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PARTIAL DIFFERENTIAL EQUATIONS I
TUTORIAL SHEET 1

WS 2016/17
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T 1. Let $u : \mathbb{R}^n \to \mathbb{R}$. Determine the order of the following partial differential equations. Moreover, find a function $F : \mathbb{R}^{n^k} \times \ldots \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that, after suitable choice of coordinates, the differential equation reads

$$F(D^k u, \ldots, Du, u, x) = 0.$$

Finally, decide whether the differential equations below are linear, semilinear, quasilinear or fully nonlinear.

a) $-\Delta u = 0$,

b) $-\Delta u = \lambda u$,

c) $u_t + \sum_{i=1}^{n-1} (b^i u)_{x_i} = 0$,

d) $u_t - \Delta u = 0$,

e) $|Du| = 1$,

f) $-\Delta u = f(u)$,

g) $\operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = 0$

h) $\operatorname{div}(|Du|^{p-2} Du) = 0$,

i) $\det(D^2 u) = f(x, u)$.

T 2. Determine whether the following partial differential equations on $\mathbb{R}^2$ are elliptic, parabolic or hyperbolic:

a) $3u_{xx} + 4u_{xy} + u_{yy} = 0$,

b) $9u_{xx} + 12u_{xy} + 4u_{yy} = 0$,

c) $2u_{xx} + u_{yy} + 2u_x = 0$,

d) $(x + y)^2(u_{xx} + u_{yy}) - 2(x - y)^2 u_{xy} = 0$.

T 3. Denote by $B = B(0, 1)$ the unit ball in $\mathbb{R}^n$ and let $p \geq 1$, $0 < s < 1$. In this exercise we consider the function

$$u : B \to \mathbb{R}, u(x) = |x|^{-s}.$$

a) Prove that $u \in L^p(B)$ holds for $p < \frac{n}{s}$.

b) Calculate the partial derivatives u_{x_j}, $j = 1 \ldots d$ of u on the set $B \setminus \{(0, \ldots, 0)\}$.

c) Let moreover $p < \frac{n}{s+1}$. Prove that in this case $\int_B |u_{x_i}|^p \, dx < \infty$ holds.

The aim of the following exercise is to extend the results from exercise 4 on homework sheet 1 to open sets $U \subseteq \mathbb{R}^n$. We start with some definitions.

a) For $U \subset \mathbb{R}^n$ open and $\varepsilon > 0$ we define $U_\varepsilon := \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}$, where dist denotes the distance function.

b) A function $0 \leq \eta \in C_c^\infty(\mathbb{R}^n)$ with $\|\eta\|_1 = 1$ and $\operatorname{supp}(\eta) \subseteq \overline{B(0,1)}$ is called a mollifier. Given a mollifier η we define for $\varepsilon > 0$ the functions $\eta_\varepsilon(\cdot) := \varepsilon^{-n}\eta(\varepsilon^{-1}\cdot)$.

c) For $U \subset \mathbb{R}^n$ open, $f \in L^1(U)$, $\varepsilon > 0$ and η a mollifier we define the function $f^\varepsilon \in L^1(U_\varepsilon)$ as

$$f^\varepsilon(x) := (\eta_\varepsilon * \tilde{f})(x) = \int_U \eta_\varepsilon(x-y) f(y)\,\mathrm{d}y \qquad (x \in U_\varepsilon).$$

Here the function $\tilde{f} \in L^1(\mathbb{R}^n)$ is defined via $\tilde{f}|_U = f$ and $\tilde{f}|_{\mathbb{R}^n \setminus U} = 0$.

T 4. Let $U \subseteq \mathbb{R}^n$ open and $f \in L^1(U)$

a) Prove that $f^\varepsilon \in C^1(U_\varepsilon)$ and argue that $f^\varepsilon \in C^\infty(U_\varepsilon)$ holds.

b) Prove that the pointwise convergence $f^\varepsilon \xrightarrow{\varepsilon \searrow 0} f$ holds (Lebesgue-) almost everywhere on U. Also prove that if $f \in C(U)$ is continuous then the above convergence is uniform on compact subsets of U.