42. Chern Classes

Version 1.1

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We now concentrate on the principal fibre bundles with structure group $G = GL(n,\mathbb{C})$ in order to define the Chern classes of vector bundles of rank n.

<u>(42.1) DEFINITION</u>: Let Q_k $(0 \leq k \leq n)$ be defined by

$$\det\left(tI_n - \frac{1}{2\pi i}X\right) = \sum_{k=0}^{n} Q_k(X)\, t^{n-k}, \quad Q_k(X) \in \mathbb{C},$$

$X \in \mathfrak{gl}(n,\mathbb{C})$. Here, the usual description of X as an $n\times n$ array $X = (X^{ij})$ is the unique description $X = X^{ij}e_{ij}$ with respect to the standard basis e_{ij} consisting of the matrices with 1 in exactly one component (position i,j) and 0 in the other components.

Consequently the Q_k are k-homogeneous polynomials and from the definitions we conclude

$$Q_k(X) = \left(\frac{-1}{2\pi i}\right)^k \sum_{1\leq i_1 < \dots < i_k \leq n} \det\begin{pmatrix} X^{i_1 i_1} & \dots & X^{i_1 i_k} \\ \vdots & & \vdots \\ X^{i_k i_1} & \dots & X^{i_k i_k}\end{pmatrix}$$

In particular, $Q_0 = 1$.

(42.2) PROPOSITION: $Q_k \in P^\bullet_G(\mathfrak{g})$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$, and for block matrices $X = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, $A \in \mathfrak{gl}_i(\mathbb{C})$, $B = \mathfrak{gl}_{n-i}(\mathbb{C})$:

$$Q_k(X) = \sum_{j=0}^{k} Q_j(A)\, Q_{k-j}(B)$$

Pf. For $g \in G$ and $X \in \mathfrak{gl}_n(\mathbb{C})$: $\mathrm{Ad}_g(X) = gXg^{-1}$. det is Ad invariant, hence $Q_k \in P^k_G(\mathfrak{g})$. The formula follows from

$$\det\left(tI_n - \tfrac{1}{2\pi i}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) = \det\left(tI_i - \tfrac{1}{2\pi i}A\right)\det\left(tI_{n-i} - \tfrac{1}{2\pi i}B\right). \qquad \square$$

(42.3) PROPOSITION: The restrictions Q_k to $\mathfrak{u}(n) = \mathrm{Lie}\, U(n)$ are $U(n)$-invariant and real-valued:

$$Q_k\big|_{\mathfrak{u}(n)} \in P^\bullet_{U(n)}(\mathfrak{u}(n)).$$

They are algebraically independent and they generate $P^\bullet_{U(n)}(\mathfrak{u}(n))$.

Pf: $$\sum \overline{Q_k(X)\, t^{n-k}} = \overline{\det\left(tI_n - \tfrac{1}{2\pi i}X\right)} = \det\left(tI_n + \tfrac{1}{2\pi i}\overline{X}\right)$$

$$= \det\left(tI_n - \tfrac{1}{2\pi i}X^T\right) = \det\left(tI_n - \tfrac{1}{2\pi i}X\right) = \sum Q_k(X)\, t^{n-k},$$

hence Q_k is real-valued. Of course Q_k is Ad-invariant with respect to $U(n) \subset GL(n,\mathbb{C})$ since it is Ad-invariant

with respect to $GL(n,\mathbb{C})$.
That the Q_k generate the full algebra $P^{\bullet}_{U(n)}(\mathfrak{u}(n))$ will be proven using the elementary symmetric polynomials.
We recall that each $X \in \mathfrak{u}(n)$ is conjugate to a diagonal matrix $\mathrm{diag}(\lambda_1, \dots \lambda_n)$ with λ_j the eigenvalues of X. Consequently, every invariant polynomial is a symmetric polynomial in the eigenvalues $\lambda_1, \dots \lambda_n$.
Moreover,

$$\det\left(tI_n - \tfrac{1}{2\pi i} X\right) = \det\left(tI_n - \tfrac{1}{2\pi i}\,\mathrm{diag}(\lambda_1, \dots \lambda_n)\right)$$

$$= \prod_{j=1}^{n} \left(t - \tfrac{1}{2\pi i}\lambda_j\right).$$

We conclude

$$Q_k\big|_{\mathfrak{u}(n)}(X) = \left(\frac{-1}{2\pi i}\right)^k \sigma_k(\lambda_1, \dots \lambda_n)$$

where

$$\sigma_k(\lambda_1, \dots \lambda_n) = \sum_{1 \le i_1 < \dots < i_k \le n} \lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_k}$$

is the k-the elementary symmetric polynomial. The result now follow from the fact, that every symmetric polynomial P is a polynomial in the σ_i, i.e

$$P(\lambda_1, \dots \lambda_n) = q(\sigma_1, \dots \sigma_n)\ , \quad q \in \mathbb{C}[\sigma_1, \dots \sigma_n].$$

$\square$

After all these preliminaries we now come to the Chern classes of a complex vector bundle E:

Let $E \to M$ be a complex vector bundle of rk n. Let $R = GL(E)$ be the frame bundle of E. One can reduce R to the unitary frame bundle $P = \mathcal{U}(E)$: There exists a hermitian metric on E, and P is the bundle of unitary frames of E_a, $a \in M$. E is then isomorphic to the associated bundles

$$E \cong R \times_{GL(n,\mathbb{C})} \mathbb{C}^n \cong P \times_{U(n)} \mathbb{C}^n .$$

Let Q_k as above, $Q_k \in P^{\cdot}_G(\mathfrak{gl}(n,\mathbb{C}))$, $G = GL(n,\mathbb{C})$, and $Q_k|_{\mathfrak{u}(n)} \in P^{\cdot}_{U(n)}(\mathfrak{u}(n))$. It is easy to check

$$W_R(Q_k) = W_P(Q_k|_{\mathfrak{u}(n)}) .$$

This shows that $W_P(Q_k)$ is a real-valued cohomology class and that $W_P(Q_k|_{\mathfrak{u}(n)})$ does not depend on chosen metric on E.

(42.4) DEFINITION: Let E be a complex vector bundle on M of rank n. For $k \in \{1, 2, \dots n\}$

$$\boxed{c_k(E) := W_R(Q_k) = [Q_k(\Omega^\omega)] \in H^{2k}_{dR}(M, \mathbb{R})}$$

is the k-th Chern class of E. Here, R is the frame bundle $R = GL(E)$ of E and ω is any connection form on R. Moreover, $c_0(E) := 0$, and

$$c(E) := c_0(E) + c_1(E) + \dots + c_n(E) \quad \text{(total Chern class)}.$$

Notation: $\quad c(E) = \det\left(tI_n - \frac{1}{2\pi i}\Omega^\omega\right)$.

Let $\Omega^\omega = (F^{ij})$ with respect to the standard base of $gl(n, \mathbb{C})$ (see above). Each F^{ij} is a real-valued 2-form. Hence

$$c_k(E) = \left(\frac{-1}{2\pi i}\right)^k \sum_{1 \le i_1 < \dots < i_k \le n} \quad \sum_{\sigma = \binom{i_1 \dots i_k}{j_1 \dots j_k} \in S_k} \operatorname{sign}\sigma \; F^{i_1 j_1} \wedge \dots \wedge F^{i_k j_k}$$

In particular:

(42.5) PROPOSITION:

$1°$ $c_1(E) = \left[-\frac{1}{2\pi i} \sum F^{jj}\right] = \left[-\frac{1}{2\pi i} \operatorname{Tr}(\Omega)\right]$

2° $c_2(E) = \left[-\frac{1}{4\pi^2} \sum_{1\le i<j\le n} (F^{ii}\wedge F^{jj} - F^{ij}\wedge F^{ji})\right]$

$= \left[\frac{1}{8\pi^2}\left(Tr(\Omega\wedge\Omega) - Tr(\Omega)\wedge Tr(\Omega)\right)\right],$

since $Tr(\Omega)\wedge Tr(\Omega) = 2\sum_{i<j} F^{ii}\wedge F^{jj}$ and $Tr(\Omega\wedge\Omega) = 2\sum_{i<j} F^{ij}\wedge F^{ji}$.

3° $c_n(E) = \left(-\frac{1}{2\pi i}\right)^n \det\Omega$ ($n = \text{rank } E$).

<u>(42.6) EXAMPLES:</u> 1° For a trivial vector bundle E:

$c(E) = 1$, i.e. $c_k(E) = 0$ for all $k \geq 1$.

And the same holds for vector fields which admit a flat connection.

2° Let E be induced from a $SU(n)$-pfb P, i.e. $E \cong E_\varrho = P\times_\varrho \mathbb{C}^n$ with a representation $\varrho : SU(n) \to GL(n,\mathbb{C})$. Then

$c_1(E) = 0$ (by 46.5.1°) and

$c_2 = \left[\frac{1}{8\pi^2} Tr(\Omega\wedge\Omega)\right]$ (by 46.5.2°).

3° $c_1(\det E) = c_1(E)$

A covariant derivative D on the vector bundle induces a covariant derivative D^{det} on $\det E = \Lambda^n E$ through

$$D^{det}(s_1 \wedge \ldots \wedge s_n) := \sum_{j=1}^{n} s_1 \wedge \ldots \wedge Ds_j \wedge \ldots \wedge s_n$$

The corresponding curvatures satisfy

$$F^{D^{det}}(s_1 \wedge s_2 \wedge \ldots \wedge s_n) = D^{det} D^{det}(s_1 \wedge \ldots \wedge s_n)$$

$$= \sum_{j=1}^{n} s_1 \wedge s_2 \wedge \ldots \wedge F^D s_j \wedge \ldots \wedge s_n$$

$$= \mathrm{Tr}(F^D)\, s_1 \wedge \ldots \wedge s_n \, .$$

4^0 For the tautological line bundle $T \to P_1(\mathbb{C})$: $c_1(T) = -1$. This assertion needs an explanation: The Riemann sphere $P_1(\mathbb{C})$ is a compact twodimensional manifold. For any 2dim. compact manifold we have $H^2_{dR}(M, \mathbb{R}) \cong \mathbb{R}$ and such an isomorphism is given by the evaluation

$$H^2_{dR}(M, \mathbb{R}) \to \mathbb{R}, \quad \alpha \mapsto \int_M \alpha .$$

In this sense, $c_1(T) = -1$, i.e. we have to show that for any connection ω on the frame bundle R of T:

$$-\frac{1}{2\pi i}\int_{\mathbb{P}_1}\Omega^\omega = -1 \text{ , or } \int_{\mathbb{P}_1}\Omega^\omega = 2\pi i .$$

Let us recall the structure of the tautological bundle. $\mathbb{P}_1$ is the projective line, i.e. the quotient manifold of $\mathbb{C}^2\setminus\{0\} =: R$ with respect to the equivalence relation $\xi \sim \xi' \iff \exists \lambda \in \mathbb{C} : \xi = \lambda\xi'$, i.e. $\mathbb{P}_1$ is the space of complex lines in $\mathbb{C}^2$ through 0.

$$R \xrightarrow{\gamma} \mathbb{P}_1 := R/_\sim , \quad \gamma(z_0,z_1) =: (z_0 : z_1)$$

A description of the equivalence relation in the spirit of group action is given by the right action of $\mathbb{C}^\times$ on R:

$$R\times\mathbb{C}^\times , \ (\xi,\lambda) \longmapsto \xi\lambda .$$

The orbit space $R/_{\mathbb{C}^\times}$ exists and is (isomorphic) to $\mathbb{P}_1$. Now, the tautological complex line bundle $T \xrightarrow[\pi]{} \mathbb{P}_1$ is the bundle where the fibre T_a over each „line" $a \in \mathbb{P}_1$ is the line a itself : $a = T_a$. It can be defined as the following subbundle of the trivial bundle $\mathbb{P}_1\times\mathbb{C}^2 \longrightarrow \mathbb{P}_1$

$$T = \{ (a,(w_0,w_1)) \in \mathbb{P}_1\times\mathbb{C}^2 \mid (w_0,w_1) = 0 \text{ or } (w_0,w_1) \in a \} .$$

The frame bundle $R = R(T)$ of T is the bundle

$$R = \mathbb{C}^2 \setminus \{0\} \xrightarrow{\gamma} \mathbb{P}_1$$

which we have used already.
With respect to the standard charts

$$U_0 := \{ (z_0 : z_1) \mid z_0 \neq 0 \} \xrightarrow{\varphi_j} \mathbb{C} , \quad (z_0, z_1) \longmapsto \frac{z_1}{z_0} ,$$

$$U_1 := \{ (z_0 : z_1) \mid z_1 \neq 0 \} \longrightarrow \mathbb{C} , \quad (z_0, z_1) \longmapsto \frac{z_0}{z_1} ,$$

the change of frame $\varphi_{01} : \varphi_1^{-1}(U_{01}) = \mathbb{C}^\times \to \mathbb{C}^\times = \varphi_0^{-1}(U_{01})$

$$\varphi_{01} : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times , \quad w \longmapsto \frac{1}{w} .$$

On $U_0 (U_1)$ the tautological bundle T has the local trivialization

$$\psi_0 : \pi^{-1}(U_0) \longrightarrow U_0 \times \mathbb{C} , \quad \big((z_0 : z_1), (w_0, w_1)\big) \longmapsto \big((z_0 : z_1), w_0\big) ,$$

$$\psi_1 : \pi^{-1}(U_1) \longrightarrow U_1 \times \mathbb{C} , \quad \big((z_0 : z_1), (w_0, w_1)\big) \longmapsto \big((z_0 : z_1), w_1\big) .$$

with $\psi_1^{-1}((z_0 : z_1), \lambda) = ((z_0 : z_1), (\frac{z_0}{z_1} \lambda, \lambda))$, hence

$$\psi_0 \circ \psi_1^{-1}((z_0 : z_1), \lambda) = \psi_0 \big((z_0 : z_1), (\tfrac{z_0}{z_1}\lambda, \lambda)\big) = \big((z_0 : z_1), \tfrac{z_0}{z_1}\lambda\big)$$

Therefore the transition function is

$$g_{01} : U_{01} \longrightarrow \mathbb{C}^\times \quad \text{is} \quad (z_0 : z_1) \longmapsto \frac{z_0}{z_1}.$$

Similarly it can be seen that the transition function for $R \xrightarrow[\gamma]{} \mathbb{P}_1$ is $g_{01}(z_0 : z_1) = \frac{z_0}{z_1}$ as well. This shows that $R \xrightarrow[\gamma]{} \mathbb{P}_1$ is in fact the frame bundle with structure group $\mathbb{C}^\times = GL(1, \mathbb{C})$

Note that T is the complex tangent bundle!

We prefer to calculate in the corresponding $U(1)$-bundle of unitary frames which is

$$(S^3, \gamma, \mathbb{P}_1, U(1)) \qquad (\text{included in } R : S^3 \subset R)$$

We consider the 1-form $\omega \in \mathcal{A}^1(S^3, i\mathbb{R})$, $i\mathbb{R} = \mathrm{Lie}\, U(1)$:

$$\omega = \frac{1}{2}\left(\bar{w}_0 dw_0 - w_0 d\bar{w}_0 + \bar{w}_1 dw_1 - w_1 d\bar{w}_1 \right)$$

We want to check that ω is a connection form. Let $g \in U(1)$ and $X \in T_\xi S^3$ $\quad (\xi = (w_0, w_1)$ with $|w_0|^2 + |w_1|^2 = 1)$. Then X is of the form $X = [\, w_0(t), w_1(t) \,]_\xi$.

$$(\Psi_g^*\omega)_\xi(X) = \omega_{\xi g}\left([w_0(t)g, w_1(t)g]_{\xi g}\right) =$$

$$= \frac{1}{2}\left(\overline{w_0 g}\, w_0'(0)g - w_0 g\, \overline{w_0'(0)g} + \overline{w_1 g}\, w_1'(0)g - w_1 g\, \overline{w_1'(0)g}\right)$$

$$= \frac{1}{2}\left(\overline{w_0}\, w_0'(0) - w_0 \overline{w_0'(0)} + \overline{w_1}\, w_1'(0) - w_1 \overline{w_1'}(0)\right)$$

$$= \omega_\xi(X) = g^{-1}\omega_\xi(X)\, g .$$

Hence, ω satisfies $(\omega 2)$.

The fundamental vector field X^* of $X = ix \in i\mathbb{R}$ in $\xi = (w_1, w_2)$ is:

$$X^*(\xi) = [w_0 e^{itx}, w_1 e^{itx}],$$

and

$$\omega_\xi(X^*(\xi)) = \frac{1}{2}\left(\overline{w_0}\, w_0\, ix - w_0\, \overline{w_0}(-ix) + \overline{w_1} w_1 ix - w_1 \overline{w_1}(-ix)\right)$$
$$= \frac{1}{2}\left(ix\left(2(|w_0|^2 + |w_1|^2)\right)\right) = X^*,$$

hence $(\omega 1)$.

Now, $\Omega^\omega = d\omega$ (since $[\omega,\omega] = 0$), i.e.

$$\Omega^\omega = -\left(dw_0 \wedge d\overline{w_0} + dw_1 \wedge d\overline{w_1}\right).$$

With respect to the chart U_1 we see

$$\Omega^{\omega} = \gamma^* \circ \varphi_1^*(F)$$

with the form

$$F = - \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} \qquad \text{on } \mathbb{C}$$

Now, $c_1(T) = -\frac{1}{2\pi i} \int_{\mathbb{C}} F = \frac{1}{2\pi i} \int \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$

$$\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} \underset{z=x+iy}{=} -\frac{1}{\pi} \int \frac{dx \wedge dy}{(1+|x|^2+|y|^2)^2}$$

$$\Big((dx+idy) \wedge (dx-idy) = i\,dy \wedge dx - i\,dx \wedge dy = -2i\,dx \wedge dy \Big)$$

Hence,

$$c_1(T) = -\frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \frac{r}{(1+r^2)^2}\, dr$$

$$= -\frac{1}{\pi} \cdot 2\pi \; \frac{-2}{1+r^2} \Big|_0^{\infty} = -1$$

(42.6) PROPOSITION: Properties of Chern classes.

1° For isomorphic vector bundles $E_1 \cong E_2$:

$$c(E_1) = c(E_2) \quad , \text{ i.e. } \forall k \in \mathbb{N}: c_k(E_1) = c_k(E_2)$$

2° For smooth $f: N \to M$ and complex vector bundles $E \to M$ one has

$$c(f^*E) = f^* c(E).$$

3° For complex vector bundles E_1, E_2 over M:

$$c(E_1 \oplus E_2) = c(E_1)\, c(E_2), \text{ i.e.}$$

$$c_k(E_1 \oplus E_2) = \sum_{j=0}^{k} c_{k-j}(E_1)\, c_j(E_2)$$

4° Let $E^\vee$ be the dual bundle of E

$$c_k(E^\vee) = (-1)^k c_k(E).$$

Pf. 1° & 2° follow from the good properties of the Weil homomorphism (cf. section 41).

3° is a consequence of 42.2: Let E_1 be a vb of rk n_1 and E_2 be a vb of rk n_2, with frame bundles R_1 resp R_2. Then $R_1 \times R_2$ is a pfb with structure group $GL(n_1, \mathbb{C}) \times GL(n_2, \mathbb{C}) \overset{\varrho}{\subset} GL(n_1+n_2, \mathbb{C})$ and $E_1 \oplus E_2$ is isomorphic to $(R_1 \times R_2) \times_\varrho \mathbb{C}^{n_1+n}$. $R_1 \times R_2$ is a reduction of the frame bundle $R(E_1 \times E_2)$ to the subgroup $GL(n_1, \mathbb{C}) \times GL(n_2, \mathbb{C}) \overset{\varrho}{\subset} GL(n_1+n_2, \mathbb{C})$.

As a consequence, $c_k(E_1 \oplus E_2) = W_P(Q_k|_{g_1 \oplus g_2})$ with $g_i = gl(n_i, \mathbb{C})$, $i=1,2$, where $g_1 \oplus g_2 \subset gl(n_1+n_2, \mathbb{C})$ by $A+B \longmapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

For connection forms ω_1 on R_1 and ω_2 on R_2 we can check that

$$\omega := pr_1^* \omega_1 + pr_2^* \omega_2$$

is a connection on $R_1 \times R_2$. We confirm that

$$\Omega^\omega = pr_1^* \Omega^{\omega_1} + pr_2^* \Omega^{\omega_2} .$$

Furthermore, $Q_k(pr_i^* \Omega^{\omega_i}) = Q_k(\Omega^{\omega_i})$ on M. From 42.2 finally:

$$Q_k(\Omega^\omega) = \sum_{j=0}^{k} Q_{k-j}(\Omega^{\omega_1}) \, Q_j(\Omega^{\omega_2}) .$$

This implies the product formula 3°.

4° Let R be the frame bundle of E and $R^\vee$ the frame bundle of $E^\vee$. Let $\Phi : R \to R^\vee$ the smooth map which maps a basis in E to the dual basis in $R^\vee$. We obtain

$$\Phi(pg) = \Phi(p)(g^{-1})^T , \quad (p,g) \in P \times G ,$$

Let ω be a connection form on R. Then

$$\omega^\vee := -(\Phi^{-1})^*(\omega^T)$$

is a connection form on $R^\vee$ with

$$\Omega^{\omega^\vee} = -(\Phi^{-1})^*(\Omega^\omega)^T.$$

Finally, on M:

$$Q_k(\Omega^{\omega^\vee}) = Q_k\left(-(\Phi^{-1})^*(\Omega^\omega)^T\right) = Q_k(-\Omega^{\omega^T})$$

$$= Q_k(-\Omega^\omega) = (-1)^k Q_k(\Omega^\omega)$$

$\square$

<u>(42.7) EXAMPLE:</u> The chern classes of the complex projective space $\mathbb{P}_n = \mathbb{P}_n(\mathbb{C})$, i.e. $c(T\mathbb{P}_n)$, are given by

$$c(T\mathbb{P}_n) = (1+\alpha)^{n+1}$$

where $\alpha = c_1(T^\vee)$, $T \to \mathbb{P}_n$ the tautological bundle.

<u>Pf.</u> $K = T^\vee$ is called the canonical bundle, $K = \Lambda^n T\mathbb{P}_n$. For the trivial line bundle $\Theta = \mathbb{P}_n \times \mathbb{C}$ one can show the decomposition

$$T\mathbb{P}_n \oplus \mathcal{O} \cong \underbrace{K \oplus K \oplus \dots \oplus K}_{(n+1) \text{ times}}$$

As a consequence,

$$c(T\mathbb{P}_n) = c(T\mathbb{P}_n)\, c(\mathcal{O}) = c(T\mathbb{P}_n \oplus \mathcal{O}) = c(K)\, c(K) \dots c(K)$$
$$= (1+\alpha)^{n+1}$$

To obtain the decomposition, define

$$T^\perp := \{ (\gamma(z), w) \in \mathbb{P}_n \times \mathbb{C}^{n+1} : w \perp z \}$$

$$\left(T = \{ (\gamma(z), w) \in \mathbb{P}_n \times \mathbb{C}^{n+1} : w = 0 \text{ or } w \in \gamma(z) \} \right)$$

Evidently, $T \oplus T^\perp \cong \mathbb{P}_n \times \mathbb{C}^{n+1}$. We need the fact

$$\mathrm{Hom}(T, T^\perp) \cong T\mathbb{P}_n .$$

This isomorphism is given by ($z \in \mathbb{C}^{n+1} \setminus \{0\}$):

$$\mathrm{Hom}(T, T^\perp)_{\gamma(z)} \to T_{\gamma(z)} \mathbb{P}_n , \quad h_{\gamma(z)} \longmapsto T_z \gamma (h_{\gamma(z)}(z)).$$

More precisely, $T_{\gamma(z)} = \{ (\gamma(z), \lambda z) : \lambda \in \mathbb{C} \} \subset \mathbb{P}_n \times \mathbb{C}^{n+1}$, hence

$$h_{\gamma(z)} : T_{\gamma(z)} \to T^\perp_{\gamma(z)}$$

is completely determined by $h_{\gamma(z)}(z) = (\gamma(z), w(h, z))$, $w(h, z) \perp z$, which is interpreted as the tangent vector $\big(z, w(h,z)\big) \in \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ at $z \in \mathbb{C}^{n+1}$!

Now, the decomposition follows:

$$
\begin{aligned}
T\mathbb{P}_n \oplus \mathcal{O} &\cong \mathrm{Hom}(T, T^{\perp}) \oplus \mathcal{O} \\
&\cong \mathrm{Hom}(T, T^{\perp}) \oplus \mathrm{Hom}(T, T) \\
&\cong \mathrm{Hom}(T, T \oplus T^{\perp}) \\
&\cong \mathrm{Hom}(T, \mathbb{P}_n \times \mathbb{C}^{n+1}) \\
&\cong \mathrm{Hom}(T, \underbrace{\mathcal{O} \oplus \ldots \oplus \mathcal{O}}_{(n+1)\text{-times}}) \\
&\cong \underbrace{T^{\vee} \oplus \ldots \oplus T^{\vee}}_{(n+1)\text{ times}}
\end{aligned}
$$

$\square$

We come to the splitting principle which has its importance for practical calculations but also for theoretical considerations

(42.8) PROPOSITION: Let E be a complex vector bundle over the manifold M. Then there exists a smooth map $f: M' \to M$ with:

1° $f^*: H^k_{dR}(M, \mathbb{R}) \to H^k_{dR}(M', \mathbb{R})$ is injective.

2° The pullback $f^*E \to M'$ of E is isomorphic to a sum of line bundles $L_j \to M'$:

$$f^*E \cong L_1 \oplus \ldots \oplus L_n$$

As a consequence $c(f^*E) = \prod_{j=1}^{n}(1+c_1(L_j)) = f^*(c(E))$, and to calculate $c_k(E)$ we need to calculate $c_1(L_j)$ for line bundles only.
Moreover, regarding the properties in 42.6, we see that the theory of Chern classes for complex vector bundles could be based on the 1st Chern classes of line bundles.

The proof of the splitting 42.8 can be found in many texts, in particular in the book „Characteristic Classes" by Milnor / Stasheff (Princeton 1974) where the whole subject of characteristic classes is treated in a particularly good way.

We conclude the section and the course with some general remarks:

Further characteristic classes

1° Pontrjagin Classes

Given a real vector space E over M of rank n the complexification $E^{\mathbb{C}} := E \otimes \mathbb{C}$ is a complex vector space of rank n. Define:

$$p_k(E) = (-1)^k c_{2k}(E^{\mathbb{C}}) \in H^{4k}(M, \mathbb{R})$$

(The odd Chern classes $c_{2k+1}(E^{\mathbb{C}})$ vanish!)

2° Euler Class

Related to Euler characteristic of cpt M

$$\chi(M) := \sum (-1)^j \dim_{\mathbb{R}} H^j_{dR}(M, \mathbb{R}$$

through:

$$\int_M e(M,g) = \chi(M) \qquad (\text{GAUSS-BONNET})$$

for (M,g) cpt Riemannian manifold.

In general, for $E \to M$ real with Riemannian metric g of rank $2m$ one defines the Euler class $e(M,g) \in H^{2m}_{dR}(M,\mathbb{R})$ such that in particular

$$e(E,g)\cdot e(E,g) = p_m(E) = c_{2m}(E^{\mathbb{C}}).$$

3° Power series of characteristic classes

Let E be a complex vb of rank n with $E \cong L_1 \oplus L_2 \oplus \dots \oplus L_n$. Then

$$c(E) = \prod c(L_j) = \prod (1+c_1(L_j)) = \prod_{j=1}^{n}(1+x_j)$$

In general: $c(E) = \prod_{j=1}^{n} (1+x_j)$ formally

Chern character:

$$ch(E) = \sum_{j=1}^{n} e^{x_j} = \sum_{j=1}^{n}\sum_{\nu=1}^{\infty} \frac{x_j^{\nu}}{\nu!} \quad \in H^{\bullet}_{dR}(M,\mathbb{R})$$

$$td(E) = \sum_{j=1}^{n} \frac{x_j}{1-e^{-x_j}}$$

$\rightsquigarrow$ Theorem of Riemann-Roch-Hirzebruch

$\rightsquigarrow$ Theorem of Atiyah-Singer