LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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PARTIAL DIFFERENTIAL EQUATIONS I
HOMEWORK SHEET 4

WS 2016/17
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Exercise 1 (Properties of analytic functions; 5 points)**.**

(a) Let $U \subseteq \mathbb{R}^n$ be open and connected and let $u, v \in C^2(U)$ be harmonic functions. Prove that, if there exist $x_0 \in U$ and $r > 0$ such that

$$u(x) = v(x) \quad \text{for all } x \in B_r(x_0)$$

holds, then $u(x) = v(x)$ holds for all $x \in U$.

(b) Let $U \subseteq \mathbb{R}^n$ and $u \in C^2(U)$ be as above. Prove that, if there exists $x_0 \in U$ such that

$$\forall k \in \mathbb{N}_0 : \quad \lim_{x \to x_0} \Big(|x - x_0|^{-k} u(x) \Big) = 0$$

holds, then $u(x) = 0$ readily holds for all $x \in U$.

(c) Assume that u is a harmonic function on $\mathbb{R}^n$ such that $|u(x)| \leq A(1 + |x|^p)$ holds for all $x \in \mathbb{R}^n$, where $A, p \geq 0$ are suitable constants. Prove that in this case u is a polynomial of degree $\deg(u) \leq p$.

In the following exercise we abbreviate $\max u := \max_{x \in K} u(x)$ and $\min u := \min_{x \in K} u(x)$ for a function $u : K \to \mathbb{R}$, $K \subset \mathbb{R}^n$ compact.

Exercise 2 (Minimum Principle, part 1; 5 points)**.**

(a) Let $U \subset \mathbb{R}^n$ be an open and bounded set, $u \in C^2(U) \cap C^0(\overline{U})$ and $a : U \to \mathbb{R}$ such that $a(x) > 0$ for all $x \in U$. Assume that u satisfies

$$-\Delta u(x) + a(x) u(x) = 0 \quad \text{for all } x \in U.$$

Prove that this implies for the function $g := u|_{\partial U}$ that

$$\min\{0, \min g\} \leq u(x) \leq \max\{0, \max g\} \quad \text{for all } x \in \overline{U}$$

(b) Show that, in general, $\min g \leq u(x) \leq \max g$ is wrong in the framework of exercise (a).
Hint: You may for instance consider the special case of the ordinary differential equation $-u'' + u = 0$.

Exercise 3 (Minimum Principle, part 2; 5 points)**.** For the whole exercise $U \subseteq \mathbb{R}^n$ is an open set.

(a) Let $b = (b_1, \dots, b_n)$ be a vector of functions, where $b_i \in C^0(\overline{U}), i = 1, \dots, n$. Prove that, if for the function $u \in C^2(U)$

$$\Delta u(x) + b(x) \cdot \nabla u(x) < 0 \quad \text{for all } x \in U$$

holds, then u cannot attain a local minimum on the set U. Moreover, if U is bounded and $u \in C^2(U) \cap C^0(\overline{U})$, then $\min_{\overline{U}} u = \min_{\partial U} u$.

(b) Let U be bounded and b be the vector of functions defined in (a) above. Prove that, if $u \in C^2(U) \cap C^0(\overline{U})$ is such that

$$\Delta u(x) + b(x) \cdot \nabla u(x) \leq 0 \text{ for all } x \in U$$

holds, then $\min_{\overline{U}} u = \min_{\partial U} u$.
Hint: You may adapt the proof of the weak minimum principle as known from lecture. Note that, due to the extra terms involving lower-order derivatives, it is convenient to consider a modified auxiliary function.

Exercise 4 (Minimum Principle, part 3; 5 points)**.**

(a) Let A, B be two real-valued symmetric and positive semi-definite matrices. Prove that this implies $\text{tr}(AB) \geq 0$.

(b) Let U be open and bounded. For real-valued functions $a_{ij} \in C^0(\overline{U})$,$i, j = 1, \dots, n$ we define the matrix-valued function $A : U \to M(n \times n; \mathbb{R})$ via $A(x) := (a_{ij}(x))_{i,j=1}^n$, $x \in U$. We assume that the matrix $A(x)$ is symmetric for all $x \in U$ and that A is uniformly positive definite, i.e. there exists a constant $C > 0$ such that for all vectors $v \in \mathbb{R}^n$

$$\sum_{i,j=1}^n v_i a_{i,j}(x) v_j \geq C \sum_{i=1}^n v_i^2 \qquad \text{for all } x \in U$$

holds. Prove that, if $u \in C^2(U) \cap C^0(\overline{U})$ is such that

$$-\text{tr}(A(x)D^2u(x)) = -\sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j}(x) \geq 0 \quad \text{for all } x \in U$$

holds, then $\min_{\overline{U}} u = \min_{\partial U} u$.
Hint: You can again adapt the proof of the weak minimum principle.

You can drop your homework solutions until **Monday, November 14** at **16 o'clock** into the appropriate letterbox on the first floor near the library.