36. The Trivial Case

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This is an introductory section at the beginning of Chap. X. Geometry of PFB, with the aim to describe the geometry of pfb locally and show how this geometry is a tool to formulate classical field theories (gauge field theories) as the counterparts of quantum field theories.

Ingrediants:

- M n-dim. base manifold (e.g. Minkowski space $\cong \mathbb{R}^4$)
- $G \subset GL(k,\mathbb{R})$ matrix Lie group
- $P := M \times G$ product; interpretation as a pfb
- $\mathfrak{g} = \mathrm{Lie}\, G \subset \mathfrak{gl}(k,\mathbb{R})$ matrix Lie algebra
- $A_\mu : \mathcal{U} \to \mathfrak{g}$ smooth, $\mu = 1, \dots n$, $\mathcal{U} \subset M$ open

  $A = (A_1, \dots A_n)$ gauge potential, $\alpha = A_\mu dq^\mu$ 1-form
- $\nabla_\mu = \partial_\mu + A_\mu$ Operator, $\partial_\mu = \frac{\partial}{\partial q^\mu}$
- $\varrho : G \to GL(r,\mathbb{K})$ representation
- E_ϱ associated vector bundle of rank $r \cong M \times \mathbb{K}^r$
- ∇_μ acts on sections $\phi \in \Gamma(\mathcal{U}, E_\varrho) \cong \mathcal{E}(\mathcal{U},\mathbb{K})^r$

(36.1) $\nabla_\mu \phi = \partial_\mu \phi + \rho_*(A_\mu)\phi \quad , \quad \phi \in \mathcal{E}(\mathcal{U})^r .$

Here, $\rho_*(A_\mu) = T_e\rho \circ A_\mu : \mathcal{U} \to \mathfrak{gl}(r, \mathbb{K})$.

$\nabla = (\nabla_1, \dots \nabla_n)$ cov. derivative, connection, ... on E_ρ with curvature F given by

(36.2) $F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \in \mathcal{E}(\mathcal{U}, \mathfrak{g})$

$[A_\mu, A_\nu] \neq 0$ in general if G not abelian.

(36.3) Gauge transformations are maps $\gamma \in \mathcal{E}(\mathcal{U}, G)$

$$A_\mu \longmapsto A'_\mu = A^\gamma_\mu = \gamma^{-1} A_\mu \gamma + \gamma^{-1}\partial_\mu \gamma \qquad \text{(cf. 23.8)}$$
$$F_{\mu\nu} \longmapsto F'_{\mu\nu} = F^\gamma_{\mu\nu} = \gamma^{-1} F_{\mu\nu} \gamma$$

In particular, for $G = U(1)$ and $\gamma(q) = \exp(i\varphi(q))$, $\varphi : \mathcal{U} \to \mathbb{R}$. $\alpha = A_\mu dq^\mu$, $F = F_{\mu\nu}\, dq^\mu \wedge dq^\nu$

(36.4) $\alpha^\gamma = \alpha + i d\varphi \quad , \quad A^\gamma_\mu = A_\mu + i \frac{\partial \varphi}{\partial q^\mu}$

$F^\gamma = F$ gauge independent force field

Now, let $\xi = (P, \pi, M, G)$ be a (trivial or non-trivial) pfb with a matrix Lie group G as structure group.

(36.5) DEFINITION: (M-connection) Let ξ be given by a cocycle (g_{ij}), $g_{ij} : U_{ij} \to G$, with respect to an open covering (U_j) of the base manifold M. An M-connection on ξ ("M" for "matrix) is a family (α_j) (of local gauge potentials)

$$\alpha_j \in \Omega^1(U_j, \mathfrak{g})$$

such that

$$\boxed{(MZ) \quad \alpha_j|_{U_{ij}} = g_{ij}^{-1} \alpha_i|_{U_{ij}} g_{ij} + g_{ij}^{-1} dg_{ij},} \qquad i,j \in I.$$

In the following (section 37): We present the concept of a connection on ξ in five different ways, all of which are equivalent to 36.5, whenever G is a matrix group.