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PARTIAL DIFFERENTIAL EQUATIONS I
TUTORIAL SHEET 7

WS 2016/17
December 5, 2016

T 1. Show that the heat kernel Φ is a solution to the heat equation, i.e.

$$\partial_t\Phi(t,x) - \Delta\Phi(t,x) = 0 \qquad \text{for all } (t,x) \in (0,\infty)\times\mathbb{R}^n.$$

The heat kernel on $(0,\infty)\times\mathbb{R}^n$ is given by

$$\Phi(t,x) := \frac{1}{(4\pi t)^{n/2}}\exp\left(-\frac{|x|^2}{4t}\right).$$

T 2. Let $g \in \mathcal{C}(\mathbb{R}^n)\cap \mathrm{L}^\infty(\mathbb{R}^n)$, Φ be the heat kernel and let $u : [0,\infty)\times\mathbb{R}^n \to \mathbb{R}$ be defined by:

$$u(t,x) := \int_{\mathbb{R}^n} \Phi(t,x-y)g(y)\,\mathrm{d}y. \tag{1}$$

i) Show that

$$|u(t,x)| \le \frac{1}{(4\pi t)^{n/2}}\,\|g\|_{\mathrm{L}^1(\mathbb{R}^n)}, \quad \forall\, x\in\mathbb{R}^n,\ t>0, \tag{2}$$

under the condition that $g \in \mathrm{L}^1(\mathbb{R}^n)$.

ii) Show that

$$|u(t,x)| \le \frac{\|g\|_{\mathrm{L}^1(\mathbb{R}^n)}}{(4\pi t)^{n/2}} e^{-\frac{(1-\varepsilon)|x|^2 - C_\varepsilon}{4t}}, \quad \forall\, x\in\mathbb{R}^n,\ t>0, 0<\varepsilon<1, \tag{3}$$

whereas g is a function with compact support. The constant C_ε may only depend on ε and the support of g.

T 3 (Laplace-Transformation)**.** Let $\Omega\subset\mathbb{R}^n$ be open and bounded, $f\in C^2(\Omega)$ and let $v \in C^2([0,\infty)\times\Omega)\cap L^\infty([0,\infty)\times\Omega)$ with $\|\nabla_{t,x}v\|_{L^\infty([0,\infty)\times\Omega)}, \|D_x^2 v\|_{L^\infty([0,\infty)\times\Omega)} < \infty$ be a solution of the heat equation

$$\begin{cases} v_t - \Delta v = 0 & \text{in } (0,\infty)\times\Omega \\ v(0,\cdot) = f & \text{in } \Omega. \end{cases} \tag{4}$$

The Laplace transform in the time component $v^\#$ of v is given by

$$v^\#(s,x) := \int_0^\infty e^{-st}v(t,x)\,\mathrm{d}t, \quad x\in\Omega, s>0. \tag{5}$$

Show that for a fixed $s>0$ the function $u := v^\#(s,\cdot)$ solves the *resolven equation*

$$-\Delta u + su = f \text{ in } \Omega. \tag{6}$$