \equiv \langle s_i, \mathrm{InL}(u_i')\rangle$. Hence $u_i := {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}} x_i f_i \equiv s_i :: u_i'$, and the claim follows.

Case R. Then we have s_1, s_2, x_1', x_2' with $s_1 \sim_{\mathbb{D}} s_2$ and $x_1' \doteq_{\alpha} x_2'$ such that $f_i x_i \equiv \langle s_i, \mathrm{InR}(x_i')\rangle$. Hence $u_i := {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}} x_i f_i \equiv s_i :: u_i'$ with $u_i' := {}^{\mathrm{co}}\mathcal{R}^{\alpha}_{\mathbb{S}} x_i' f_i$, and again the claim follows. □

We now prove compatibility of extracted terms with pointwise equality w.r.t. the cotype of the formula proved. For a convenient formulation we assume two more fixed assignments $u \mapsto z_u', z_u''$ of object variables to assumption variables.

THEOREM 4.3.5 (Compatibility of extracted terms). *Let $M\colon A$ be a proof of a c.r. formula A and $u_i\colon C_i$ $(i = 1, \dots, n)$ all free c.r. assumptions whose associated object variable z_{u_i} is free in $\mathrm{et}(M)$. Then we can find a proof of*

$$\mathrm{et}(M)(z'_{u_1}, \dots z'_{u_n}) \doteq_A \mathrm{et}(M)(z''_{u_1}, \dots z''_{u_n})$$

from assumptions $z'_{u_i} \doteq_{C_i} z''_{u_i}$ for $i = 1, \dots, n$.

PROOF. By induction on M. *Case* $u\colon C$. Immediate. *Case* $c\colon A$ an axiom. This is clear in case the extracted term is the identity. For the axioms $I^{\pm}$ and ${}^{\mathrm{co}}I^{\pm}$ it was proved in Lemmas 4.3.1, 4.3.2, 4.3.3 and 4.3.4.

Case $(\lambda_{u^A} M^B)^{A\to B}$ with A c.r. For simplicity assume that u is the only assumption variable whose z_u is free in $\mathrm{et}(M)$. By IH we have a proof of $\mathrm{et}(M)(z_u') \doteq_A \mathrm{et}(M)(z_u'')$ from an assumption $z_u' \doteq_A z_u''$. We want a proof of $\mathrm{et}(\lambda_u M) \doteq_{A\to B} \mathrm{et}(\lambda_u M)$, i.e., $\lambda_{z_u}\mathrm{et}(M)(z_u) \doteq_{A\to B} \lambda_{z_u}\mathrm{et}(M)(z_u)$, which is defined to be

$$\forall_{z_u', z_u''}(z_u' \doteq_A z_u'' \to \mathrm{et}(M)(z_u') \doteq_B \mathrm{et}(M)(z_u'')).$$

Apply $\to^+$ and twice $\forall^+$ to the proof given by IH. In case A n.c. the extracted term $\mathrm{et}(\lambda_u M)$ is $\mathrm{et}(M)$ and the claim is immediate.

Case $M^{A\to B} N^A$ with A c.r. For simplicity assume that there no assumption variables whose associated object variable is free in $\mathrm{et}(MN)$. By IH_M we have a proof of $\mathrm{et}(M) \doteq_{A\to B} \mathrm{et}(M)$, i.e.,

$$\forall_{z_u', z_u''}(z_u' \doteq_A z_u'' \to \mathrm{et}(M) z_u' \doteq_B \mathrm{et}(M) z_u'').$$

By IH_N we have a proof of $\mathrm{et}(N) \doteq_A \mathrm{et}(N)$. Applying an instance of the first proof to the second gives $\mathrm{et}(M)\mathrm{et}(N) \doteq_B \mathrm{et}(M)\mathrm{et}(N)$, as required. In case A n.c. the extracted term $\mathrm{et}(MN)$ is $\mathrm{et}(M)$ and the claim is immediate.

The cases $\lambda_x M$ and Mt are obvious since for them the extracted term does not change. □

COROLLARY 4.3.6 (Extensionality of extracted terms). *Let $M\colon A$ be a proof of a c.r. formula A and $u_i\colon C_i$ $(i = 1, \dots, n)$ all free c.r. assumptions*

whose associated object variable z_{u_i} *is free in* $\mathrm{et}(M)$*. Then we can find a proof of* $\mathrm{et}(M) \doteq_A \mathrm{et}(M)$ *from assumptions* $z_{u_i} \doteq_{C_i} z_{u_i}$ *for* $i = 1, \ldots, n$*.*

PROOF. In the proof constructed in Theorem 4.3.5 we only need to substitute z'_{u_i}, z''_{u_i} by z_{u_i}. □