35. Associated Bundles

Version 1.1

<u>(35.1) PROPOSITION:</u> Let $\xi = (P, \pi, M, G)$ a principal fibre bundle and let $\varrho : G \times F \to F$ be a left action of G on another manifold F. Then the right action

$$(P \times F) \times G \longrightarrow P \times F \ , \ ((p,y),g) \longmapsto (pg, g^{-1}y),$$

on $P \times F$ is free and the orbit space $P \times F/_G$ exists.
Notation: $P \times_G F := P \times F/_G$ or $P \times_\varrho F := P \times F/_G$ is the associated bundle (associated to the action ϱ).

<u>Proof</u>: If $(p,y) = (pg, g^{-1}y)$ for $(p,y) \in P \times F$ and $g \in G$ we conclude $g = e$ from $p = pg$, since the action $(p,g) \mapsto pg$ on P is free. Hence the above action of G on $P \times F$ is free. To apply 33.8 we could show that $R_G \subset (P \times F) \times (P \times F)$ is a closed submanifold. Instead, the direct construction:
Let $a \in M$ and $\varphi : P_U \to U \times G$ a local trivialization of the pfb. Then $\varphi(p) = (\pi(p), h(p))$ with $h : U \to G$ smooth and $h(pg) = h(p)g$ for $(p,g) \in P_U \times G$. We want to define a "bundle chart" on $P \times F/_G$ induced by φ.

Let $[p,y]$ denote the equivalence class determined by (p,y), i.e. $[p,y] = \{(pg, g^{-1}y) : g \in G\} =: \tau(p,y)$ and let $\tau : P \times F \to P\times F/_G$ the projection. We are more interested in the projection

$$\pi_F : P\times F/_G \longrightarrow M \ , \ [(p,y)] \mapsto \pi p$$

π_F is well-defined and continuous, since $\pi_F \circ \tau(p,y) = \pi(p)$. We want to define the "bundle chart"

$$\tilde{\varphi} : P\times F/_G|_U = \pi_F^{-1}(U) \longrightarrow U \times F$$

induced by $\varphi : P_U \to U \times G$. Set

$$\tilde{\varphi}([p,y]) := (\pi(p), h(p)y) \in U\times G \ , \ (p,y) \in \pi^{-1}(U)\times F.$$

$\tilde{\varphi}$ is well-defined: When $(p,y) \sim (p',y')$ we have $g \in G$ with $p' = pg$ and $y' = g^{-1}y$. Hence

$$(\pi(p'), h(p')y') = (\pi(p), h(p)g g^{-1}y) = (\pi(p), h(p)y).$$

$\tilde{\varphi}$ is clearly bijective and the corresponding transition functions are smooth. Therefore, these bundle charts define the structure of a manifold on $P \times F/G$ such that $\tau : P \times F \to P \times F/G$ is a submersion. Note that $\tilde{\varphi} \circ \tau = pr \circ \hat{\varphi}$, where $\hat{\varphi}(p,y) := (\varphi(p), h(p)y)) = (\pi(p), h(p), h(p)y)$:

$$\tilde{\varphi} \circ \tau(p,y) = \tilde{\varphi}([p,y]) = (\pi(p), h(p)y) = pr(\hat{\varphi}(p,y)).$$

We thus have the following commutativ diagram

$$\begin{tikzcd}
P_U \times F \arrow[r, "\hat{\varphi}"] \arrow[d, "\tau"'] & (U \times G) \times F \arrow[d, "pr"] & ((a,g),y) \arrow[d, mapsto] \\
P \times F/G|_U \arrow[r] & U \times F & (a,y)
\end{tikzcd}$$

As a consequence, up to diffeomorphisms τ maps like the projection pr, which is clearly a submersion. From the discussion in section 33B (cf. 33.14) that for the existence of the orbit space $P \times F/G$ it suffices to find a manifold structure on the Hausdorff space* $P \times F/G$ such that the projection map $\pi_F : P \times F \to P \times F/G$ is a submersion. $\square$

With respect to the projection $\pi_F : P \times_G F \to M$ our

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* It remains to check that the quotient topology is Hausdorff, equivalently that R_G is closed.

associated bundle is a fibration with typical fibre F. In fact, the associated bundle belongs to the class of fibre bundles with structure group. We will not discuss the case of a general fibre F but restrict to the case of a vector space $\mathbb{K}^r = F$ on which the matrix group $GL(r,\mathbb{K})$ acts from the left or – slightly more general – where the left action of a general Lie group is given by a representation $\rho : G \to GL(r,\mathbb{K})$.

<u>(35.2) FACT:</u> In this situation, $P\times_\rho \mathbb{K}^r$ is a vector bundle of rank r.

<u>Proof</u>. We only have to check that the $\tilde{\varphi}$ are linear on the fibres: On $E_a = \pi_F^{-1}(a)$ with $E := P\times_\rho \mathbb{K}^r$ we have the vector space operations

$$[p,y] + [p,y'] := [p, y+y']$$
$$\lambda [p,y] := [p, \lambda y]$$

Now,

$$\tilde{\varphi}\big([p,y] + \lambda[p,y']\big) = \big(\pi(p), h(p).(y+\lambda y')\big) \quad \text{and}$$

$h(p) \in G$ acts as the linear map $\rho(h(p)) : \mathbb{K}^r \to \mathbb{K}^r$, i.e.

$h(p)(y+\lambda y') = h(p)y + \lambda h(p)y'$. It follows

$$\tilde{\varphi}([p,y] + \lambda[p,y']) = (\pi(p), h(p).y + \lambda h(p)y')$$

is linear, and E a vector bundle. $\square$

<u>(35.3) REMARK:</u> 1° Every principal fibre bundle P can be given by a cocycle (g_{ij}), $g_{ij} \in \mathcal{E}(U_{ij}, G)$, as is explained in 34.5. When $\rho : G \to GL(r, \mathbb{K})$ is a representation of the Lie group G in $\mathbb{K}^r$ then $P \times_\rho \mathbb{K}^r$ is given by the cocycle $\rho(g_{ij}) \in GL(r, \mathbb{K})$ (up to isomorphism of vector bundles).

2° When the vector bundle E is given by the cocycle (g_{ij}), $g_{ij} : U_{ij} \to GL(r, \mathbb{K})$ then the frame bundle $R(E)$ is given by the same cocycle (cf. 34.5, as we have seen in 34.9). If we apply now the "association" procedure 35.1 with respect to the natural left action of $GL(r, \mathbb{K})$ on $\mathbb{K}^r$ we end up with the vector bundle $R(E) \times_\rho \mathbb{K}^r$ given by (g_{ij}) and isomorphic to E. Similarly, if we start with P as in 1° with $G = GL(r, \mathbb{K}^r)$ and obtain the associated vector bundle ($\varrho = \mathrm{id}_G$) E the frame bundle $R(E)$ is isomorphic to P and for E and P we can use the same cocycle. We thus have established a 1-to-1 correspondence

$$\left\{\begin{matrix}\text{isomorphism classes of}\\ \text{pfbs with structure}\\ \text{group } GL(r,\mathbb{K}) \text{ on } M\end{matrix}\right\} \cong \left\{\begin{matrix}\text{isomorphism classes}\\ \text{of vector bundles}\\ \text{of rank } r \text{ on } M\end{matrix}\right\} \cong \begin{matrix} H^1(M, GL(r,\mathbb{K}))\\ \text{(cf. section 17)}\end{matrix}$$

3° The case of a general Lie group G with a fixed representation ϱ can be put into this picture, as well. We call a vector bundle E attached to ϱ (and G is called its structure group) if there exist a cocycle (g_{ij}), $g_{ij} : U_{ij} \to GL(r, \mathbb{K})$, defining E, which is of the form $g_{ij} = \varrho(h_{ij})$, $h_{ij} \in \mathcal{E}(U_{ij}, G)$, a cocycle in G. The isom. classes of such vector bundles are the isom. classes of associated bundles $P \times_\varrho K^r$, where P is a pfb with structure group G. [11.1.11]

(35.4) REMARK: Because of the universal property of the quotient $P \times_G F = P \times F/_G$ we have a natural bijection

$$\mathcal{E}(P\times_G F) \cong \{ f\in \mathcal{E}(P\times F) \mid f(p,y) = f(pg, g^{-1}y)\ \forall g\in G\}$$
$$g \longmapsto g\circ\tau .$$

In general, the G-invariant functions on a G-manifold Q will be denoted by $\mathcal{E}(Q)^G$ or $\mathcal{E}^G(Q)$. Hence:

$$\mathcal{E}(P\times_G F) = \mathcal{E}(P\times F)^G. \ *$$

(35.5) REMARK: In physics the pfbs occur in the following way: A configuration is given by a pfb P with structure group G (the internal symmetry group). An equivariant local trivialization

$$\psi : P_U \to U\times G$$

is called a gauge, or a choice of gauge. Transition function $\varphi\circ\psi^{-1}$ from one gauge to another is a gauge transformation.
An important example is the trivial (product) bundle over Minkowski space M

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* This description of functions on $P\times F$ has an interesting generalization to sections of the vector bundles $E_\varrho = P\times_\varrho \mathbb{K}^r$.

$$P = M \times U(1)\ ,\quad P = M \times SU(2)\ ,\quad P = M \times G$$

A (still classical) particle field with internal symmetry group G is given by a representation $\varrho : G \to GL(r, \mathbb{K})$ and leads to the vector bundle

$$E_\varrho = P \times_\varrho \mathbb{K}^r \to M\ .$$

The "wave functions" are the sections of E_ϱ, i.e.

$$\Gamma(M, E_\varrho),$$

and the dynamics is given by the geometry on E_ϱ (resp. on P).

We thus have to develop the geometry on a principal fibre bundle $\xi = (P, \pi, M, G)$ which induces a corresponding geometry on the vector bundles E_ϱ, where $\varrho : G \to GL(r, \mathbb{K})$ is a representation. We deal with these geometries in the next chapter, Chap. X.