30. The Exponential Map

Version 1.1

Notiztitel

The main tool used to prove several of the results of the last section 29 is the exponential map.

Let G be a Lie group and $X \in T_eG \cong \mathrm{Lie}\, G$ a tangent vector. The corresponding right invariant vector field $\tilde{X} \in \mathrm{Lie}\, G \subset \mathbb{D}(G)$ has a flow (cf. section 8)

$$\varphi^{\tilde{X}}(t,a) = \varphi^{X}(t,a) \quad \text{for } t \in \mathbb{R}, a \in G.$$

$\varphi^{\tilde{X}}(t,a)$ represents $\tilde{X}$ in the sense that $\tilde{X}(a) = [\varphi^X(\cdot,a)]_a$, i.e. the curve $\gamma(t) := \varphi^{\tilde{X}}(t,a)$ satisfies the ordinary differential equation:

$$\dot{\gamma} = \tilde{X}(\gamma).$$

In general, such flows need not be defined for all $t \in \mathbb{R}$, but for the $\tilde{X} \in \mathrm{Lie}\, G$ this property holds true.

We define

$$\exp X := \varphi^{\tilde{X}}(1,e) \quad \text{and}$$

$$\exp_X(t) := \exp tX := \varphi^{\tilde{X}}(t,e) \quad , \ t \in \mathbb{R}.$$

<u>(30.1) Lemma:</u> $\varphi_X : \mathbb{R} \to G$ is a Lie group homomorphism such that $\mathrm{Lie}(\exp_X) : \mathrm{Lie}\,\mathbb{R} \longrightarrow \mathrm{Lie}\,G$, $t\frac{d}{dq} \longmapsto tX$.

<u>(30.2) PROPOSITION:</u> The exponential map $\exp : T_eG \to G$ (or $\exp : \mathrm{Lie}\,G \to G$) has the following properties:

1° $\exp(tX) = \exp_X(t)$

2° $\exp(t+s)X = \exp tX \exp sX$

3° $\exp(-tX) = (\exp tX)^{-1}$

4° exp is a smooth map and

$$\mathrm{Lie}\exp = T_0 \exp : T_0(T_eG) \to T_eG$$

is the identity.

5° exp is a local diffeomorphism: There is an open $Q \subset T_eG$ such that $\exp(Q) =: \mathcal{U} \subset G$ is open and

$$\exp|_Q : Q \to \mathcal{U}$$

is a diffeomorphism.

[13.12.10]

<u>(30.3) EXAMPLE:</u> For $GL(n,\mathbb{K})$ the exponential map is

$$\exp(X) := e^{tX} = \sum_{n=0}^{\infty} \frac{1}{n!} X^n \quad , \quad X \in \mathfrak{gl}(r, \mathbb{K}),$$

the usual exponential series. It is well-known that the series converges. Moreover $\gamma(t) = e^{tX}$ satisfies

$$\dot\gamma = X\gamma = \tilde{X}(\gamma).$$

(30.4) DEFINITION: The result 30.2.5° yields special coordinates. Let $Q \subset \mathfrak{g}$ be an open and star shaped neighbourhood of $0 \in \mathfrak{g}$ such the $\exp|_Q : Q \to \exp(Q) =: \mathcal{U}$ is a diffeomorphism (30.2.5°). Then

$$\varphi := \left(\exp|_Q\right)^{-1} : \mathcal{U} \longrightarrow Q \subset \mathfrak{g} = \mathbb{R}^n$$

is a chart. $\mathcal{U}$ is called a normal neighbourhood and the coordinates $\varphi = (q^1, .. q^n)$ are called normal coordinates.

Correspondingly, we obtain normal neighbourhoods and coordinates around any $g \in G$ by left (or right) translation: $\mathcal{U}(g) := \mathcal{L}_g(\mathcal{U})$ and

$$\varphi := \exp^{-1} \circ \mathcal{L}_{g^{-1}} : \mathcal{U}(g) \to Q \subset \mathfrak{g}$$

(30.5) REMARK: As an example, let us describe how most of the results of the second half of the last section are proven, by establishing 29.8 in the case of $G = GL(n, \mathbb{K})$ (which is all we needed for the construction of $\hat{\mathfrak{g}}$):
Given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{gl}(n, \mathbb{K}) = \mathfrak{g}$ we have to determine a Lie subgroup $H \subset G$ such that $\operatorname{Lie} H = \mathfrak{h}$.
Now, H can be defined by

$$H := \{ e^X : X \in \mathfrak{h} \} = \exp(\mathfrak{h}).$$

Using normal neighbourhoods it is easy to see that H is a submanifold. Moreover, H is a subgroup (and then, of course, the group operations are smooth with respect to the submanifold structure) which follows from the next proposition.

Note, that $\exp(X)\exp(Y) \neq \exp(X+Y)$, in general, but in a normal neighourhood $\mathcal{U} = \exp(Q)$ we have: For $X, Y \in Q$ with $\exp(X+Y) \in \mathcal{U}$ there exists a unique $X * Y \in Q$ with

$$\exp(X * Y) = \exp(X+Y).$$

(30.6) PROPOSITION: (Campbell-Hausdorff formula).

$$X * Y = X + Y + \frac{1}{2}[X,Y] - \frac{1}{12}\left([[X,Y],X] + [[Y,X],Y]\right) + \sum_{r_k=4}^{\infty} r_k(X,Y)$$

where the r_k are (k-multilinear) expressions in certain combination of Lie brackets where only X, Y are involved.

As a result, $\exp(X+Y)$ differs from $\exp X \exp Y$ only be second order terms, and $X * Y \in \mathfrak{h}$ if $X, Y \in \mathfrak{h}$ for a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

The exponential mapping is also the key ingredient in the proof of the following structure result:

(30.7) PROPOSITION: A continuous homomorphism between Lie groups is already a Lie group homomorphism.