CHAPTER 1

Logic

The main subject of Mathematical Logic is mathematical proof. In this chapter we deal with the basics of formalizing such proofs and analysing their structure. The system we pick for the representation of proofs is natural deduction as in Gentzen (1935). Our reasons for this choice are twofold. First, as the name says this is a *natural* notion of formal proof, which means that the way proofs are represented corresponds very much to the way a careful mathematician writing out all details of an argument would go anyway. Second, formal proofs in natural deduction are closely related (via the Curry-Howard correspondence) to terms in typed lambda calculus. This provides us not only with a compact notation for logical derivations (which otherwise tend to become somewhat unmanageable tree-like structures), but also opens up a route to applying the computational techniques which underpin lambda calculus.

An essential point for Mathematical Logic is to fix the formal language to be used. We take implication $\to$ and the universal quantifier $\forall$ as basic. Then the logic rules correspond precisely to lambda calculus. The additional connectives (i.e., the existential quantifier $\exists$, disjunction $\vee$ and conjunction $\wedge$) will be added via axioms. Later we will see that these axioms are determined by particular inductive definitions. In addition to the use of inductive definitions as a unifying concept, another reason for that change of emphasis will be that it fits more readily with the more computational viewpoint adopted there.

This chapter does not simply introduce basic proof theory, but in addition there is an underlying theme: to bring out the constructive content of logic, particularly in regard to the relationship between minimal and classical logic. It seems that the latter is most appropriately viewed as a subsystem of the former.

1.1. Natural deduction

Rules come in pairs: we have an introduction and an elimination rule for each of the logical connectives. The resulting system is called *minimal logic*; it was introduced by Kolmogorov (1932), Gentzen (1935) and Johansson

(1937). First we only consider implication $\to$ and universal quantification $\forall$. Note that no negation is yet present. If we go on and require *ex-falso-quodlibet* for a distinguished propositional variable $\bot$ ("falsum") we can embed *intuitionistic logic* with negation $\neg A$ defined as $A \to \bot$. To embed classical logic, we need to go further and add as an axiom schema the principle of *indirect proof*, also called *stability* ($\forall_{\vec{x}}(\neg\neg R\vec{x} \to R\vec{x}\,)$ for relation symbols R), but then it is appropriate to restrict to the language based on $\to$, $\forall$, $\bot$ and $\wedge$. The reason for this restriction is that we can neither prove $\neg\neg\exists_x A \to \exists_x A$ nor $\neg\neg(A \vee B) \to A \vee B$ (the former one for decidable A is Markov's scheme). However, we can prove them for the classical existential quantifier and disjunction defined by $\neg\forall_x\neg A$ and $\neg A \to \neg B \to \bot$. Thus we need to make a distinction between two kinds of "exists" and two kinds of "or": the classical ones are "weak" and the non-classical ones "strong" since they have constructive content. We mark the distinction by writing a tilde above the weak disjunction and existence symbols thus $\tilde{\vee}, \tilde{\exists}$.

1.1.1. Examples of derivations. To motivate the rules for natural deduction, let us start with informal proofs of some simple logical facts.

$$(A \to B \to C) \to (A \to B) \to A \to C.$$

Informal proof. Assume $A \to B \to C$. To show: $(A \to B) \to A \to C$. So assume $A \to B$. To show: $A \to C$. So finally assume A. To show: C. Using the third assumption twice we have $B \to C$ by the first assumption, and B by the second assumption. From $B \to C$ and B we then obtain C. Then $A \to C$, cancelling the assumption on A; $(A \to B) \to A \to C$ cancelling the second assumption; and the result follows by cancelling the first assumption. □

$$\forall_x(A \to B) \to A \to \forall_x B, \quad \text{if } x \notin \mathrm{FV}(A).$$

Informal proof. Assume $\forall_x(A \to B)$. To show: $A \to \forall_x B$. So assume A. To show: $\forall_x B$. Let x be arbitrary; note that we have not made any assumptions on x. To show: B. We have $A \to B$ by the first assumption. Hence also B by the second assumption. Hence $\forall_x B$. Hence $A \to \forall_x B$, cancelling the second assumption. Hence the result, cancelling the first assumption. □

A characteristic feature of these proofs is that assumptions are introduced and eliminated again. At any point in time during the proof the free or "open" assumptions are known, but as the proof progresses, free assumptions may become cancelled or "closed" because of the implies-introduction rule.

We reserve the word *proof* for the informal level; a formal representation of a proof will be called a *derivation*.

An intuitive way to communicate derivations is to view them as labelled trees each node of which denotes a rule application. The labels of the inner nodes are the formulas derived as conclusions at those points, and the labels of the leaves are formulas or terms. The labels of the nodes immediately above a node k are the *premises* of the rule application. At the root of the tree we have the conclusion (or end formula) of the whole derivation. In natural deduction systems one works with *assumptions* at leaves of the tree; they can be either *open* or *closed* (cancelled). Any of these assumptions carries a *marker*. As markers we use *assumption variables* denoted $u, v, w, u_0, u_1, \dots$. The variables of the language previously introduced will now often be called *object variables*, to distinguish them from assumption variables. If at a node below an assumption the dependency on this assumption is removed (it becomes closed) we record this by writing down the assumption variable. Since the same assumption may be used more than once (this was the case in the first example above), the assumption marked with u (written $u\colon A$) may appear many times. Of course we insist that distinct assumption formulas must have distinct markers. An inner node of the tree is understood as the result of passing from premises to the conclusion of a given rule. The label of the node then contains, in addition to the conclusion, also the name of the rule. In some cases the rule binds or closes or cancels an assumption variable u (and hence removes the dependency of all assumptions $u\colon A$ thus marked). An application of the $\forall$-introduction rule similarly binds an object variable x (and hence removes the dependency on x). In both cases the bound assumption or object variable is added to the label of the node.

Definition. A formula A is called *derivable* (in *minimal logic*), written $\vdash A$, if there is a derivation of A (without free assumptions) using the natural deduction rules. A formula B is called derivable from assumptions $A_1, \dots, A_n$, if there is a derivation of B with free assumptions among $A_1, \dots, A_n$. Let Γ be a (finite or infinite) set of formulas. We write $\Gamma \vdash B$ if the formula B is derivable from finitely many assumptions $A_1, \dots, A_n \in \Gamma$.

We now formulate the rules of natural deduction.

1.1.2. Introduction and elimination rules for $\to$ and $\forall$. First we have an assumption rule, allowing to write down an arbitrary formula A together with a marker u:

$$u\colon A \qquad \text{assumption.}$$

The other rules of natural deduction split into introduction rules (I-rules for short) and elimination rules (E-rules) for the logical connectives which, for the time being, are just $\to$ and $\forall$. For implication $\to$ there is an introduction rule $\to^+$ and an elimination rule $\to^-$ also called *modus ponens*. The left premise $A \to B$ in $\to^-$ is called the *major* (or *main*) premise, and the right premise A the *minor* (or *side*) premise. Note that with an application of the $\to^+$-rule *all* assumptions above it marked with $u\colon A$ are cancelled (which is denoted by putting square brackets around these assumptions), and the u then gets written alongside. There may of course be other uncancelled assumptions $v\colon A$ of the same formula A, which may get cancelled at a later stage.

$$\dfrac{\begin{array}{c}[u\colon A]\\ \mid M\\ B\end{array}}{A \to B}\to^+ u \qquad \dfrac{\begin{array}{c}\mid M\\ A \to B\end{array} \quad \begin{array}{c}\mid N\\ A\end{array}}{B}\to^-$$

For the universal quantifier $\forall$ there is an introduction rule $\forall^+$ (again marked, but now with the bound variable x) and an elimination rule $\forall^-$ whose right premise is the term t to be substituted. The rule $\forall^+ x$ with conclusion $\forall_x A$ is subject to the following *(eigen-)variable condition*: the derivation M of the premise A should not contain any open assumption having x as a free variable.

$$\dfrac{\begin{array}{c}\mid M\\ A\end{array}}{\forall_x A}\forall^+ x \ (\text{var.cond.}) \qquad \dfrac{\begin{array}{c}\mid M\\ \forall_x A(x)\end{array} \quad t}{A(t)}\forall^-$$

We now give derivations of the two example formulas treated informally above. Since in many cases the rule used is determined by the conclusion, we suppress in such cases the name of the rule.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{u\colon A \to B \to C \quad w\colon A}{B \to C} \quad \dfrac{v\colon A \to B \quad w\colon A}{B}}{C}}{A \to C}\to^+ w}{(A \to B) \to A \to C}\to^+ v}{(A \to B \to C) \to (A \to B) \to A \to C}\to^+ u$$

For the second example we obtain

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{u\colon \forall_x(A \to Px) \quad x}{A \to Px} \quad v\colon A}{Px}}{\forall_x Px}\forall^+ x}{A \to \forall_x Px}\to^+ v}{\forall_x(A \to Px) \to A \to \forall_x Px}\to^+ u$$

Note that the variable condition is satisfied: x is not free in A (and also not free in $\forall_x(A \to Px)$).

1.1.3. Negation, disjunction, conjunction and existence. Recall that negation is defined by $\neg A := (A \to \bot)$. The following can easily be derived.

$$A \to \neg\neg A,$$
$$\neg\neg\neg A \to \neg A.$$

However, $\neg\neg A \to A$ is in general *not* derivable (without stability – we will come back to this later on). The derivation of $\neg\neg\neg A \to \neg A$ is

$$\dfrac{\dfrac{\dfrac{u\colon ((A \to \bot) \to \bot) \to \bot \qquad \dfrac{\dfrac{w\colon A \to \bot \qquad v\colon A}{\bot}}{(A \to \bot) \to \bot}\to^+ w}{\dfrac{\bot}{A \to \bot}\to^+ v}}{}}{(((A \to \bot) \to \bot) \to \bot) \to A \to \bot}\to^+ u$$

Derivations for the following formulas are left as exercises.

$$(A \to B) \to \neg B \to \neg A,$$
$$\neg(A \to B) \to \neg B,$$
$$\neg\neg(A \to B) \to \neg\neg A \to \neg\neg B,$$
$$(\bot \to B) \to (\neg\neg A \to \neg\neg B) \to \neg\neg(A \to B),$$
$$\neg\neg\forall_x A \to \forall_x \neg\neg A.$$

For disjunction the introduction and elimination axioms are

$$\vee_0^+\colon A \to A \vee B,$$
$$\vee_1^+\colon B \to A \vee B,$$
$$\vee^-\colon A \vee B \to (A \to C) \to (B \to C) \to C.$$

For conjunction we have

$$\wedge^+\colon A \to B \to A \wedge B, \qquad \wedge^-\colon A \wedge B \to (A \to B \to C) \to C$$

and for the existential quantifier

$$\exists^+\colon A \to \exists_x A, \qquad \exists^-\colon \exists_x A \to \forall_x (A \to B) \to B \quad (x \notin \mathrm{FV}(B)).$$

REMARK. All these axioms can be seen as special cases of a general schema, that of an *inductively defined predicate*, which is defined by some introduction rules and one elimination rule. Later we will study this kind of definition in full generality.

It is easy to see that for each of the connectives $\vee$, $\wedge$, $\exists$ the axioms and the following rules are equivalent over minimal logic; this is left as an exercise. For disjunction the introduction and elimination rules are

$$\frac{\begin{array}{c}\mid M\\ A\end{array}}{A \vee B}\,\vee_0^+ \qquad \frac{\begin{array}{c}\mid M\\ B\end{array}}{A \vee B}\,\vee_1^+ \qquad \frac{\begin{array}{c}\\ \mid M\\ A \vee B\end{array}\quad \begin{array}{c}[u\colon A]\\ \mid N\\ C\end{array}\quad \begin{array}{c}[v\colon B]\\ \mid K\\ C\end{array}}{C}\,\vee^- u, v$$

For conjunction we have

$$\frac{\begin{array}{c}\mid M\\ A\end{array}\quad \begin{array}{c}\mid N\\ B\end{array}}{A \wedge B}\,\wedge^+ \qquad \frac{\begin{array}{c}\\ \mid M\\ A \wedge B\end{array}\quad \begin{array}{c}[u\colon A]\quad [v\colon B]\\ \mid N\\ C\end{array}}{C}\,\wedge^- u, v$$

and for the existential quantifier

$$\frac{t \quad \begin{array}{c}\mid M\\ A(t)\end{array}}{\exists_x A(x)}\,\exists^+ \qquad \frac{\begin{array}{c}\\ \mid M\\ \exists_x A\end{array}\quad \begin{array}{c}[u\colon A]\\ \mid N\\ B\end{array}}{B}\,\exists^- x, u \text{ (var.cond.)}$$

Similar to $\forall^+ x$ the rule $\exists^- x, u$ is subject to an *(eigen-)variable condition*: in the derivation N the variable x (i) should not occur free in the formula of any open assumption other than $u\colon A$, and (ii) should not occur free in B.

We collect some easy facts about derivability; $B \leftarrow A$ means $A \to B$.

LEMMA 1.1.1. *The following are derivable.*

$$(A \wedge B \to C) \leftrightarrow (A \to B \to C),$$
$$(A \to B \wedge C) \leftrightarrow (A \to B) \wedge (A \to C),$$

$$(A \vee B \to C) \leftrightarrow (A \to C) \wedge (B \to C),$$
$$(A \to B \vee C) \leftarrow (A \to B) \vee (A \to C),$$

$$(\forall_x A \to B) \leftarrow \exists_x (A \to B) \quad \textit{if } x \notin \mathrm{FV}(B),$$
$$(A \to \forall_x B) \leftrightarrow \forall_x (A \to B) \quad \textit{if } x \notin \mathrm{FV}(A),$$

$$(\exists_x A \to B) \leftrightarrow \forall_x (A \to B) \quad \textit{if } x \notin \mathrm{FV}(B),$$
$$(A \to \exists_x B) \leftarrow \exists_x (A \to B) \quad \textit{if } x \notin \mathrm{FV}(A).$$

PROOF. A derivation of the final formula is

$$\dfrac{\dfrac{\dfrac{u\colon \exists_x(A\to B) \qquad \dfrac{x \qquad \dfrac{w\colon A\to B \qquad v\colon A}{B}}{\exists_x B}}{\exists_x B}\ \exists^- x, w}{A\to\exists_x B}\ \to^+ v}{\exists_x(A\to B)\to A\to\exists_x B}\ \to^+ u$$

The variable condition for $\exists^-$ is satisfied since the variable x (i) is not free in the formula A of the open assumption $v\colon A$, and (ii) is not free in $\exists_x B$. The rest of the proof is left as an exercise. □

1.2. Embedding intuitionistic and classical logic

As already mentioned, we distinguish two kinds of "or" and "exists": the "weak" or classical ones and the "strong" or constructive ones. In the present context both kinds occur together and hence we must mark the distinction; we shall do this by writing a tilde above the weak disjunction and existence symbols thus

$$A \,\tilde{\vee}\, B := \neg A \to \neg B \to \bot, \qquad \tilde{\exists}_x A := \neg\forall_x \neg A.$$

These weak variants of disjunction and the existential quantifier are no stronger than the proper ones (in fact, they are weaker):

$$A \vee B \to A \,\tilde{\vee}\, B, \qquad \exists_x A \to \tilde{\exists}_x A.$$

This can be seen easily by putting $C := \bot$ in $\vee^-$ and $B := \bot$ in $\exists^-$.

REMARK. Since $\tilde{\exists}_x\tilde{\exists}_y A$ unfolds into a rather awkward formula we extend the $\tilde{\exists}$-terminology to lists of variables:

$$\tilde{\exists}_{x_1,\dots,x_n} A := \forall_{x_1,\dots,x_n}(A \to \bot) \to \bot.$$

Nothing is lost here, since the omitted double negations could be eliminated. Moreover let

$$\tilde{\exists}_{x_1,\dots,x_n}(A_1 \,\tilde{\wedge} \dots \tilde{\wedge}\, A_m) := \forall_{x_1,\dots,x_n}(A_1 \to \dots \to A_m \to \bot) \to \bot.$$

This allows to stay in the $\to, \forall$ part of the language. Notice that $\tilde{\wedge}$ only makes sense in this context, i.e., in connection with $\tilde{\exists}$.

1.2.1. Intuitionistic and classical derivability. In the definition of derivability falsity $\bot$ plays no role. We may change this and require *ex-falso-quodlibet* axioms, of the form

$$\forall_{\vec{x}}(\bot \to R\vec{x}\,)$$

with R a relation symbol distinct from $\bot$. Let Efq denote the set of all such axioms. A formula A is called *intuitionistically derivable*, written $\vdash_i A$, if $\mathrm{Efq} \vdash A$. We write $\Gamma \vdash_i B$ for $\Gamma \cup \mathrm{Efq} \vdash B$.

We may even go further and require *stability* axioms, of the form

$$\forall_{\vec{x}}(\neg\neg R\vec{x} \to R\vec{x}\,)$$

with R again a relation symbol distinct from $\bot$. Let Stab denote the set of all these axioms. A formula A is called *classically derivable*, written $\vdash_c A$, if $\mathrm{Stab} \vdash A$. We write $\Gamma \vdash_c B$ for $\Gamma \cup \mathrm{Stab} \vdash B$.

It is easy to see that intuitionistically (i.e., from Efq) we can derive $\bot \to A$ for an *arbitrary* formula A, using the introduction rules for the connectives. A similar generalization of the stability axioms is only possible for formulas in the language not involving $\vee, \exists$. However, it is still possible to use the substitutes $\tilde{\vee}$ and $\tilde{\exists}$.

THEOREM 1.2.1 (Stability, or principle of indirect proof).

(a) $\vdash (\neg\neg A \to A) \to (\neg\neg B \to B) \to \neg\neg(A \wedge B) \to A \wedge B$.
(b) $\vdash (\neg\neg B \to B) \to \neg\neg(A \to B) \to A \to B$.
(c) $\vdash (\neg\neg A \to A) \to \neg\neg\forall_x A \to A$.
(d) $\vdash_c \neg\neg A \to A$ *for every formula A without* $\vee, \exists$.

PROOF. (a) is left as an exercise.

(b) For simplicity, in the derivation to be constructed we leave out applications of $\to^+$ at the end.

$$\dfrac{u\colon \neg\neg B \to B \qquad \dfrac{\dfrac{v\colon \neg\neg(A \to B) \qquad \dfrac{\dfrac{u_1\colon \neg B \qquad \dfrac{u_2\colon A \to B \qquad w\colon A}{B}}{\bot}}{\neg(A \to B)}\,\to^+ u_2}{\bot}}{\neg\neg B}\,\to^+ u_1}{B}$$

(c)

$$\dfrac{u\colon \neg\neg A \to A \qquad \dfrac{\dfrac{v\colon \neg\neg\forall_x A \qquad \dfrac{\dfrac{u_1\colon \neg A \qquad \dfrac{u_2\colon \forall_x A \qquad x}{A}}{\bot}}{\neg\forall_x A}\,\to^+ u_2}{\bot}}{\neg\neg A}\,\to^+ u_1}{A}$$

(d) Induction on A. The case $R\vec{t}$ with R distinct from $\bot$ is given by Stab. In the case $\bot$ the desired derivation is

$$\dfrac{u\colon (\bot\to\bot)\to\bot \qquad \dfrac{v\colon \bot}{\bot\to\bot}\,\to^+ v}{\bot}$$

In the cases $A\wedge B$, $A\to B$ and $\forall_x A$ use (a), (b) and (c), respectively. □

Using stability we can prove some well-known facts about the interaction of weak disjunction and the weak existential quantifier with implication. We first prove a more refined claim, stating to what extent we need to go beyond minimal logic.

LEMMA 1.2.2. *The following are derivable.*

$$(\tilde{\exists}_x A\to B)\to\forall_x(A\to B) \quad \textit{if } x\notin \mathrm{FV}(B), \tag{1}$$

$$(\neg\neg B\to B)\to \quad \forall_x(A\to B)\to\tilde{\exists}_x A\to B \quad \textit{if } x\notin \mathrm{FV}(B), \tag{2}$$

$$(\bot\to B[x{:=}c])\to(A\to\tilde{\exists}_x B)\to\tilde{\exists}_x(A\to B) \quad \textit{if } x\notin \mathrm{FV}(A), \tag{3}$$

$$\tilde{\exists}_x(A\to B)\to A\to\tilde{\exists}_x B \quad \textit{if } x\notin \mathrm{FV}(A). \tag{4}$$

The last two items can also be seen as simplifying a weakly existentially quantified implication whose premise does not contain the quantified variable. In case the conclusion does not contain the quantified variable we have

$$(\neg\neg B\to B)\to \quad \tilde{\exists}_x(A\to B)\to\forall_x A\to B \quad \textit{if } x\notin \mathrm{FV}(B), \tag{5}$$

$$\forall_x(\neg\neg A\to A)\to(\forall_x A\to B)\to\tilde{\exists}_x(A\to B) \quad \textit{if } x\notin \mathrm{FV}(B). \tag{6}$$

PROOF. (1)

$$\dfrac{\tilde{\exists}_x A\to B \qquad \dfrac{\dfrac{\dfrac{u_1\colon \forall_x\neg A \quad x}{\neg A}\qquad A}{\bot}}{\neg\forall_x\neg A}\,\to^+u_1}{B}$$

(2)

$$\dfrac{\neg\neg B\to B \qquad \dfrac{\dfrac{\neg\forall_x\neg A \qquad \dfrac{\dfrac{\dfrac{u_2\colon\neg B \qquad \dfrac{\dfrac{\forall_x(A\to B)\quad x}{A\to B}\qquad u_1\colon A}{B}}{\bot}}{\neg A}\,\to^+u_1}{\forall_x\neg A}}{\bot}}{\neg\neg B}\,\to^+u_2}{B}$$

(3) Writing B_0 for $B[x{:=}c]$ we have

$$\dfrac{\dfrac{\forall_x\neg(A\to B)\quad c}{\neg(A\to B_0)}\qquad \dfrac{\dfrac{\bot\to B_0\qquad \dfrac{\dfrac{A\to\tilde{\exists}_x B\quad u_2\colon A}{\tilde{\exists}_x B}\qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall_x\neg(A\to B)\quad x}{\neg(A\to B)}\qquad\dfrac{u_1\colon B}{A\to B}}{\bot}}{\neg B}\,{\to^+}u_1}{\forall_x\neg B}}{\bot}}{B_0}}{A\to B_0}\,{\to^+}u_2}{\bot}$$

(4)

$$\dfrac{\tilde{\exists}_x(A\to B)\qquad \dfrac{\dfrac{\dfrac{\dfrac{\forall_x\neg B\quad x}{\neg B}\qquad \dfrac{u_1\colon A\to B\quad A}{B}}{\bot}}{\neg(A\to B)}\,{\to^+}u_1}{\forall_x\neg(A\to B)}}{\bot}$$

(5)

$$\dfrac{\neg\neg B\to B\qquad \dfrac{\dfrac{\tilde{\exists}_x(A\to B)\qquad \dfrac{\dfrac{\dfrac{u_2\colon\neg B\qquad \dfrac{u_1\colon A\to B\qquad \dfrac{\forall_x A\quad x}{A}}{B}}{\bot}}{\neg(A\to B)}\,{\to^+}u_1}{\forall_x\neg(A\to B)}}{\bot}}{\neg\neg B}\,{\to^+}u_2}{B}$$

(6) We derive $\forall_x(\bot\to A)\to(\forall_x A\to B)\to\forall_x\neg(A\to B)\to\neg\neg A$. Writing Ax, Ay for $A(x), A(y)$ we have

$$\dfrac{\dfrac{\dfrac{\forall_x\neg(Ax\to B)\quad x}{\neg(Ax\to B)}\qquad \dfrac{\dfrac{\forall_x Ax\to B\qquad \dfrac{\dfrac{\dfrac{\forall_y(\bot\to Ay)\quad y}{\bot\to Ay}\qquad \dfrac{u_1\colon\neg Ax\quad u_2\colon Ax}{\bot}}{Ay}}{\forall_y Ay}}{B}}{Ax\to B}\,{\to^+}u_2}{\bot}}{\neg\neg Ax}\,{\to^+}u_1$$

Using this derivation M we obtain

$$\dfrac{\dfrac{\forall_x\neg(Ax\to B)\quad x}{\neg(Ax\to B)}\qquad \dfrac{\dfrac{\forall_x Ax\to B\qquad \dfrac{\dfrac{\dfrac{\forall_x(\neg\neg Ax\to Ax)\quad x}{\neg\neg Ax\to Ax}\qquad \begin{matrix}\mid M\\ \neg\neg Ax\end{matrix}}{Ax}}{\forall_x Ax}}{B}}{Ax\to B}}{\bot}$$

Since clearly $\vdash(\neg\neg A\to A)\to\bot\to A$ the claim follows. □

Remark. An immediate consequence of (6) is the classical derivability of the "drinker formula" $\tilde{\exists}_x(Px\to\forall_x Px)$, to be read "in every non-empty bar there is a person such that, if this person drinks, then everybody drinks". To see this let $A := Px$ and $B := \forall_x Px$ in (6).

Corollary 1.2.3.

$\vdash_c (\tilde{\exists}_x A\to B)\leftrightarrow\forall_x(A\to B)$ *if* $x\notin \mathrm{FV}(B)$ *and* B *without* $\vee,\exists$,

$\vdash_i (A\to\tilde{\exists}_x B)\leftrightarrow\tilde{\exists}_x(A\to B)$ *if* $x\notin \mathrm{FV}(A)$,

$\vdash_c \tilde{\exists}_x(A\to B)\leftrightarrow(\forall_x A\to B)$ *if* $x\notin \mathrm{FV}(B)$ *and* A,B *without* $\vee,\exists$.

There is a similar lemma on weak disjunction:

Lemma 1.2.4. *The following are derivable.*

$$\begin{aligned}
&& (A\,\tilde{\vee}\,B\to C)&\to(A\to C)\wedge(B\to C),\\
(\neg\neg C\to C)&\to& (A\to C)&\to(B\to C)\to A\,\tilde{\vee}\,B\to C,\\
(\bot\to B)&\to& (A\to B\,\tilde{\vee}\,C)&\to(A\to B)\,\tilde{\vee}\,(A\to C),\\
&& (A\to B)\,\tilde{\vee}\,(A\to C)&\to A\to B\,\tilde{\vee}\,C,\\
(\neg\neg C\to C)&\to& (A\to C)\,\tilde{\vee}\,(B\to C)&\to A\to B\to C,\\
(\bot\to C)&\to& (A\to B\to C)&\to(A\to C)\,\tilde{\vee}\,(B\to C).
\end{aligned}$$

Proof. The derivation of the final formula is

$$\dfrac{\neg(B\to C)\qquad \dfrac{\dfrac{\bot\to C\qquad \dfrac{\neg(A\to C)\qquad \dfrac{\dfrac{\dfrac{A\to B\to C\quad u_1\colon A}{B\to C}\qquad u_2\colon B}{C}}{A\to C}\,{\to^+}u_1}{\bot}}{C}}{B\to C}\,{\to^+}u_2}{\bot}$$

The other derivations are similar to the ones above, if one views $\tilde{\exists}$ as an infinitary version of $\tilde{\vee}$. □

Corollary 1.2.5.

$$\vdash_c (A \mathbin{\tilde{\vee}} B \to C) \leftrightarrow (A \to C) \wedge (B \to C) \quad \textit{for } C \textit{ without } \vee, \exists,$$

$$\vdash_i (A \to B \mathbin{\tilde{\vee}} C) \leftrightarrow (A \to B) \mathbin{\tilde{\vee}} (A \to C),$$

$$\vdash_c (A \to C) \mathbin{\tilde{\vee}} (B \to C) \leftrightarrow (A \to B \to C) \quad \textit{for } C \textit{ without } \vee, \exists.$$

It is easy to see that weak disjunction and the weak existential quantifier satisfy the same axioms as the strong variants, if one restricts the conclusion of the elimination axioms to formulas without $\vee, \exists$. In fact, we have

Lemma 1.2.6.

$$\vdash A \to \tilde{\exists}_x A,$$

$$\vdash_c \tilde{\exists}_x A \to \forall_x (A \to B) \to B \quad (x \notin \mathrm{FV}(B),\ B \textit{ without } \vee, \exists),$$

$$\vdash A \to A \mathbin{\tilde{\vee}} B, \quad \vdash B \to A \mathbin{\tilde{\vee}} B,$$

$$\vdash_c A \mathbin{\tilde{\vee}} B \to (A \to C) \to (B \to C) \to C \quad (C \textit{ without } \vee, \exists).$$

Proof. The derivations of the second and the fourth formula are

$$\AxiomC{$\neg\neg B \to B$} \AxiomC{$\neg\forall_x \neg A$} \AxiomC{$u_1\colon \neg B$} \AxiomC{$\forall_x(A \to B)$} \AxiomC{x} \BinaryInfC{$A \to B$} \AxiomC{$u_2\colon A$} \BinaryInfC{B} \BinaryInfC{$\bot$} \RightLabel{$\to^+ u_2$} \UnaryInfC{$\neg A$} \UnaryInfC{$\forall_x \neg A$} \BinaryInfC{$\bot$} \RightLabel{$\to^+ u_1$} \UnaryInfC{$\neg\neg B$} \BinaryInfC{B} \DisplayProof$$

and

$$\AxiomC{$\neg\neg C \to C$} \AxiomC{$\neg A \to \neg B \to \bot$} \AxiomC{$u_1\colon \neg C$} \AxiomC{$A \to C$} \AxiomC{$u_2\colon A$} \BinaryInfC{$C$} \BinaryInfC{$\bot$} \RightLabel{$\to^+ u_2$} \UnaryInfC{$\neg A$} \BinaryInfC{$\neg B \to \bot$} \AxiomC{$u_1\colon \neg C$} \AxiomC{$B \to C$} \AxiomC{$u_3\colon B$} \BinaryInfC{$C$} \BinaryInfC{$\bot$} \RightLabel{$\to^+ u_3$} \UnaryInfC{$\neg B$} \BinaryInfC{$\bot$} \RightLabel{$\to^+ u_1$} \UnaryInfC{$\neg\neg C$} \BinaryInfC{$C$} \DisplayProof$$

□

1.2.2. Gentzen translation. Classical derivability $\Gamma \vdash_c B$ was defined in Section 1.2.1 by $\Gamma \cup \mathrm{Stab} \vdash B$. This embedding of classical logic into minimal logic can be expressed in a somewhat different and very explicit form, namely as a syntactic translation $A \mapsto A^g$ of formulas such that A is derivable in classical logic if and only if its translation A^g is derivable in minimal logic.

Definition (Gentzen translation A^g).

$$\begin{aligned}
(R\vec{t}\,)^g &:= \neg\neg R\vec{t} \quad \text{for } R \text{ distinct from } \bot,\\
\bot^g &:= \bot,\\
(A \vee B)^g &:= A^g \mathrel{\tilde{\vee}} B^g,\\
(\exists_x A)^g &:= \tilde{\exists}_x A^g,\\
(A \circ B)^g &:= A^g \circ B^g \quad \text{for } \circ = \to, \wedge,\\
(\forall_x A)^g &:= \forall_x A^g.
\end{aligned}$$

Lemma 1.2.7. $\vdash \neg\neg A^g \to A^g$.

Proof. Induction on A.

Case $R\vec{t}$ with R distinct from $\bot$. We must show $\neg\neg\neg\neg R\vec{t} \to \neg\neg R\vec{t}$, which is a special case of $\vdash \neg\neg\neg B \to \neg B$.

Case $\bot$. Use $\vdash \neg\neg\bot \to \bot$.

Case $A \vee B$. We must show $\vdash \neg\neg(A^g \mathrel{\tilde{\vee}} B^g) \to A^g \mathrel{\tilde{\vee}} B^g$, which is a special case of $\vdash \neg\neg(\neg C \to \neg D \to \bot) \to \neg C \to \neg D \to \bot$:

$$\dfrac{\neg\neg(\neg C \to \neg D \to \bot) \qquad \dfrac{\dfrac{\dfrac{u_1\colon \neg C \to \neg D \to \bot \qquad \neg C}{\neg D \to \bot} \qquad \neg D}{\bot}}{\neg(\neg C \to \neg D \to \bot)}\to^+ u_1}{\bot}$$

Case $\exists_x A$. In this case we must show $\vdash \neg\neg\tilde{\exists}_x A^g \to \tilde{\exists}_x A^g$, but this is a special case of $\vdash \neg\neg\neg B \to \neg B$, because $\tilde{\exists}_x A^g$ is the negation $\neg\forall_x\neg A^g$.

Case $A \wedge B$. We must show $\vdash \neg\neg(A^g \wedge B^g) \to A^g \wedge B^g$. By induction hypothesis $\vdash \neg\neg A^g \to A^g$ and $\vdash \neg\neg B^g \to B^g$. Now use part (a) of Theorem 1.2.1 (on stability).

The cases $A \to B$ and $\forall_x A$ are similar, using parts (b) and (c) of Theorem 1.2.1 instead. □

Theorem 1.2.8. (a) $\Gamma \vdash_c A$ *implies* $\Gamma^g \vdash A^g$.
(b) $\Gamma^g \vdash A^g$ *implies* $\Gamma \vdash_c A$ *for* Γ, A *without* $\vee, \exists$.

Proof. (a) Use induction on $\Gamma \vdash_c A$. For a stability axiom $\forall_{\vec{x}}(\neg\neg R\vec{x} \to R\vec{x}\,)$ we must derive $\forall_{\vec{x}}(\neg\neg\neg\neg R\vec{x} \to \neg\neg R\vec{x}\,)$, which is easy (as above). For the rules $\to^+$, $\to^-$, $\forall^+$, $\forall^-$, $\wedge^+$ and $\wedge^-$ the claim follows immediately from the induction hypothesis, using the same rule again. This works because the Gentzen translation acts as a homomorphism for these connectives. For the rules $\vee_i^+$, $\vee^-$, $\exists^+$ and $\exists^-$ the claim follows from the induction hypothesis and Lemma 1.2.6. For example, in case $\exists^-$ the induction hypothesis gives

$$\begin{array}{c} \mid M \\ \tilde{\exists}_x A^g \end{array} \quad \text{and} \quad \begin{array}{c} u\colon A^g \\ \mid N \\ B^g \end{array}$$

with $x \notin \mathrm{FV}(B^g)$. Now use $\vdash (\neg\neg B^g \to B^g) \to \tilde{\exists}_x A^g \to \forall_x (A^g \to B^g) \to B^g$. Its premise $\neg\neg B^g \to B^g$ is derivable by Lemma 1.2.7.

(b) First note that $\vdash_c (B \leftrightarrow B^g)$ for B without $\vee, \exists$ (induction on B). From $\Gamma^g \vdash A^g$ we obtain $\Gamma \vdash_c A$ as follows. We argue informally. Assume Γ. Then Γ^g by the note, hence A^g because of $\Gamma^g \vdash A^g$, hence A again by the note. □

1.3. The Curry-Howard correspondence

Clearly the tree structure of logical derivations of any complexity at all can be quite cumbersome, and the availability of some alternative representation therefore becomes increasingly important, especially when we wish to operate on derivations. The Curry-Howard correspondence provides a neat, computationally inspired alternative. The underlying idea is that if we have a derivation $M(x)$ of $A(x)$ then any means of (universally) binding the x should then represent a derivation of $\forall_x A(x)$. The notation chosen for binding the x is $\lambda_x M(x)$, denoting the function $x \mapsto M(x)$. On the side of $\to$ a derivation M of B from some assumptions A, each of which must now in addition have a label u, is then represented as $\lambda_u M(u)$, denoting the function $u \mapsto M(u)$. This requires the labelling of assumptions so that all assumptions discharged by an application of $\to^+$ must have the same label.

More precisely, we represent natural deduction derivations as typed "derivation terms", where the derived formula is the "type" of the term (and displayed as a superscript). This representation goes under the name of *Curry-Howard correspondence*. It dates back to Curry (1930) and somewhat later Howard, published only in (1980), who noted that the types of the combinators used in combinatory logic are exactly the Hilbert style axioms for minimal propositional logic. Subsequently Martin-Löf (1984) transferred these ideas to a natural deduction setting where natural deduction proofs of formulas A now correspond exactly to lambda terms with type A. This representation of natural deduction proofs will henceforth be used consistently.

We give an inductive definition of such derivation terms for the $\to, \forall$-rules in Table 1 where for clarity we have written the corresponding derivations to the left. One can also define derivation terms covering the rules for $\vee$, $\wedge$ and $\exists$, but we shall not do so here.

To see the usefulness of derivation terms consider the problem of eliminating "detours" in logical derivations. Such a detour occurs if the main premise of an elimination rule (for $\to$ or $\forall$) is derived by an introduction

Derivation	Term
$u\colon A$	u^A
$\dfrac{\begin{matrix}[u\colon A] \\ \mid M \\ B\end{matrix}}{A \to B}\, \to^+ u$	$(\lambda_{u^A} M^B)^{A \to B}$
$\dfrac{\begin{matrix}\mid M \\ A \to B\end{matrix} \qquad \begin{matrix}\mid N \\ A\end{matrix}}{B}\, \to^-$	$(M^{A \to B} N^A)^B$
$\dfrac{\begin{matrix}\mid M \\ A\end{matrix}}{\forall_x A}\, \forall^+ x$ (with var.cond.)	$(\lambda_x M^A)^{\forall_x A}$ (with var.cond.)
$\dfrac{\begin{matrix}\mid M \\ \forall_x A(x)\end{matrix} \qquad t}{A(t)}\, \forall^-$	$(M^{\forall_x A(x)} t)^{A(t)}$

TABLE 1. Derivation terms for $\to$ and $\forall$

rule. One can then eliminate this detour be a "conversion". We write them in tree notation and also as derivation terms.

$\to$-conversion.

$$\dfrac{\dfrac{\begin{matrix}[u\colon A] \\ \mid M \\ B\end{matrix}}{A \to B}\, \to^+ u \qquad \begin{matrix}\mid N \\ A\end{matrix}}{B}\, \to^- \quad \mapsto_\beta \quad \begin{matrix}\mid N \\ A \\ \mid M \\ B\end{matrix}$$

or written as derivation terms

$$(\lambda_u M(u^A)^B)^{A \to B} N^A \mapsto_\beta M(N^A)^B.$$

The reader familiar with λ-calculus should note that this is nothing other than β-conversion.

$\forall$-*conversion.*

$$\dfrac{\dfrac{\begin{matrix} \mid M(x) \\ A(x) \end{matrix}}{\forall_x A(x)}\,\forall^+ x \qquad t}{A(t)}\,\forall^- \qquad \mapsto_\beta \qquad \begin{matrix} \mid M(t) \\ A(t) \end{matrix}$$

or written as derivation terms

$$(\lambda_x M(x)^{A(x)})^{\forall_x A(x)} t \mapsto_\beta M(t)^{A(t)}.$$

Every derivation term carries a formula as its type. However, we shall usually leave these formulas implicit and write derivation terms without them. The two β-conversions above then appear as

$$\begin{aligned} (\lambda_u M(u))N &\mapsto_\beta M(N), \\ (\lambda_x M(x))t &\mapsto_\beta M(t). \end{aligned}$$

Remark (Normalization). One can show that every reduction sequence given by internal β-conversions terminates after finitely many steps, and that the resulting "normal form" is uniquely determined. For time reasons we refer to the literature for proofs of these facts.