33B. Homogeneous Spaces: Orbit Space

The objective of this section is to complement Section 33 by providing a proof of theorem 33.8. To reformulate the theorem (see below) let G be a Lie group with a left action

$$G \times M \longrightarrow M, \quad (g,a) \longmapsto ga,$$

on a manifold M. The action induces the equivalence relation

$$R = \{ (a,b) \in M \times M \mid \exists g \in G : b = ga \} = R_G$$

with the orbits Ga, $a \in M$, as equivalence classes. Let $M/G = M/R$ be the set of equivalence classes with the projection $\pi : M \to M/G$, $a \mapsto Ga = \pi(a)$. On M/G there always exists the quotient topology with the properties:

- $\pi : M \to M/G$ is continuous
- A map $f : M/G \to Y$ with values in any topological space Y is continuous if and only if $f \circ \pi$ is continuous.

$$\begin{tikzcd} M \arrow[d, "\pi"'] \arrow[dr, "f\circ\pi"] & \\ M/G \arrow[r, "f"'] & Y \end{tikzcd}$$

> (33.8) THEOREM: The orbit space exists $\Leftrightarrow$ $R \subset M \times M$ is a closed submanifold of $M \times M$.

Recall that in our terminology the orbit space $M/_R = M/_G$ exists if and only if the quotient $M/_R$ can be endowed with a manifold structure such that this manifold is the quotient manifold (cf. the discussion after 33.6 in section 33) and such that the projection map $\pi : M \to M/_R$ is a submersion.

We have already seen in section 33 that „$\Rightarrow$" is true. In order to prove „$\Leftarrow$" we have to show:

1. The topological quotient $M/_R$ is a Hausdorff space,
2. $M/_R$ admits the structure of a quotient manifold,
3. The projection $\pi : M \to M/_R$ is a submersion.

In order to give a complete picture we recall the following characterization of a submersion:

(33.18) PROPOSITION: For a smooth map $p : M \to N$ the following properties at a point $a \in M$ are equivalent:

1° p is a submersion at a,

2° $T_a p : T_a M \to T_a N$ is surjective,

3° There exist an open neighbourhood U of a, an open neighbourhood V of $p(a)$, an open neighbourhood $W \subset \mathbb{R}^d$ of $0 \in \mathbb{R}^d$ (with $d = \dim M - \dim N$), and a diffeomorphism $\psi : U \to V \times W$ such that $p|_U = pr_1 \circ \psi$.

$$\begin{tikzcd} U \arrow[r, "\psi", "\sim"'] \arrow[dr, "p|_U"'] & V \times W \arrow[d, "pr_1"] \\ & V \end{tikzcd}$$

4° There exist an open neighbourhood V of $p(a)$ and a smooth map $s : V \to M$ such that $p \circ s = id_V$.

<u>Proof</u>: Implicit mapping theorem or constant rank theorem („Satz vom Rang").

<u>(33.14) COROLLARY:</u> Let $R \subset M \times M$ be an equivalence relation such that M/R is a Hausdorff space*. Assume that we have found a manifold structure on M/R such that $\pi : M \to M/R$ is a submersion. Then the manifold structure is the quotient structure.

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* This is equivalent to $R \subset M \times M$ being closed in the product topology.

<u>Proof</u>: We have to show: If $f: M/R \to Y$ is a map into a manifold Y such that $f \circ \pi$ is smooth, then f is smooth as well. Let $\pi a \in M/R$ be an arbitrary point in M/R. According to 13.13.4° there exist an open neighbourhoods V of πa and a smooth map $s: V \to M$ with $\pi \circ s = id_V$. Now, $f|_V = f|_V \circ (\pi \circ s) = (f \circ \pi)|_{\pi^{-1}(\pi V)} \circ s$ is smooth as a composition of smooth maps, hence f is smooth. □

<u>(13.15) COROLLARY</u>: A surjective submersion $p: M \to N$ is an open map.

Pf. This follows from 13.13.3° since pr_1 is open. □

<u>PROOF OF THEOREM 33.8</u>:

According to the corollary 13.14 it is enough to determine a manifold structure on M/R such that π is a submersion. This is done by describing a suitable atlas on M/R with the aid of a the following seven assertions I. – VII.

I: $pr_1|_R : R \to M$ and $pr_2|_R : R \to M$ are submersions.

Pf. For $a \in M$ and $p := (a,b) \in R$ we have $b = ga$ for a suitable g. Any $X \in T_a M$ is represented by a curve γ, $\gamma(0) = a$: $X = [\gamma]_a$. $\beta(t) = (\gamma(t), g\gamma(t))$ is a curve in R with $\beta(0) = p$. And $T_p pr_1([\beta]_p) = [pr_1 \circ \beta]_a = [\gamma]_a$. Hence $T_p(pr_1|_R)$ is surjective for every $p \in R$. Analogously, $T_p(pr_2|_R)$ is surjective, and hence $pr_1|_R$, $pr_2|_R$ are submersions.

$\square$

II. $\pi : M \to M/_R$ is an open map.

Pf. Let $U \subset M$ be an open subset of M. We have to show that $\pi(U)$ is open, i.e. we have to show that $\pi^{-1}(\pi(U)) = \{ b \in M \mid \exists a \in U : (a,b) \in R \}$ is open, since a subset $V \subset M/_R$ is open if there exists an open $W \subset M$ with $\pi(W) = V$. Now, $\{ b \in M \mid \exists a \in U : (a,b) \in R \} = pr_2(pr_1^{-1}(U) \cap R)$ is open since any submersion is an open map (cf. 33.15) and $pr_2|_R$ is a submersion by I.

$\square$

III. $M/_R$ is a Hausdorff space.

Pf. Let $a, b \in M$ with $\pi a \neq \pi b$. Then $(a,b) \notin R$ and there is an open $W \subset M \times M$ with $(a,b) \in W$ and $W \cap R = \emptyset$,

since R is closed. By the definition of the product topology there are open subset $U, V \subset M$ with $(a,b) \in U \times V \subset W$. By $(U \times V) \cap R = \emptyset$, $\pi(U) \cap \pi(V) = \emptyset$. Since π is an open map we have found disjoint open neighbourhoods $\pi(U)$ of πa and $\pi(V)$ of πb:
M/R is Hausdorff! $\square$

After these preleminaries we now approach the difficult part of the proof.
We want to show that $\pi : M \to M/R$ is a submersion.
But π being a submersion means according to 13.13.4° that for every $a \in M$ there are local sections $s : \pi U \to U$ with $a \in s \circ \pi(U) =: S \subset U$ such that $s \circ \pi : U \to S$ has some good properties which we summarize in the next assertion*

<u>IV.</u> For every $a \in M$ there exist an open neighbourhood $U \subset M$ of a, a submanifold $S \subset U$ and a surjective submersion $k : U \to S$, $k^2 = k$, with the property:

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* $S \subset U$ is sometimes called a "slice" („Scheibe" in German), and the assertion <u>IV</u> is a version of the "slice theorem" („Scheibensatz").

$(*)\ (x,y) \in R \iff kx = ky.$

We prove IV below. From IV we immediately obtain:

V. To every $a \in M$ there correspond a chart $\varphi : \mathcal{U} \to V \times W \subset \mathbb{R}^{n-d} \times \mathbb{R}^d$ $(V \subset \mathbb{R}^{n-d}, W \subset \mathbb{R}^d$ open$)$ and a homeomorphism $\psi : \pi\mathcal{U} \to V$ with $\psi \circ \pi = pr_1 \circ \varphi$. That is, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\varphi", "\sim"'] \arrow[d, "\pi"'] & V \times W \arrow[d, "pr_1"] \\ \pi\mathcal{U} \arrow[r, "\sim", "\psi"'] & V \end{tikzcd}$$

Pf. In IV. $\mathcal{U}$ & S can be made small enough so that S is diffeomorphic to an open subset $V \subset \mathbb{R}^{n-d}$:

$$\mathcal{U} \xrightarrow{k} S \simeq V \subset \mathbb{R}^{n-d}$$

After shrinking $\mathcal{U}$ (and S) again (if necessary) we can assume that k factors through $S \times W$ for an open subset $W \subset \mathbb{R}^d$, i.e. $k = pr_1 \circ \lambda$ with a diffeomorphism $\lambda : \mathcal{U} \to S \times W$

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "k"] \arrow[dr, "\sim", "\lambda"'] & & S \\ & S \times W \arrow[ur, "pr_1"'] & \end{tikzcd}.$$

Because of $S\times W \cong V\times W$ we obtain a chart $\varphi : U \to V\times W$.

$$\begin{tikzcd}
U \arrow[r, "k"] \arrow[dr, "\sim"] \arrow[drr, bend right, "\varphi"'] & S \arrow[r, "\sim"] & V \subset \mathbb{R}^{n-d} \\
 & S\times W \arrow[u, "pr_1"'] \arrow[r, "\cong"] & V\times W \arrow[u, "pr_1"']
\end{tikzcd}$$

Now $\psi(\pi c)$ for $c\in U$ is defined to be $pr_1(\varphi(c))$. This is well defined, since for $\pi c = \pi c'$, i.e. $(c,c')\in R$ we know $kc = kc'$ by $(*)$, hence $pr_1(\varphi(c)) = pr_1(\varphi(c'))$ by the diagram above. We obtain the commutative diagram

$$\begin{tikzcd}
U \arrow[r, "\varphi"] \arrow[d, "\pi"'] & V\times W \arrow[d, "pr_1"] \\
\pi U \arrow[r, dashed, "\psi"'] & V
\end{tikzcd}$$

By construction, ψ is bijective. Moreover, since pr_1 and π (cf. $\underline{\mathrm{II}}$.) are open maps, ψ is a homeomorphism.

$\square$

The charts $\varphi : U \to V\times W$ of $\underline{\mathrm{V}}$ yield the manifold structure on M/R by taking the maps $\psi : \pi U \to V$

to be the charts of M/R:

<u>VI.</u> To every $a \in M$ there correspond $\varphi_a : U_a \xrightarrow{\sim} V_a \times W_a$ and ψ_a as in <u>V</u>. As a matter of fact, the collection

$$(\psi_a : \pi U_a \xrightarrow{\sim} V_a \mid a \in M)$$

is a smooth atlas on M/R such that π is a submersion.

Pf. One only has to check that the charts are compatible. Then the induced structure makes π a submersion (cf. the preceding diagram). and for a submersion π the universal property holds (cf. 13.14).

Let $p \in \pi U_a \cap \pi U_{a'}$ with $y := \psi_a(p) \in V_a$, $y' := \psi_{a'}(p) \in V_{a'}$. In particular, $y' = \psi_{a'}(\psi_a^{-1}(y))$. There are $c \in U_a$, $c' \in U_{a'}$ with $\pi c = p = \pi c'$, hence $pr_1 \circ \varphi_a(c) = \psi_a \circ \pi(c) = y$, $pr_1 \circ \varphi_{a'}(c') = y'$. Because of $\pi c = \pi c'$ there exists $g \in G$ with $c' = gc = \Psi_g(c)$. Let $Z \subset U_a$ be an open neighbourhood of c with $Z' := \Psi_g(Z) \subset U_{a'}$ (Ψ_g is cont.). Then (with $\Psi_g : M \to M$ being the map $a \mapsto ga$)

$$\varphi_{a'} \circ \Psi_g \circ \varphi_a^{-1} : \varphi_a Z \to \varphi_{a'} Z'$$

is smooth and $\varphi_a Z \subset V_a \times W_a$, $\varphi_{a'} Z' \subset V_{a'} \times W_{a'}$. The first component of $\varphi_{a'} \circ \Psi_g \circ \varphi_a^{-1}$ is $\psi_{a'} \circ \psi_a^{-1}$. This shows that $\psi_{a'} \circ \psi_a^{-1}$ is smooth and the atlas is compatible. A diagram for the various maps involved:

$$
\begin{tikzcd}
V_a \times W_a \supset \varphi_a Z \arrow[d, "pr_1"'] & Z \arrow[l, "\varphi_a"'] \arrow[rr, "\Psi_g"] \arrow[dr, "\pi"'] & & Z' \arrow[r, "\varphi_{a'}"] \arrow[dl, "\pi"] & \varphi_{a'} Z' \subset V_{a'} \times W_{a'} \arrow[d, "pr_1"] \\
V_a \supset \psi_a(\pi U_a \cap \pi U_{a'}) & & \pi U_a \cap \pi U_{a'} \arrow[ll, "\psi_a"] \arrow[rr, "\psi_{a'}"'] & & \psi_{a'}(\pi U_a \cap \pi U_{a'}) \subset V_{a'}
\end{tikzcd}
$$

$\square$

In order to prove the crucial property IV. we show:

VII. Every $a \in M$ has an open neighbourhood $U \subset M$ with a submanifold $S \subset U$, $a \in S$, such that for $Y := R|_U \cap (S \times U)$ the restrictions

$$pr_2|_Y : Y \longrightarrow U$$

is a diffeomorphism.

Before proving VII. let us show how VII implies IV.:

„ $\overrightarrow{\text{VII}}$. $\Rightarrow$ $\overrightarrow{\text{IV}}$ ": $Y = R|_{\mathcal{U}} \cap p^{-1}(S)$ with $p := pr_1|_{\mathcal{U}} : \mathcal{U}\times\mathcal{U} \to \mathcal{U}$.

Set
$$k := p \circ (pr_2|_Y)^{-1} : \mathcal{U} \to S .$$
Evidently, k is a submersion (essentially p) and k is surjective. Moreover,
$$(pr_2|_Y)^{-1}(b) = (kb, b), \ b \in \mathcal{U}.$$

Now, let $b \in S$. Then $(b,b) \in Y$ and $(b,b) = (pr_2|_Y)^{-1}(b)$ $= (kb, b)$. Hence, $kb = b$ and $k^2 = k$.

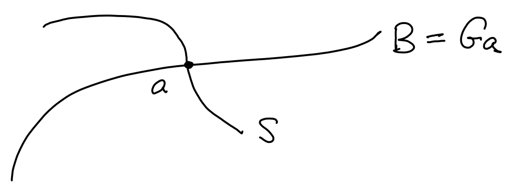
Finally, let $(c,c') \in R|_{\mathcal{U}}$.

1) If $c, c' \in S$, then $(c',c) \in Y$ and $(c,c) \in Y$, hence $c = c'$ & $kc = kc'$.

2) If $c, c' \in \mathcal{U}$ and not $c, c' \in S$, then $kc \sim kc'$, hence $kc = kc'$ according to 1). $\square$

<u>Proof of $\overline{\text{VII}}$:</u> Since $pr_1|_R : R \to M$ is a submersion, $pr_1^{-1}(a) \cap R = \{a\} \times Ga$ is a submanifold. Hence, every orbit $B = Ga \subset M$ is a submanifold. Applying the local description of submanifolds we see that locally, at a given point $a \in M$, there exists a submanifold $S \subset \mathcal{U}$ in an open neighbourhood $\mathcal{U}$ of a

with $a \in S$ such that $T_aM = T_aB \oplus T_aS$, i.e. S is "transversal" to B at a:



It remains to show that with

$$Y := R|_{\mathcal{U}} \cap (S \times \mathcal{U}) \subset R|_{\mathcal{U}} = R \cap (\mathcal{U} \times \mathcal{U})$$

the tangent map of $pr_2|_Y : Y \to \mathcal{U}$ at (a,a),

$$T_{(a,a)}(pr_2|_Y) : T_{(a,a)}Y \to T_aM ,$$

is an isomorphism (then we need only to shrink $\mathcal{U}$, if necessary, to obtain $\overline{\text{VII}}$.). For a tangent vector $X \in T_aS$ we have $X = [\sigma]$ with σ a curve in S, $\sigma(0) = a$: $X = [\sigma]_a$. Hence $\hat{\sigma}(t) = (\sigma(t), \sigma(t)) \in S \times S \subset Y$, $\hat{\sigma}(0) = (a,a)$, yields $[\hat{\sigma}]_{(a,a)} \in T_{(a,a)}Y$ with $T_{(a,a)}(pr_2|_Y)[\hat{\sigma}]_{(a,a)} = [\sigma]_a = X$. If $Z \in T_aB$ we have $Z = [g(t)a]_a$, g a curve in G, $g(0) = e$. Hence, $\hat{g}(t) = (a, g(t)a) \in Y$, $\hat{g}(0) = (a,a)$ and $T_{(a,a)}(pr_2|_Y)[\hat{g}]_{(a,a)} = [g(t)a]_a = Z$. As a result, $T_{(a,a)}pr_2|_Y$ is surjective. $T_{(a,a)}pr_2|_Y$ is injective since $\dim M = \dim Y$. $\square$

As a result $\mathcal{U} \cong S \times (B \cap \mathcal{U})$.