QFT in curved space-time

1 Introduction

1.1 The Einstein-Hilbert action (31.01.2010)

Before we turn to quantum field theory in curved space-times, we shortly review the classical theory of gravity. Einstein's equations of classical general relativity can be derived from the Einstein-Hilbert action:

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{g}(R - 2\Lambda).$$

with R the Ricci curvature and the cosmological constant Λ. Under a variation δg_{ab}, the variation of the action is given by:

$$\delta S = \frac{1}{16\pi G} \int d^4x \left(\delta\left(\sqrt{g}\right)(R - 2\Lambda) + \sqrt{g}\delta\left(R_{ab}\right) g^{ab} + \sqrt{g} R_{ab} \delta\big(g^{ab}\big) \right).$$

To analyze this expression, use:

$$\begin{aligned} \delta g &= \det\left(g_{ab} + \delta g_{ab}\right) - \det g_{ab} = \det g_{ab} \left[\det\left(g^b{}_c + \delta g^b{}_c \right) - 1 \right] = \\ &= \det g_{ab} \cdot \mathrm{Tr} \delta g^b{}_c = g g^{ab} \delta g_{ab} = -g g_{ab} \delta g^{ab} \\ g^{ab} \delta R_{ab} &= g^{ab} \left(\partial_c \delta \Gamma^c_{ab} - \partial_b \delta \Gamma^c_{ac} + \Gamma \delta \Gamma \right) = g^{ab} \left(\nabla_c \delta \Gamma^c_{ab} - \nabla_b \delta \Gamma^c_{ac} \right) \\ &= \nabla_c \left(g^{ab} \delta \Gamma^c_{ab} - g^{ab} \delta \Gamma^c_{bc} \right) \end{aligned}$$

For the last derivation, we used normal coordinates around some point p, i.e. $\Gamma|_p = 0$ to get rid of terms quadratic in the Christoffel symbols. Next, we replaced the partial derivatives by covariant derivatives for the same reasons. Since the equation is now tensorial again, it holds everywhere. In the last step, we used the property of the Levi-Cività connection, $\nabla_a g^{bc} = 0$. A slightly different derivation is given in [9], appendix E.

The total derivative contributes only a boundary term, which can be eliminated adding an additional piece $2\int_{\partial M} K$ where K is the extrinsic curvature, $K = h^a{}_b \nabla_a n^b$, where $h_{ab} = g_{ab} \pm n_a n_b$ is the induced metric on the boundary, and n^a is a normal vector on the boundary. Ignoring these subtleties (for example for a manifold without boundary), we get:

$$\delta S = \frac{1}{16\pi G} \int d^4x \sqrt{g} \left(R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} \right) \delta g^{ab}$$

resulting in Einstein's equations in vacuum after setting the variation to zero.

To add matter simply add another term to the action, p.e. a field strength term of Yang-Mills gauge theory or electromagnetism:

$$S_M = \int d^4x \sqrt{g} \mathcal{L}_m = \frac{1}{4g^2} \int \mathrm{Tr}\, F \wedge \star F = \frac{1}{4g^2} \int d^4x \sqrt{g}\, \mathrm{Tr}\, F^{ab} F_{ab}.$$

Define the energy-momentum tensor by $T_{ab} = \frac{2}{\sqrt{g}} \frac{\delta\left(\sqrt{g}\mathcal{L}_m\right)}{\delta g^{ab}}$ and get Einstein's equations

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}.$$

1.2 Attempts to quantize general relativity

General relativity is a classical field theory, with metric field g_{ab}, which plays a dual rôle, both being the dynamical variable in the game, and describing the background. Over the last decades there were various attempts to find a quantum theory of gravity. One of the major lines are the following:

1. Covariant perturbation method: $g_{ab} = g_{ab}^{(0)} + h_{ab}$, where the first part solves the classical Einstein equations. Write down Feynman rules. The problem of this attempt is that the perturbation theory is non-renormalizable. Let's try to understand this.
   In natural units $c = \hbar = 1$, the action is dimensionless. In dimension n, R has dimension L^{-2}, while the measure contributes L^n, so Newton's constant G_N has dimension L^{n-2}. In fact, $G_N = l_p^{n-2}$ in these units, where l_p is the n dimensional Planck length.
   So our coupling parameter is:

$$\frac{1}{\kappa^2} = \frac{1}{16\pi G_N} = \frac{1}{16\pi l_p^{n-2}} = \frac{M_p^{n-2}}{16\pi}$$

   with Planck mass M_p. So at a distance X-times smaller (or at momentum X-times larger), G_N seems X^{n-2}-times larger, so $G_N \sim p^{n-2}$ and the critical dimension is 2. So in 4d, gravity seems non-renormalizable, i.e. gets stronger at shorter distances. In 4 dimensions $G_N = l_p^2$.
   Pure GR is finite to one-loop order (i.e. the divergences cancel due to an analogue of the Ward identity) but not in higher orders or if coupled to matter.
   Next step: supergravity. Make the theory supersymmetric, add a gravitino (spin 3/2), to cancel some divergences. Highest dimension (with one time dimension) is 11, so this is the most natural setup. Get solutions in 4 dimensions by dimensional reduction (like in Kaluza-Klein theories).
   Finally: Perturbative String theory.

2. Canonical quantization method: Wheeler-DeWitt equation: $\widehat{H} |\Psi\rangle = 0$. Background independance as a principle.
   This approach leads to Loop Quantum Gravity after introducing Ashtekar's variables.

3. Path integral method: Try a generating functional:

$$Z = \int \mathcal{D}[g_{ab}] \, \mathrm{e}^{-S}$$

   with the Einstein-Hilbert action S as above. It is not known how to define a measure on the moduli space of metrics. Another problem is again non-renormalizability. In LQG, the moduli space is reduced, one ends up with the spin foam formulation.

4. Alternate routes: Twistor theory, gravitational OR (Roger Penrose), non-commutative geometry (Alain Connes).

1.3 One step back: QFT in curved space-time

Lacking a full theory of gravity, let's see how far we get without it. We saw above, that (reinserting $\hbar$) the Einstein Hilbert action in four dimensions is proportional to $\frac{\hbar}{l_p^2}$. Looking at the formal path integral

$$\int D[g_{ab}]\,\mathrm{e}^{-S} = \int D[g_{ab}]\,\mathrm{e}^{-\frac{\hbar}{l_p^2}\int R}$$

it seems that quantum effects of gravity become important only at the order of the Planck length. A first step towards quantum electrodynamics is to consider matter in a classical background of an electromagnetic field. In analogy, we might hope to get some insight to quantum effects of gravity by considering quantized fields in a classical background, i.e. in a classical curved space. Still there are some issues with this idea. The principle of general covariance states that matter and mass couple equally strongly to gravity.
Nevertheless, if we consider for example a free field theory, only one-loop 1PI diagrams are possible (lacking interaction vertices). Up to order $\hbar$, we can therefore truncate gravity also to first-loop order.
For an interacting theory of matter fields, higher loops of these fields are possible, so we should also better include higher graviton loops. Recall the very different nature of the coupling constant in, say, electromagnetism, and in Einstein gravity. In the first case, it is given by the dimensionless fine-structure constant $\alpha \simeq \frac{1}{137}$, while for gravity, the coupling is dimensionfull, being proportional to the Planck length. While one single graviton loop is of zeroth order in G_N, higher loops come from graviton interactions, contributing powers of G_N. Hence for length scales l satisfying $\frac{G_N}{l^2} = \frac{l_p^2}{l^2} \ll \alpha = \frac{e^2}{4\pi}$, quantum effects of matter dominate, and we can truncate gravity at one-loop order. It seems that this should give reasonable results, though there is still much uncertainty about the domain of validity.
To sum up, there are several steps forward towards a quantum theory of gravity:

1. Quantum field theory in curved space-time: the background space-time is classical, meaning we work in zeroth order in $\hbar$. We ignore the back-reaction of the matter on space-time.

2. Semi-classical gravity: still treat the background as classical, but now take the back-reaction into account. One important element at this stage is the expectation value of the energy-momentum tensor (being quantized) in some state ψ. It contributes to Einstein's equation:
$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G \left\langle \hat{T}_{ab} \right\rangle_\psi .$$

3. Include also higher graviton loops.

We concentrate here on the first two stages, which already lead to interesting results, like particle creation, Unruh effect, and Hawking radiation.

2 Particle creation in a time-varying background (01.02.2010, cf. [9])

To do perturbative QFT (at least in the classical fashion), one needs a notion of particle states. Given such states, the S-matrix yields probabilities for a transition between a certain in-state as an element of some Fock space into an out-state in another Fock space.
In particular, we want to define a vacuum state (a state without particle excitations). The

notion of a particle is no longer automatic in a curved background. As we will see later on, different observers might well disagree over the number of excitations. Only under certain assumptions, it is possible to define meaningful particle states. Possible setups are:

- asymptotically flat space-time, or
- non-vanishing curvature only in a compact region, or
- globally hyperbolic (space-time possesses a Cauchy hypersurface), or
- asymptotically stationary space-time (time-like Killing vector field gives rise to a split into positive and negative frequencies, see below), or
- Feynman propagator (i.e. Green's function) exists (propagating positive frequency in-states into positive frequency out-states).

To define positive and negative frequencies, we want a positive-definite inner product. For concreteness, consider a Klein-Gordon field. There exists a conserved current, which gives rise to an inner product, the so-called Klein-Gordon product. We will come back to details later on. If we restrict to fields of positive frequency (the Fourier transform $\widetilde{\Phi}(\omega, \vec{x})$ vanishes for $\omega < 0$), the product is positive-definite. The Hilbert space of possible states is given by the symmetric Fock space $\mathfrak{F}_S(\mathcal{H})$ with vacuum state $|0\rangle = (1, 0, 0, \ldots)$, where the i-th component in the infinite dimensional vector is in $\bigodot^{i-1} \mathcal{H}$. The field operator can now be given (in a distributional sense) by

$$\hat{\Phi}(x) = \sum_i \left(v_i^*(x) \hat{a}^- + v_i^*(x) \hat{a}^+ \right)$$

where a^+, a^- are creation and annihilation operators, and v_i denotes an orthonormal basis of $\mathcal{H}$. Specify Hilbert spaces $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$. The S-matrix describes, how a state evolves, p.e. for an in-state $|0\rangle \in \mathfrak{F}_S(\mathcal{H}_{\text{in}})$, the out-state $S\,|0\rangle \in \mathfrak{F}_S(\mathcal{H}_{\text{out}})$ tells about spontaneous creation of particle due to a time-dependant gravitational field. The latter state can be calculated. It vanishes for an odd tensor product of Hilbert spaces, hence particles can only be created in pairs. Particle creation only occurs, if some negative-frequency part is picked up.

3 Quantum fluctuations (13.02.2010)

We will from now on closely follow Mukhanov's book [8].
Let's start with an harmonic oscillator in a flat background with Hamiltonian $H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$.
The classical equations of motion are $\ddot{x} + \omega^2 x = 0$.
The Schrödinger equation is: $\left(-\frac{\hbar^2}{2m}\Delta + \frac{m\omega^2}{2} \right) |\psi\rangle = E\,|\psi\rangle$. Solve for the wavefunction:

$$\psi_n(x) = \sqrt{\frac{1}{2^n n!}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar} \right) \cdot H_n\left(\sqrt{\frac{m\omega}{\hbar}} x \right), \qquad n = 0, 1, 2, \ldots.$$

In particular, the vacuum state is given by[1]:

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar} \right). \tag{3.1}$$

[1] alternative derivation: $a\,|0\rangle = 0$, where $a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{i}{m\omega} p \right)$. This leads to: $x\psi_0(x) + \frac{\hbar}{m\omega} \frac{d\psi_0(x)}{dx} = 0$.

The vacuum state has expectation value $\langle x \rangle = 0$. Nevertheless, there is a fluctuation $\delta q \sim \sqrt{\frac{\hbar}{m\omega}}$.
Now consider the field theory case. Here we have "one oscillator at each point". In momentum space: $\ddot{\Phi}_{\mathbf{k}} + (k^2 + m^2)\Phi_{\mathbf{k}} = 0$, leading to a wave functional[2]:

$$\Psi[\Phi] \sim \exp\left(-\frac{1}{2}\int d^3\mathbf{k}\ |\Phi_{\mathbf{k}}|^2\,\omega_{\mathbf{k}}\right).$$

In analogy, there occur vacuum fluctuations

$$\delta\Phi_{\mathbf{k}} = \sqrt{\langle|\Phi_{\mathbf{k}}|\rangle} \sim \frac{1}{\sqrt{\omega_{\mathbf{k}}}}.$$

If we consider fluctuations of a field in a box with length L, the fluctuations occur at $k \sim \frac{1}{L}$ and are given by:

$$\delta\Phi_L \sim \sqrt{(\delta\Phi_{\mathbf{k}})^2\,k^3} \sim \begin{cases} L^{-1} & \text{for } L \ll \frac{1}{m} \\ L^{-3/2} & \text{for } L \gg \frac{1}{m} \end{cases}.$$

One consequence of these fluctuations is the Casimir effect (see below).

4 Driven harmonic oscillator

In QFTCS, a time-varying background geometry leads to particle excitations. A lower-dimensional analogy is a harmonic oscillator coupled to a time-varying source term. Take the Lagrangian:

$$L = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + J(t)q.$$

Assume further that $J(t)$ is non-vanishing for $t \in [0,T]$ only[3]. Now a^- satisfies:

$$\dot{a}^- = -i\omega a^- + \frac{i}{\sqrt{2\omega}}J(t),$$

which can be integrated. If we define

$$J_0 = \frac{i}{\sqrt{2\omega}}\int_0^T dt\ \mathrm{e}^{i\omega t}\,J(t),$$

a straight forward calculation shows, that the occupation number operator has expectation values[4]:

$$\langle 0_{\text{in}}|\hat{N}(t)|0_{\text{in}}\rangle = |J_0|^2$$
$$\langle 0_{\text{out}}|\hat{N}(t)|0_{\text{in}}\rangle = 0.$$

Hence the energy expectation value gets shifted due to the interactions:

$$\langle 0_{\text{in}}|\hat{H}(t)|0_{\text{in}}\rangle = \left(\frac{1}{2} + |J_0|^2\right)\omega.$$

[2] compare equation (3.1), with $m = 1$, $\hbar = 1$, $\omega \to \omega_{\mathbf{k}}$, $x \to \Phi_{\mathbf{k}}$.
[3] This corresponds later on to a time-dependant gravitational field, which is stationary in the beginning and the end, to get notions of particles.
[4] We are in the Heisenberg picture. If we start with the vacuum $|0_{\text{in}}\rangle$, the state stays in $|0_{\text{in}}\rangle$. So the first line indicates an exictation of the vacuum. However, after the interaction, the true vacuum is given by $|0_{\text{out}}\rangle$. This is reflected in the second equation.

5 Quantization of fields

Now we turn to quantum field theory. As stated above, there is "one harmonic oscillator at each point":

$$S[\phi] = \frac{1}{2}\int d^4x\,\dot{\phi}^2 - \int d^4x\,d^4y\,\phi(\mathbf{x})M(\mathbf{x},\mathbf{y})\phi(\mathbf{y}).$$

M gets diagonalized to $M(\mathbf{x},\mathbf{y}) = \left(-\Delta_{\mathbf{x}} + m^2\right)\delta(\mathbf{x}-\mathbf{y})$. Due to the Laplacian, the oscillators are coupled. We can decouple them by a Fourier transformation. Next we perform a mode expansion in eigenfunctions in momentum space.
The quantized solution for the field operator $\hat{\phi}$ is given by:

$$\hat{\phi}(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}}\left(v_{\mathbf{k}}^*(t)\,\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\,\hat{a}_{\mathbf{k}}^- + v_{\mathbf{k}}(t)\,\mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}\,\hat{a}_{\mathbf{k}}^+\right), \tag{5.1}$$

where the mode functions satisfy:

$$\ddot{v}_{\mathbf{k}} + \omega_k^2(t)v_{\mathbf{k}} = 0.$$

In the case of a free field in flat space, one gets:

$$v_{\mathbf{k}}(t) = \frac{1}{\sqrt{\omega_k}}\,\mathrm{e}^{i\omega_k t}, \tag{5.2}$$

with $\omega_k = \sqrt{k^2+m^2}$. In particular, the mode functions are isotropic (independent of the direction of $\mathbf{k}$).
The vacuum energy is in general infinite. Each point is an oscillator by itself and therefore contributes $\frac{1}{2}\omega_{\mathbf{k}}$. In the free field case in flat space, one can renormalize the Hamiltonian by imposing a normal-ordering condition, thus substracting the vacuum energy.
Another way to understand the appearence of mode functions goes as follows (see [4]):

1. Start with a Lagrangian, p.e.

   $$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \xi R\phi^2\right)$$

   $\xi = 0$ gives the standard Klein-Gordon equation and therefore easy equations of motion. For $\xi = 0$ the field is conformally coupled, so the action is invariant under $g_{\mu\nu} \to \mathrm{e}^\lambda\, g_{\mu\nu}$. For the special case of conformally flat metrics (like FLRW metrics), the field decouples from gravity, since its action is equivalent to an action in flat space.

2. Derive the field equations[5]:

   $$\Box\phi + m^2\phi + \xi R\phi = 0.$$

   Define an inner product on solutions. To this end, pick a Cauchy surface Σ with normal unit vector $\mathbf{n}^\mu$, and set:

   $$(v_1, v_2) = i\int v_2^*\,\overleftrightarrow{\partial_\mu}\,v_1\,d\Sigma\,\mathbf{n}^\mu = i\int\left(v_2^*\partial_\mu v_1 - v_1\partial_\mu v_2^*\right)\,d\Sigma\,\mathbf{n}^\mu,,$$

   where v_1 and v_2 are solutions to the equations of motion.
   By the equations of motion, the definition does not depend on the specific choice of Cauchy surface. Therefore, we get a well-defined inner product, the Klein-Gordon product.

[5] Here $\Box = \nabla^\mu\nabla_\mu$. For scalar fields: $\Box\phi = \nabla^\mu\partial_\mu\phi$.

3. Now proceed with canonical quantization:

$$\left[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)\right] = i\delta(\mathbf{x}',\mathbf{x}), \text{ where}$$
$$\pi = \frac{\delta L}{\delta\dot{\phi}},$$
$$\int \delta(\mathbf{x}',\mathbf{x})\, d\Sigma = 1.$$

If (v_i) is a complete set of positive norm ("positive energy" in the flat case), then as in conventional QFT, ϕ gets operator valued, and can be expanded as:

$$\hat{\phi} = \sum_i \left(v_i^* \hat{a}^- + v_i \hat{a}_i^+\right),$$

where a^+, a^- are creation and annihilation operators. In the non-discrete case, we are back in (5.1).

The problem of curved space-times is the ambiguity in a choice of basis v_i. While in the flat case, one picks positive frequency solutions (5.2), there is no canonical choice in the curved case.

5.1 Fields in FLRW models

Now consider a field ϕ with Lagrangian ($\xi = 0$ above)

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right)$$

in a FLRW universe with metric

$$ds^2 = dt^2 - a^2(t)d\mathbf{x}^2 = a^2(\eta)\left(dt^2 - d\mathbf{x}^2\right).$$

It is equivalent to a field $\chi = a\phi$ in flat spacetime, where χ obeys the equation

$$\chi'' - \Delta\chi + m_{\text{eff}}^2\chi = 0,$$

where $m_{\text{eff}}^2 = a^2m^2 - \frac{a''}{a}$ and the prime denotes taking the derivative with respect to the conformal time η. For ξ non-zero, $m_{\text{eff}}^2 = a^2\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right]$.
Again, the oscillator decouple after a Fourier transformation. Now the mode functions satisfy

$$v_k'' + \omega_k^2(\eta)v_k,$$

with $\omega_k(\eta) = \sqrt{k^2 + m_{\text{eff}}^2(\eta)}$. This equation has two linear independent solutions, which we interpret as a complex solution v_k. Making use of the Wronskian, we normalize:

$$\Im(v_k'v_k^*) = \frac{v_k'v_k^* - {v_k'}^*v_k}{2i} = \frac{W[v_k, v_k^*]}{2i} = 1.$$

Note that the normalization is independent of η due to the equations of motion.
The normalization is equivalent to

$$(v_k, v_{k'}) = \delta_{kk'},$$

with the scalar product defined above.

5.2 Bogolyubov transformations

For general manifolds, two isotropic mode functions are related via

$$v_k^* = \alpha_k u_k^* + \beta_k u_k,$$

with Bogolyubov coefficients α_k, β_k. Normalization is kept, if $|\alpha_k|^2 - |\beta_k|^2 = 1$. The operators are related via the Bogolyubov transformation:

$$\hat{b}_{\mathbf{k}}^- = \alpha_k \hat{a}_{\mathbf{k}}^- + \beta_k^* \hat{a}_{-\mathbf{k}}^+,$$
$$\hat{b}_{\mathbf{k}}^+ = \alpha_k^* \hat{a}_{\mathbf{k}}^+ + \beta_k \hat{a}_{-\mathbf{k}}^-.$$

The b vacuum can be expressed in terms of the a vacuum:

$$|0_b\rangle = \left[\prod_{\mathbf{k}} \frac{1}{\sqrt{|\alpha_{\mathbf{k}}|}} \exp\left(-\frac{\beta_{\mathbf{k}}^*}{2\alpha_{\mathbf{k}}} \hat{a}_{\mathbf{k}}^+ \hat{a}_{-\mathbf{k}}^+\right)\right] |0_a\rangle .$$

The product converges, if $|\beta_{\mathbf{k}}|^2 \to 0$ faster than $\frac{1}{k^3}$. This is the mean density of b-particles in the mode $\chi_{\mathbf{k}}$ in the a-vacuum, since $\langle 0_a|\hat{N}_b|0_a\rangle = |\beta_{\mathbf{k}}|^2 \delta^{(3)}(0)$. The complete mean density of b-particles is $\int d^3\mathbf{k}\, |\beta_k|^2$ which converges under the same condition as above.

5.3 Choice of vacua

One possible vacuum is the instantaneous lowest energy vacuum: minimize the energy at a given instant of time η_0. A solution exists, if $\omega_k^2(\eta_0) > 0$. The initial conditions for the mode functions in this case are[6]:

$$v_{\mathbf{k}}(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}},$$
$$v_{\mathbf{k}}'(\eta_0) = i\sqrt{\omega_k(\eta_0)},$$

which turn out to be isotropic. The Hamiltonian is:

$$\hat{H}(\eta_0) = \int d^3\mathbf{k}\, \omega_k(\eta_0) \left(\hat{a}_{\mathbf{k}}^+ \hat{a}_{\mathbf{k}}^- + \frac{1}{2}\delta^{(3)}(0)\right).$$

$|0_{\eta_0}\rangle$ is the vacuum of instantaneous diagonalization, since the Hamiltonian is diagonal in its exited states. At later times, the vacuum usually turns into a superposition of excited states. In general, there is no unique prescription to define a vacuum. Several problems may occur:

- It makes sense to talk about a particle with momentum p, if $\Delta p \ll p$ and hence the spatial size $\lambda \gg \frac{1}{p}$. But in this region, the spatial curvature may change. For high wave numbers and small curvature, this effect might become irrelevant, even on a cosmological scale.
- In general the 4d curvature counts, not only the spatial curvature. For example for a FLRW universe, the frequency $\omega_k(\eta) = \sqrt{k^2 + m_{\text{eff}(\eta)}^2}$ can become imaginary. In this case, the modes become growing and decaying. No lowest energy state exists.

[6] Compare also with the free field solution (5.2).

- Further the definition depends on the chosen reference frame.
- Even for small deviations in the curvature and on a small time interval, the particle production from $|0_{\eta_1}\rangle$ to $|0_{\eta_2}\rangle$ can be infinite.

Another possible vacuum choice which avoids the last difficulty is the adiabatic vacuum. If the variation of ω_k is only small in some range of η one can use a WKB approximation. In this adiabatic regime, at fixed η_0 one solves for the mode function of the adiabatic vacuum, s.t. $v_k(\eta_0)$ and $v_k'(\eta_0)$ coincide with the WKB approximation

$$v_k^{\mathrm{WKB}}(\eta) = \frac{1}{\sqrt{\omega_k(\eta)}} \exp\left[i \int_{\eta_0}^{\eta} d\eta \, \omega_k(\eta)\right]$$

at η_0.

6 Particle creation

In the Heisenberg picture, the states are time-independant, while the operator change in time. Let us prepare our system in the state $|0_a\rangle$, the vacuum corresponding to operators $\hat{a}_{\mathbf{k}}^{\pm}$. These operators can be chosen in a natural way, if the universe is static at initial time (see section 2). After an interaction with the gravitational field, let us assume, that our universe is again in a static state. But now the natural operators are $\hat{b}_{\mathbf{k}}^{\pm}$, with vacuum $|0_b\rangle$. Since our universe is still in the state $|0_a\rangle$, but particles are counted with $\hat{N}_b$, the mean density of b-particles is

$$\int d^3\mathbf{k} \, |\beta_k|^2 \,,$$

as we have seen above. The energy density is

$$\int d^3\mathbf{k} \, \omega_k \, |\beta_k|^2 \,.$$

6.1 Example

Consider a massless scalar field in a FLRW universe which is static in the past and future. We want to solve

$$\chi_k'' - \Delta \chi_k + m_{\mathrm{eff}}^2 \chi_k = 0,$$
$$m_{\mathrm{eff}}^2 = a^2 \left(\xi - \frac{1}{6}\right) R,$$

with initial condition (in-state)

$$\chi_k^{(\mathrm{in})}(\eta) = \frac{\mathrm{e}^{-i\omega\eta}}{\sqrt{2\omega}}.$$

We can reformulate the problem as an integral equation

$$\chi_k(\eta) = \chi_k^{(\mathrm{in})}(\eta) + \frac{1}{\omega} \int_{-\infty}^{\eta} d\tilde{\eta} \, \chi_k(\tilde{\eta}) \, m_{\mathrm{eff}}(\tilde{\eta}) \sin \omega(\eta - \tilde{\eta}),$$

which can be solved perturbatively by iteration.
One application is particle creation due to gravitational interactions during inflation. After

a deSitter expansion ($p = -\rho$), the universe enters a matter dominated phase ($p = 0$). The energy contribution can be estimated by

$$\rho \simeq (1 - 6\xi)^2 \rho_{\text{Vac}} \frac{\rho_{\text{Vac}}}{\rho_{\text{Planck}}}.$$

Here ρ_{Vac} is the density driving inflation and ρ_{Pl} is the Planckian density. For GUT scale inflation, this contribution is negligible, if reheating is efficient:

$$\rho_{\text{Vac}} \simeq \left(10^{11}\ \text{GeV}\right)^4$$
$$\rho_{\text{Planck}} \simeq \left(10^{15}\ \text{GeV}\right)^4$$

7 Fluctuations

First we calculate fluctuations in a vacuum state. There are two ways to calculate fluctuations, leading to the same result:

- Equal time correlation functions:

$$\langle 0|\hat{\chi}(\mathbf{x},\eta)\,\hat{\chi}(\mathbf{y},\eta)|0\rangle \sim \int d^3\mathbf{k}\ |v_k(\eta)|^2 \frac{\sin kL}{kL} \sim k^3 |v_k|^2, \text{ at } k \sim \frac{1}{L},$$

  where $L = |\mathbf{x} - \mathbf{y}|$ is the comoving distance (the physical distance being $a(\eta)L$).

- Fluctuations over averaged fields, with window-function (bump-function) W_L of order 1 at a scale L:

$$\langle 0| \left(\int d^3\mathbf{x}\, W_L(\mathbf{x})\, \hat{\chi}(\mathbf{x},\eta) \right)^2 |0\rangle \sim \int d^3\mathbf{k}\, |v_k|^2 |\widetilde{W}_1(\mathbf{k}L)|^2.$$

  Since the Fourier transformed window-function $\widetilde{W}_1(\mathbf{k}L)$ is of order 1 at scale $|\mathbf{k}| \sim \frac{1}{L}$, the result is the same as above, up to factors of order 1.

For the free field, one gets:

$$\delta\chi_L(\eta) \sim \frac{k^{3/2}}{(k^2 + m^2)^{1/4}} \sim \begin{cases} k^{3/2} & \text{for large } L \\ k & \text{for small } L \end{cases}$$

In a non-vacuum state, the fluctuations become:

$$(\delta\chi_L^{(b)})^2 = (\delta\chi_L)^2 \left(1 + 2|\beta_k|^2\right).$$

Another oscillating term occurs, which vanishes, averaging over large times, but can in general yield a factor even smaller than 1.

7.1 Example

For the effective mass:

$$m_{\text{eff}}^2(\eta) = \begin{cases} m_0^2 & \eta < 0 \text{ and } \eta > \eta_1 \\ -m_0^2 & 0 < \eta < \eta_1 \end{cases}$$

one gets particle densities

$$|\beta_k|^2 \sim \begin{cases} \left(\frac{m_0}{k}\right)^4 & \text{for large } k \\ \mathrm{e}^{2m_0\eta_1} & \text{for small } k \end{cases}$$

The fluctuations now are given by

$$\delta\chi_L(\eta) \sim \frac{k^{3/2}}{(k^2+m^2)^{1/4}} \sim \begin{cases} \mathrm{e}^{m_0\eta_1}\, k^{3/2} & \text{for large } L \\ k & \text{for small } L \end{cases}$$

hence there is an enhancement of the fluctuations of $\mathrm{e}^{m_0\eta_1}$ on large scales.

8 Fields in de Sitter spacetime

A de Sitter universe is a FLRW universe with $a = a_0\,\mathrm{e}^{Ht}$, corresponding to $\epsilon = -p$, leading to $\epsilon = \text{const}$. The usual coordinate patch

$$ds^2 = dt^2 - a^2 d\mathbf{x}^2$$

covers only a finite part of de Sitter space-time[7], which is nevertheless enough for cosmological applications. In cosmology, a de Sitter universe is used for inflation, with a large Hubble parameter H. Due to dark matter and dark energy, also our current universe seems to resemble de Sitter on large scales, but with a tiny Hubble parameter. So the expansion of our universe increases exponentially.
In a de Sitter universe, there exists a horizon with scale $a(t_0)\, r_{\max}(t_0) = \frac{1}{H}$ which can only be reached asymptotically by lightlike particles.
Switch to conformal coordinates[8]:

$$ds^2 = a^2(d\eta^2 - d\mathbf{x}^2)$$

Now consider

$$k|\eta| \sim \frac{1}{L}\frac{1}{aH} = \frac{L_{\text{horizon}}}{L_{\text{phys}}},$$

with physical wavelength $L_{\text{phys}} = a(\eta)L$. Small values give superhorizon modes, heavily affected by gravity. In de Sitter the modes go like a power of η. Large values correspond to subhorizon modes, almost unaffected by gravity, so one gets flat modes. At a time when

[7] This follows from the fact, that for a massive observer $-\infty < t < 0$ corresponds to finite eigentime.

[8] Coordinate transformation:

$$\eta = -\frac{1}{H}\,\mathrm{e}^{-Ht}, \qquad a(\eta) = -\frac{1}{H\eta}.$$

Here $-\infty < \eta < 0$ corresponds to $-\infty < t < \infty$.

$L_{\text{horizon}} = L_{\text{phys}}$, horizon crossing occurs. Superhorizon modes do not have a particle interpretation. Only correlation functions make sense.
The effective frequencies in de Sitter are imaginary for $k|\eta|$ small enough (superhorizon modes). Hence instantaneous vacuum states cannot be defined. Instead define the Bunch-Davies vacuum by applying the Minkowski vacuum prescription as $\eta \to -\infty$:

$$v_k \to \frac{1}{\omega_k} \mathrm{e}^{i\omega_k \eta}, \qquad v_k' \to i\omega_k v_k.$$

An explicit solution can be given in terms of Bessel functions. In inflation, de Sitter space-time is usually applied for times $\eta_i < \eta < \eta_f$. Therefore in this setup the Bunch-Davies vacuum can only be chosen for subhorizon modes, $k|\eta_i| \gg 1$.
The fluctuation look as following:

$$\delta\Phi_{L_{\text{phys}}}(\eta) = \begin{cases} \text{unknown}, & \eta < \eta_i, \text{ or } L_{\text{phys}} \gtrsim L_{\max} = H^{-1}\frac{\eta_i}{\eta} \\ L_{\text{phys}}^{-1}, & L_{\text{phys}} < H^{-1} \\ H|L_{\text{phys}}H|^{n-3/2}, & L_{\text{phys}} > H^{-1} \end{cases}$$

with $n = 3/2 - \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \simeq 1$ for small masses. Since $L_{\max} \sim \mathrm{e}^{Ht}$, the unknown regime gets shifted to high physical scales rather quickly. A de Sitter inflation therefore helps to iron out unkown fluctuations. On small scales, the fluctuations are like in Minkowski space $\sim L_{\text{phys}}^{-1}$, while on large scales they approach a constant value H. These fluctuations account for inhomogeneities in the early universe in the inflation model.

9 Some physical effects

9.1 Lamb shift

Energy splitting between $2p$ and $2s$: different geometry of electron cloud $\longrightarrow$ different interaction with vacuum fluctuations.

9.2 spontaneous radiation of hydrogen atoms

$2p \to 1s$, due to interactions of electrons with vacuum fluctuations of electromagnetic field (otherwise: $2p$ is stable).

9.3 Schwinger effect

e^+e^- pair production in an electromagnetic field. If virtual pairs get separated, they can become real, if $leE \geq 2m_e$. Probability:

$$P \sim \exp\left(-\frac{m_e^2}{eE}\right).$$

9.4 Pair production due to gravity

Works only for time-dependant (non-static) gravitational field, otherwise the virtual pair does not get apart.

9.5 Casimir effect

Consider two uncharged plates in a distance L in vacuum in $1 + 1$ dimensions. The field has now discrete frequencies due to boundary conditions. The shift in vacuum energy is[9]

$$\epsilon - \epsilon_0 = \frac{\pi}{2L^2} \sum_{n=1}^{\infty} n - \lim_{L \to \infty} \frac{\pi}{2L^2} \sum_{n=1}^{\infty} n = -\frac{\pi}{24L^2}.$$

Here ϵ denotes the energy per length L In $3 + 1$ dimensions:

$$\frac{F}{A} = -\frac{\pi^2}{240L^4}.$$

10 The Unruh effect

10.1 Rindler space-time

We follow [3], section 4.5, with slightly different metrics and sign conventions.
The Unruh effect deals with an accelerated observer in flat Minkowski space-time. For simplicity, we only look at the 2-dimensional case. The relevant metric is Rindler space-time:

$$ds_R^2 = (a\rho)^2 d\tau^2 - d\rho^2.$$

The metric is the analogue of polar coordinates for Minkowski space. A transformation to $ds_M^2 = dt^2 - dx^2$ can be given via

$$\begin{aligned} x &= \rho \cosh a\tau \\ t &= \rho \sinh a\tau. \end{aligned} \tag{10.1}$$

In Euclidean space, the lines of constant radius are circles, while in the Minkowski case, lines of constant ρ are hyperbolas $x^2 - t^2 = \text{const}$. But in special relativity, an observer that is uniformally accelerated with acceleration a follows a hyperbola exactly parametrized by (10.1). Hence Rindler space-time describes uniformal acceleration, as sketched in Figure 10.1. Null geodesics in flat Minkowski space satisfy

$$0 = ds_M^2 = dt^2 - dx^2.$$

So out-going geodesics (moving in positive x-direction) are described by constant u_M, while in-going geodesics correspond to constant v_M, where

$$\begin{aligned} u_M &= t - x, \\ v_M &= t + x \end{aligned}$$

are the null coordinates. The same works for Rindler space-time. Here geodesics satisfy

$$0 = ds_R^2 = (a\rho)^2 d\tau^2 - d\rho^2,$$

[9] using zeta-function regularization:

$$\sum n \to \zeta(-1) = -\frac{1}{12}.$$

The result can also be derived with a cut-off function.

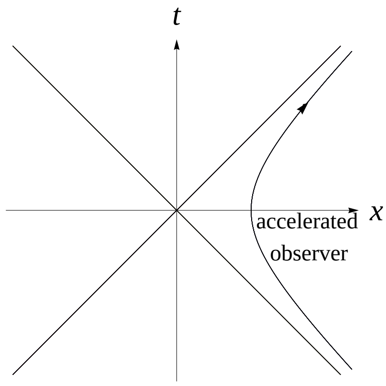


Figure 1: Rindler space-time

and null coordinates are

$$u_R = a\tau - \log\rho,$$
$$v_R = a\tau + \log\rho.$$

In null coordinates, the metrics read:

$$ds_M^2 = du_M\,dv_M,$$
$$ds_R^2 = \mathrm{e}^{v_R - u_R}\,du_R\,dv_R.$$

Written in this way, the Rindler metric can be seen to be conformally flat (as is any metric in two dimensions).
An observer following the hyperbola as sketched in Figure 10.1 can only receive signals from the lower quarter, while it can only send signals to the upper quarter. Hence there is no way to synchronize clocks in these cases and hence no notion of distance. The Rindler metric therefore only covers one quarter of Minkowski space. In particular there is a horizon, corresponding to the boundary of the Rindler wedge $x > |t|$ in Minkowski space time. The horizon consists of $u_R = \infty$ (or $u_M = 0$) and $v_R = -\infty$ (or $v_M = 0$).
Next we add a second wedge, which we get by reflecting the old one in the origin. The old one is called $\mathfrak{R}$ (right), the new one is called $\mathfrak{L}$ (left). The remaining wedges are called $\mathfrak{F}$ (future) and $\mathfrak{P}$ (past).
We want to find out the physical implications of the acceleration. Let us quantize a scalar field as usual, that obeys the Klein-Gordon equation

$$\left(\partial_t^2 - \partial_x^2\right)\phi = \partial_{u_M}\partial_{v_M}\phi = 0.$$

Since the equation of motion is conformally invariant, in Rindler space it is simply

$$\partial_{u_R}\partial_{v_R}\phi = 0.$$

The solutions are the same as given above in equation (5.2). In Minkowski space, they are:

$$v_k^M = \frac{1}{\sqrt{\omega_k}}\,\mathrm{e}^{i\omega t}\,.$$

In Rindler space, we can solve in $\mathfrak{R}$ or $\mathfrak{L}$ respectively:

$$v_k^{R(\mathfrak{R})} = \begin{cases} \frac{1}{\sqrt{\omega_k}}\, \mathrm{e}^{i\omega(v_R-u_R)} & \text{in } \mathfrak{R} \\ 0 & \text{in } \mathfrak{L} \end{cases}$$
$$v_k^{R(\mathfrak{L})} = \begin{cases} 0 & \text{in } \mathfrak{R} \\ \frac{1}{\sqrt{\omega_k}}\, \mathrm{e}^{-i\omega(v_R-u_R)} & \text{in } \mathfrak{L} \end{cases}$$

Now $v_k^{R(\mathfrak{R})}$ is a complete set of solutions in $\mathfrak{R}$, while $v_k^{R(\mathfrak{L})}$ is complete in $\mathfrak{L}$. Furthermore, one can show, that together they are complete in the whole Minkowski space.
The fields can be expanded in either set:

$$\phi = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}} \left((v_k^M)^* \, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \, \hat{a}_{\mathbf{k}}^- + v_{\mathbf{k}}^M \, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} \, \hat{a}_k^+ \right) =$$
$$= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}} \left((v_k^{R(\mathfrak{R})})^* \, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \, \hat{b}_{\mathbf{k}}^{\mathfrak{R}-} + v_{\mathbf{k}}^{R(\mathfrak{R})} \, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} \, \hat{b}_k^{\mathfrak{R}+} + (v_k^{R(\mathfrak{L})})^* \, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \, \hat{b}_{\mathbf{k}}^{\mathfrak{L}-} + v_{\mathbf{k}}^{R(\mathfrak{L})} \, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} \, \hat{b}_k^{\mathfrak{L}+} \right)$$

Now there are two vacuum states:

$$\text{Minkowski vacuum: } \hat{a}_k^- \left|0_M\right\rangle = 0,$$
$$\text{Rindler vacuum: } \hat{b}_k^{\mathfrak{R}-} \left|0_R\right\rangle = \hat{b}^{\mathfrak{L}-} \left|0_R\right\rangle = 0.$$

The system must be in the Minkowski vacuum, being the lowest energy state of an inertial observer. To prepare a state in Minkowski space in the Rindler vacuum, one needs an infinite amount of energy: if the energy is uniformly distributed in Rindler space, the energy density diverges (after zero-energy subtraction) at the horizon, which corresponds to an infinite coordinate region in Rindler space. This is not sensible, since it would lead to an infinite backreaction due to Einstein equations.
To compute the excitation number, an accelerated observer measures in the Minkowski vacuum, we have to compute generalized Bogolyubov transformations, i.e. the transformation is no longer diagonal in the momentum. Rather:

$$\hat{d}_\Omega^- = \int_{-\infty}^{\infty} d\omega \, \left(\alpha(\omega, \Omega) \, \hat{c}_\omega^- + \beta(\omega, \Omega) \, \hat{c}_\omega^+ \right) ,' \ldots$$

The Rindler mode functions must pick up some negative frequency, since they are non-smooth at the horizon, due to the sign flip in the exponent. Therefore, they cannot be superpositions of the analytic Minkowski mode functions.
To do the calculation, we construct two combinations that are analytic and bounded in the lower half planes in u^R and v^R, namely:

$$\frac{1}{\sqrt{2\sinh(\frac{\pi\omega}{a})}} \left(u_k^{R(\mathfrak{R})} + \mathrm{e}^{-\frac{\pi\omega}{a}} \, (u_{-k}^{R(\mathfrak{L})})^* \right)$$
$$\frac{1}{\sqrt{2\sinh(\frac{\pi\omega}{a})}} \left((u_{-k}^{R(\mathfrak{R})})^* + \mathrm{e}^{\frac{\pi\omega}{a}} \, u_k^{R(\mathfrak{L})} \right)$$

as one can check in a straight-forward fashion.
Expanding in these new mode functions, the corresponding operators must now annihilate the

Minkowski vacuum. By taking inner products $(\phi, u_k^{R(\mathfrak{R})})$ and $(\phi, u_k^{R(\mathfrak{L})})$ one gets the needed Bogolyubov coefficients.
Between the accelerated frame and the flat metric, one finds the mean density of particles with momentum Ω as seen from the accelerator to be:

$$n_\Omega = \frac{\exp\left(-\frac{\pi\omega}{a}\right)}{2\sinh\frac{\pi\omega}{a}} = \frac{1}{\exp\left(\frac{2\pi\omega}{a}\right)-1}.$$

This corresponds to a Bose-Einstein blackbody radiation of Unruh temperature

$$T = \frac{a}{2\pi}.$$

11 Black holes and thermodynamics

11.1 Basic facts on Black holes

Several solutions for black holes were found. They can be listed by increasing complexity:

1. The first solution found was the Schwarzschild black hole (1916):

$$ds^2 = -\left(1-\frac{r_s}{r}\right)dt^2 + \frac{1}{1-\frac{r_s}{r}}dr^2 + r^2 d\Omega^2,$$

   where $r_s = 2M = 3\frac{M}{M_\odot}$km is the Schwarzschild radius.
   It solves the Einstein equations in vacuum $R_{ab} = 0$, and is

   - stationary: there exists a time-like Killing vector field ξ^a (or a one-parameter group of isometries ϕ_t with time-like orbits).
   - spherically symmetric: the isometry group contains SO(3).
   - static[10]: the orbits of ϕ_t are integrable, meaning there exist hypersurfaces orthogonal to ϕ_{t_0} for any t_0. By Frobenius' theorem:

$$\xi^\flat \wedge d\xi^\flat = 0.$$

     Here $\xi^\flat$ is the corresponding one-form, which in abstract-index notation is written $\xi_a = g_{ab}\xi^a$.

11.2 Hawking radiation

In 1 + 1 dimensions, the situation is the same as for the Unruh effect. Kruskal coordinates give a conformal metric. The proper acceleration is given by $a = \frac{1}{4M}$, leading to the Hawking temperature:

$$T_H = \frac{\hbar c^3}{Gk}\cdot\frac{1}{8\pi M} \sim \frac{M_p^2}{M}.$$

[10]Staticity follows in fact from being spherically symmetric, which is known as Birkhoff's theorem (and generalizes a theorem by Newton). In other words, even if a mass distribution oscillates in a spherically symmetric fashion, the space-time outside is static, hence there is no emission of gravitational waves. There exist no gravitational monopoles.

With massive fields, the particle density is given by:

$$n = \frac{1}{\exp\left(\frac{E}{T_H}\right) - 1},$$

with $E = \sqrt{m^2 + k^2}$ which is only large enough for small masses. In $3+1$ dimensions, one must include a greybody factor < 1 accounting for a barrier-like potential in the Laplace equation. If one considers a black hole being formed by gravitational collapse, the radiation can be considered to stem from particles staying close to the horizon for a long time and escaping eventually.

11.3 Thermodynamics

Black holes have an entropy

$$S_{\mathrm{BH}} = \int \frac{dE}{T} = \int \frac{dM}{T} = \frac{1}{4} 4\pi (2M)^2 = \frac{1}{4} A,$$

using the black hole radius $R = 2M$.
Stefan-Boltzmann[11]:

$$L = \gamma \sigma T_H^4 A,$$

leads to:

$$M = -\int dt\, L = M_0 \left(1 - \frac{t}{t_L}\right)^{1/3}, \qquad t_L = 5120\pi \frac{M_0^3}{\gamma},$$

and hence leads to a finite life-time (without quantum gravity effects)[12].
Black holes have negative heat capacity

$$C_{\mathrm{BH}} = \frac{\partial E}{\partial T} = \frac{\partial M}{\partial T} = -\frac{1}{8\pi T^2},$$

so they become colder, if they absorb mass and energy. Only reservoirs of finite volume with a heat capacity being positive and not too large can lead to a stable equilibrium.

Hawking's original derivation of the black hole radiation [6] was somewhat different. He did not consider an eternal Schwarzschild black hole, but instead he looked at matter collapsing to a black hole. Intuitively, black hole evaporation can be understood as highly energetic particles staying close to the horizon for a long time. Eventually, they can escape and reach the detector. Though the two approaches seem to desribe different physical setups (in the first case, an eternal black hole emitting particles, in the second a black hole forming), they give the same result, namely a black body radiation with Hawking temperature.

Another, more heuristic interpretation goes as follows: consider a pair of particles, one with positive, the other with negative energy. The latter can tunnel through the event horizon. Inside the black hole, the time-like killing vector field turns space-like, and hence the particle becomes real (now propagating time-like). The particle of positive energy can propagate to infinity. The probability of this effect to happen is proportional to the surface gravity, the area of the black hole horizon.

[11] L: flux of energy, γ: degrees of freedom, $\sigma = \frac{\pi^2}{60}$: Stefan-Boltzmann constant.
[12] Only for very light black-holes, the life-time is small w.r.t. the age of the universe.

12 Path integral methods

Effective action:

$$\mathrm{e}^{i\Gamma[J]} = \int \mathcal{D}q \; \mathrm{e}^{iS[q,J]}\,.$$

Get:

$$\frac{\langle 0_{\rm out}|T\,\hat{q}(t_1)\cdots\hat{q}(t_n)|0_{\rm in}\rangle}{\langle 0_{\rm out}|0_{\rm in}\rangle} = \frac{\frac{\delta^n \, \mathrm{e}^{i\Gamma[J]}}{\delta J(t_1)\cdots\delta J(t_n)}}{\mathrm{e}^{i\Gamma[J]}}$$

$$\frac{\langle 0_{\rm in}|T\,\hat{q}(t_1)\cdots\hat{q}(t_n)|0_{\rm in}\rangle}{\langle 0_{\rm in}|0_{\rm in}\rangle} = \left.\frac{\frac{\delta^n \, \mathrm{e}^{i\Gamma[J]}}{\delta J(t_1)\cdots\delta J(t_n)}}{\mathrm{e}^{i\Gamma[J]}}\right|_{G_{\rm F}\to G_{\rm ret}}$$

where we replaced the Feynman propagator by the retarded propagator in the last equation.

12.1 Backreaction

For a system classical background plus quantum system, one gets a total effective action:

$$S_{\rm eff}[J] = S_{\rm background}[J] + \Gamma[J].$$

The equation of motion for the background now becomes:

$$\frac{\delta S_B[J]}{\delta J(t)} + \left.\frac{\delta \Gamma[J]}{\delta J(t)}\right|_{G_{\rm F}\to G_{\rm ret}} = 0.$$

The effective potential Γ leads to source terms $\langle 0_{\rm in}|\hat{q}(t)|0_{\rm out}\rangle$.
Examples:

- Electromagnetism: J is here replaced by A_μ, the equations of motion are

  $$\frac{1}{4\pi}\partial_\nu F^{\mu\nu} + \langle \hat{j}^\mu\rangle = 0,$$

  with:

  $$\frac{\delta\Gamma[J]}{\delta A_\mu} = \frac{\int \mathcal{D}\psi\, i\frac{\delta S}{\delta A_\mu}\,\mathrm{e}^{iS}}{\int \mathcal{D}\psi\;\mathrm{e}^{iS}} = \frac{\int \mathcal{D}\psi\, j^\mu\,\mathrm{e}^{iS}}{\int \mathcal{D}\psi\;\mathrm{e}^{iS}} = \langle j^\mu\rangle\,.$$

- Semi-classical Gravity: J gets replaced by $g^{\mu\nu}$, with equations of motion:

  $$-\frac{\sqrt{-g}}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{\sqrt{-g}}{2}\langle \hat{T}_{\mu\nu}\rangle = 0.$$

  with:

  $$\frac{\delta\Gamma[J]}{\delta g^{\mu\nu}} = \frac{\int \mathcal{D}\phi\,\frac{i\delta S}{\delta g^{\mu\nu}}\,\mathrm{e}^{iS}}{\int \mathcal{D}\phi\;\mathrm{e}^{iS}} = \frac{\int \mathcal{D}\phi\,\frac{\sqrt{-g}}{2}T_{\mu\nu}\,\mathrm{e}^{iS}}{\int \mathcal{D}\phi\;\mathrm{e}^{iS}} = \frac{\sqrt{-g}}{2}\langle \hat{T}_{\mu\nu}\rangle\,,$$

  leading to a vacuum polarization.

13 Heat kernel

For an action $S = \frac{1}{2}\int dt\, \phi\mathcal{O}\phi$, the effective action is given by[13]:

$$\Gamma = \frac{1}{2}\log\det\mathcal{O}.$$

Find an appropriate Hilbert space and an Hermitian operator $\hat{M}$ with the same eigenvalues λ_n. Define the zeta function of the operator $\hat{M}$ by

$$\zeta_{\hat{M}}(s) = \sum_{n=0}^{\infty}\left(\frac{1}{\lambda_n}\right)^s = \mathrm{Tr}\,\hat{M}^{-s}.$$

Now:

$$\Gamma = -\frac{1}{2}\left.\frac{d\zeta_{\hat{M}}(s)}{ds}\right|_{s=0}.$$

Define the heat kernel:

$$\hat{K}_{\hat{M}}(\tau) = \sum_n \mathrm{e}^{-\lambda_n\tau}\left|\psi_n\right\rangle\left\langle\psi_n\right|$$

The effective potential can be calculated in terms of the trace of the heat kernel, using the identity[14]:

$$\zeta_{\hat{M}}(s) = \frac{1}{\Gamma(s)}\int_0^\infty d\tau\, t^{s-1}\,\mathrm{Tr}\,\hat{K}_{\hat{M}}(\tau).$$

Solve perturbatively:

$$\hat{K}_{\hat{M}} = \hat{K}_0 + \hat{K}_1 + \ldots .$$

Writing $-\hat{M} = \Box + \hat{s}$, one gets a chain of equations:

$$\begin{aligned} \frac{d\hat{K}_0}{d\tau} &= \Box\hat{K}_0, & \hat{K}_0(0) &= \hat{1}, \\ \frac{d\hat{K}_1}{d\tau} &= \Box M\hat{K}_1 + \hat{s}\hat{K}_0, & \hat{K}_1(0) &= \hat{0}, \\ \frac{d\hat{K}_2}{d\tau} &= \Box M\hat{K}_2 + \hat{s}\hat{K}_1, & \hat{K}_2(0) &= \hat{0}, \\ &\ldots \end{aligned}$$

[13]Since in Euclidean notation:

$$\mathrm{e}^{\Gamma} = \int \mathcal{D}\phi\;\mathrm{e}^{-\frac{1}{2}\int dt\,\phi\mathcal{O}\phi} = \sqrt{\det\mathcal{O}}.$$

[14]Use:

$$\begin{aligned} \zeta_{\hat{M}}(s) &= \sum_n(\lambda_n)^{-s}, \\ \mathrm{Tr}\,\hat{K}_{\hat{M}}(\tau) &= \sum_n \mathrm{e}^{-\lambda_n\tau} \\ \Gamma(s) &= \lambda^s\int_0^\infty d\tau\;\mathrm{e}^{-\lambda\tau}\,\tau^{s-1}. \end{aligned}$$

One can either expand in curvature or in τ. The first corrections are the same. Seeley-DeWitt expansion in d dimensions is given by

$$\langle x|\hat{K}(\tau)|x\rangle = \frac{\sqrt{g}}{(4\pi\tau)^{d/2}}\left(1 + a_1(x)\tau + a_2(x)\tau^2 + \ldots\right).$$

For $\mathcal{O} = -\Box + V(x)$, the first Seeley-DeWitt coefficient is given by

$$a_1(x) = \frac{1}{6}R(x) - V(x).$$

The $\zeta_{\hat{M}}$-function diverges for $\tau \to 0$, due to UV divergences[15]. The divergences have to be regularized and renormalized by defining new physical parameters in addition to the bare parameters (like the cosmological constant, Newton's constant and a R^2 coupling constant). Finite terms can be found in the expansion up to second order in the curvature (not in τ), yielding the Polyakov action (in two dimensions)

$$\begin{aligned}\Gamma[g_{\mu\nu}] &= \frac{1}{96\pi}\int d^2x\,\sqrt{g}R\Box^{-1}R = \\ &= \frac{1}{96\pi}\int d^2x\,d^2y\,\sqrt{g(x)}\sqrt{g(y)}R(x)G(x,y)R(y).\end{aligned}$$

The variation of Γ under a conformal transformation leads to a conformal anomaly:

$$\langle 0|\hat{T}^{\mu}{}_{\mu}(x)|0\rangle = \frac{a_1(x)}{4\pi} = \frac{R(x)}{24\pi}.$$

This origins from the fact, that in path integral quantization, while S is conformally invariant, the measure $\mathcal{D}\Phi$ cannot be chosen to be generally covariant and conformally covariant at the same time.
An alternative derivation comes from looking at the Seeley-DeWitt expansion to first order.

14 C^*-algebras

Definition 14.1 (C^*-algebra)**.** A C^*-algebra A is a associative $\mathbb{C}$-algebra with norm and $\star : A \to A$, $\mathbb{C}$ anti-linear, s.t.

1. $a^{**} = a$
2. $(ab)^* = b^*a^*$
3. $\|ab\| \leq \|a\|\,\|b\|$
4. $\|a^*\| = \|a\|$ (isometry)
5. $\|aa^*\| = \|a\|^2$ (C^*-property)

[15] Another divergence for $\tau \to \infty$ only appears in the Seeley-DeWitt expansion. This is not a serious problem, since the expansion is only valid for small τ anyway.

Definition 14.2. Let A be a C^* algebra, $a \in A$.
Resolvent set:

$$r_A(a) = \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - a \text{ invertible}\}.$$

Spectrum:

$$\sigma_A(a) = \mathbb{C} \setminus r_A(a).$$

Spectral radius:

$$\rho_A(a) = \sup\{|\lambda| \mid \lambda \in \sigma_A(a)\}.$$

A self-adjoint element a is called positive, if one of the following holds[16]:

1. $a = b^2$, for some b self-adjoint,
2. $a = c^* c$, for some c,
3. $\sigma_A(a) \subset [0, \infty)$.

Lemma 14.3. *$\sigma_A(a)$ is nonempty and compact. Further:*

$$\rho_A(a) = \lim_{n \to \infty} \|a^n\|^{1/n} = \inf_{n \in \mathbb{N}} \|a^n\|^{1/n} \leq \|a\|.$$

The norm is uniquely determined by A and $\star$:

$$\|a\|^2 = \sqrt{\rho_A(a^* a)}.$$

Lemma 14.4. *Let $f : A \to B$ be an injective and unit-preserving $\star$-morphism. Then:*

$$\|f(a)\| = \|a\|.$$

15 States, observables, and operators

Definition 15.1. Let A be a C^* algebra, $\mathcal{H}$ a Hilbert space. A *representation* is a $\star$-map from A to $\mathcal{L}(\mathcal{H})$, the linear bounded operators. The *universal* representation is given by

$$\bigoplus_{\tau \in S(A)} \pi_\tau : A \to \mathcal{L}\left(\bigoplus_{\tau \in S(A)} H_\tau \right),$$

where $S(A)$ is the set of all states. $\mathcal{H}_\tau$ is the Hilbert space defined in the GNS representation below. A *state* is a positive linear functional $\tau : A \to \mathbb{C}$ with $\|\tau\| = 1$. An *observable* is a self-dual operator on $\mathcal{H}$.

- Pure vector state: $\mathcal{H}$ Hilbert space, $\mathcal{O} \in \mathcal{L}(\mathcal{H})$ linear bounded operator in $\mathcal{H}$, $\Psi \in \mathcal{H}$ vector. Set:
$$\tau(\mathcal{O}) = (\Psi, \mathcal{O}\Psi) = \langle \Psi | \mathcal{O} | \Psi \rangle .$$
In physics notation, $|\Psi\rangle$ is called state.

[16] An element can be decomposed into positive elements, $a = a_+ - a_-$ with $a_+ = \frac{1}{2}(a + |a|)$ and $a_- = \frac{1}{2}(a - |a|)$.

- Mixed state: density matrix (trace class operator). Let $\sum_i \lambda_i = 1$ and $\lambda_i > 0$. Define[17]:

$$\tau(\mathcal{O}) = \sum_i \lambda_i(\psi_i, \mathcal{O}\psi_i) = \sum_i \lambda_i \langle\psi_i|\mathcal{O}|\psi_i\rangle = \mathrm{Tr}\left[\left(\sum_i \lambda_i |\psi_i\rangle\langle\psi_i|\right)\mathcal{O}\right] = \mathrm{Tr}(\rho\mathcal{O}).$$

  If $\mathrm{Tr}(\rho^2) = 1$, the state is pure, if it is less than 1, the state is mixed.
  Note: a convex combination of states in the algebraic sense leads to mixed states (or density matrices), while a convex combination of states as vectors (in the usual QM sense) gives again a pure state.

Theorem 15.2 (GNS representation)**.** *Let τ be an arbitrary state. A scalar product is given by $\langle a, b\rangle = \tau(b^*a)$ Next we construct a Hilbert space. Therefore mod out the null space N_τ consisting of elements $a \in C^*$ with $\tau(a^*a) = 0$. Complete the pre-Hilbert coset-space A/N_τ to a Hilbert space H_τ.*
Now the Gelfand-Naimark-Segal (GNS) representation is given by:

$$\pi_\tau : A \to \mathcal{L}(H_\tau),$$
$$(\pi_\tau(a))\,[b] = [ab].$$

16 Wightman axioms

Consider the following set-up:

- States are one-dimensional subspaces of a separable complex Hilbert space $\mathcal{H}$. $\mathcal{H}$ carries a unitary representation of the Poincaré group $P = \mathrm{SL}(2,\mathbb{C}) \ltimes \mathbb{R}^{1,3}$. *Leads to notions of energy, momentum, angular momentum, center of mass.*

- $\mathcal{F}$ is a collection of fields. A field $\Phi \in F$ is a operator-valued distribution ($\mathcal{S}$ denotes the Schwartz functions), i.e.

$$\Phi : \mathcal{S}(\mathbb{R}^{1,3}) \to \mathcal{O}(\mathcal{H}),$$
$$\Phi(f) : D_{\Phi,f} \to \mathcal{H}.$$

  Here $\Phi(f)$ is defined on $D_{\Phi,f}$, a linear dense subspace of $\mathcal{H}$. Further there exists a linear dense subspace D with $D_{\Phi,f} \subset D$ for all Φ and f. *Leads to notions of locality.*

- There exists a vacuum vector $\Omega \in D$.

Now the Wightman axioms are:

[17]This has to be distinguished from a convex linear combination (superposition) of pure states (in physics sense), which would lead to

$$\tau(\mathcal{O}) = \langle \sum_i \lambda_i\psi_i|\mathcal{O}|\sum_j \lambda_j\psi_j\rangle\,,$$

and $\sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ cannot be written in terms of $\sum_i\sum_j \widetilde{\lambda}_i\widetilde{\lambda}_j |\psi_i\rangle\langle\psi_j|$.

1. Covariance: $\forall A \in P, \forall f \in \mathcal{S}, \forall \Phi \in \mathcal{F}, \forall x \in \mathbb{R}^{1,3}$, the following hold:

$$\begin{aligned}
U(A)\Omega &= \Omega, \\
U(A)D &\subset D, \\
\Phi(f)D &\subset D, \\
U(A)\Phi(f)U(A)^* &= \Phi(Af), \\
Af(x) &= f(A^{-1}x)
\end{aligned}$$

2. Spectrum condition. Let T_0 denote time-translations and $T_j \in P,\ j = 1, 2, 3$ spatial translations. Then

$$\begin{aligned}
U(T_0) &= \exp(iP_0),\ P_0 \text{ Hamiltonian}, \\
U(T_j) &= \exp(-iP_j).
\end{aligned}$$

   The joint spectrum of T_j is the forward light cone C_+.

3. Locality, microscopic causality: if $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ are causally separated, $\Phi(f)$ and $\Psi(g)$ commute (or anti-commute).

Remark 16.1. Irreducible representations of the Poincaré group P of non-negative energy, can be described by the so-called Wigner classification. The mass $m = \sqrt{P^2}$ is a Casimir invariant of P. Now the following situations can occur

- $m > 0$: in the rest frame: $P_0 = m, P_i = 0$, allowing for a representation of Spin(3).
- $m = 0, P_0 > 0$. Go to the light-cone frame $P = (k, 0, 0, -k)$, giving us a representation of the double cover of SE(2), denoting the special Euclidean group (rotations plus translations). There occur irreducible representations characterized by the helicity (multiple of $\frac{1}{2}$) and continuous spin representations.
- $m = 0, P = 0$. This is the vacuum, leading to the trivial representation.

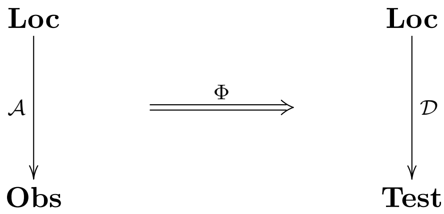
17 Algebraic Quantum Field Theory

Definition 17.1. We define three Categories:

1. **Loc**:
   - obj(**Loc**): globally hyperbolic Lorentzian manifolds, oriented and time-oriented. Or: bundles over space-time.
   - mor(**Loc**): isometric embeddings. Or: conformal transformations (for CFTs). The morphisms preserve orientation and time-orientation. Further, for $a, b \in M_1$ causally connected, also causally curves between $\psi(a), \psi(b)$ must be contained in $\mathrm{im}(\psi) \subseteq M_2$.
2. **Obs**
   - obj(**Obs**): C^*-algebras. Or: $\star$-algebras. Or: von Neumann algebras.
   - mor(**Obs**): injective, unital $\star$-homomorphisms.

3. **Test**
   - obj(**Test**): smooth, compactly supported test functions.
   - mor(**Test**): push-forward.

Now a locally covariant QFT is a covariant functor $\mathcal{A}$ between **Loc** and **Obs**. Let $\mathcal{D}$ be a covariant functor between **Loc** and **Test**. A locally covariant quantum field Φ is a natural transformation between $\mathcal{A}$ and $\mathcal{D}$.



The following conditions must be satisfied:

1. $\mathcal{A}$ is causal: if the images of two manifolds are causally separated, the images of the corresponding algebras commute.
2. $\mathcal{A}$ obeys the time-slice-axiom: if the image of a manifold M_1 contains a Cauchy surface in M_2, the image of the algebra $\mathcal{A}(M_1)$ is the complete algebra $\mathcal{A}(M_2)$, not just a subset.

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