LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Partial Differential Equations I
Tutorial Sheet 13

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T 1. For a fixed $n \in \mathbb{N}$ let $u \in C^2(\mathbb{R}^n)$ be harmonic.

(a) Show that all i-th partial derivatives $\frac{\partial u}{\partial x_i}$, $i = 1, ..., n$, are harmonic too.

(b) In addition, assume that $\left\| \frac{\partial u}{\partial x_i} \right\|_{L^\infty(\mathbb{R}^n)} < \infty$ holds for all $i = 1, ..., n$. Show that there exist constants $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that

$$u(x) = a \cdot x + b \qquad \text{for all } x \in \mathbb{R}^n.$$

T 2. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $u_0 \in C^0(\overline{\Omega})$ with $u_0 \geq 0$ and $u \in C^2((0,\infty) \times \Omega) \cap C^0([0,\infty) \times \overline{\Omega})$ be a solution of

$$\begin{cases} u_t(t,x) - \Delta u(t,x) = 4t\cos^2(u(t,x)) & \text{for } (t,x) \in (0,\infty) \times \Omega, \\ u(t,x) = 1 & \text{for } (t,x) \in [0,\infty) \times \overline{\Omega}, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega. \end{cases}$$

Show that

$$0 \leq u(t,x) \leq at^2 + b \qquad \text{for all } (t,x) \in [0,\infty) \times \overline{\Omega}$$

for constants $a, b \in \mathbb{R}$. Moreover, give explicit formulae for valid choices of the constants.
Hint: Use the comparison principle.

T 3. Determine functions $u_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R})$, such that the solution $u \in C^2([0,\infty) \times \mathbb{R})$ of

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) = 0 & \text{for } (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) & \text{for } x \in \mathbb{R} \end{cases}$$

fulfills the conditions $u(3,x) \neq 0$ for all $x \in (0,2)$ and $u(3,x) = 0$ for all $x \in (-\infty, 0] \cup [2,\infty)$.
Hint: Use that for $a, b \in \mathbb{R}$ with $a < b$ there exists a function $\eta \in C_c^\infty(\mathbb{R})$ with $\eta \neq 0$ for all $x \in (a,b)$ and supp$(\eta) = [a,b]$.

T 4. This exercise revisits some of the methods to derive explicit solutions of ordinary differential equations (as needed in for the method of characteristics). Let $I = (\alpha, \beta)$ be an interval and $a, b \in C^0(\overline{I})$.

(a) Apply the method of "separation of variables" to solve (i.e. to find all solutions for) the ordinary differential equations $y'(x) = a(x)y(x)\,(x \in I)$ and $y'(x) = x(y(x)^2 + 1)(x \in I)$.

(b) Solve the inhomogeneous equation $y'(x) = a(x)y(x) + b(x)\,(x \in I)$ via the method of "variation of constants".