26. Curvature

The most important concept in connection theory is the curvature. We approach this concept by investigating horizontal sections and path independent parallel transport. In this way we arrive at the concept of curvature in the form of the curvature operator.

Let $\pi_E : E \to M$ be again a vector bundle on M with a connection D

(26.1) PROPOSITION: For an open subset $U \subset M$ and a point $a \in U$ the assertions 2°-4° are equivalent and they imply 1°:

1° There exists a horizontal section $s \in \Gamma(U, E)$.

2° Through every $\xi \in E_a$ there exists a horizontal section $s \in \Gamma(U, E)$: $s(a) = \xi$.

3° The parallel transport $\Pi^{\gamma}_{t_0,t_1} : E_a \to E_b$, $\gamma(t_1) = b \in U$, is independent of γ for curves $\gamma : [t_0, t_1] \to U$.

4° There exists a basis $s_1, \dots s_r$ of $\Gamma(U, E)$ consisting of horizontal sections.

<u>Proof.</u> "$2° \Rightarrow 3°$" Let $a, b \in U$ and $\gamma : [t_0, t_1] \to U$ a curve connecting a and b. Let $\xi \in E_a$ and $s \in \Gamma(U, E)$ a horizontal section with $s(a) = \xi$. Then $\beta = s \circ \gamma$ is the horizontal lift of γ with $\beta(t_0) = \xi$ and $\mathbb{P}^{\gamma}_{t_0, t_1} \xi = \beta(t_1) = s \circ \gamma(t_1) = s(b)$ independent of the choice of γ.

"$3° \Rightarrow 2°$" Conversely, let $\xi \in E_a$. The various γ joining a and b inside U are all yielding the same $\eta = \mathbb{P}^{\gamma}\xi$. This defines a section $s(b) = \eta = \mathbb{P}^{\gamma}\xi$ with $s(a) = \xi$ which is horizontal: $\nabla_X s = \nabla_{\dot\gamma} s = 0$.

"$2° \Rightarrow 4°$" Let $\xi_1, \dots \xi_r$ a basis of E_a and $s_1, \dots s_r \in \Gamma(U, E)$ the corresponding horizontal sections with $s_\varrho(a) = \xi_\varrho$. Then $s_1(b), \dots s_r(b)$ is a basis of E_b since $\mathbb{P}^{\gamma} : E_a \to E_b$ is a lin. isomorphism and $\mathbb{P}^{\gamma}\xi_\varrho = s_\varrho(b)$. Hence, for an arbitrary section $s \in \Gamma(U, E)$ one finds $f_1, \dots f_\varrho : U \to \mathbb{K}$ with $s(b) = \sum_{\varrho=1}^{r} f_\varrho(b) s_\varrho(b)$. Since s & s_ϱ are smooth, the f_ϱ are smooth as well.

"$4° \Rightarrow 2°$" $\xi \in E_a$ is the unique linear combination $\xi = \sum_1^r \lambda_\varrho s_\varrho(a)$ with $\lambda_\varrho \in \mathbb{K}$. Hence, $s := \sum_{\varrho=1}^{r} \lambda_\varrho s_\varrho \in \Gamma(U, E)$ and s is horizontal since the λ_ϱ are constant. $\square$

How do we obtain horizontal sections locally if the exist at all?

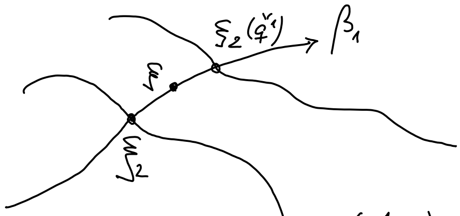
Let $U \subset M$ be a coordinate neighbourhood around $a \in U$, $\varphi : U \to Q = I_1 \times \ldots \times I_n \subset \mathbb{R}^n$ a chart with open intervals with $0 \in I_j \subset \mathbb{R}$, $a = \varphi^{-1}(0)$. Then $\partial_j = \frac{\partial}{\partial q^j}$ is a basis of $\Gamma(U, TM)$ over $\mathcal{E}(U)$.

Set $\nabla_j := \nabla_{\partial_j}$, $j = 1, \ldots n$, and choose $\xi \in E_a$.

We start with the curve $\gamma_1(t) = \varphi^{-1}(te_1)$ and observe that exists a horizontal lift β_1 with $\beta_1(0) = \xi$.

To each $q^1 \in I_1$ we set

$$\gamma_2(q^1, t) := \varphi^{-1}(q^1 e_1 + t e_2) \quad \text{and} \quad \xi_2(q^1) = \beta_1(q^1) \in E_{\gamma_1(q^1)}$$



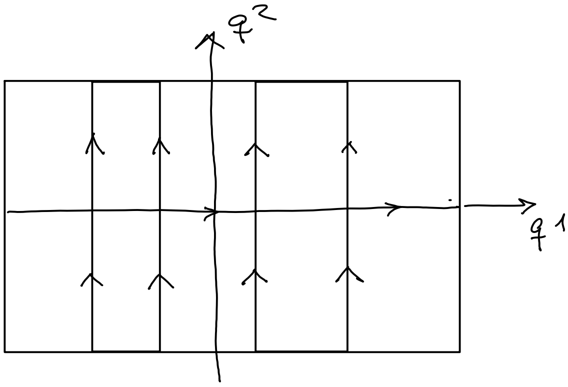
$\beta_2(\check{q}^1, \cdot)$ over $\gamma_2(\check{q}^1, \cdot)$ & $\beta_2(\check{q}^1, 0) = \xi_2(\check{q}^1)$

$\beta_2(q^1, \cdot)$ over $\gamma_2(q^1, \cdot)$ & $\beta_2(q^1, 0) = \beta_1(q^1) = \xi_2(q^1)$

This procedure yields a section along $\varphi^{-1}(I_1 \times I_2 \times \{0\} \times \ldots \times \{0\})$.

Is this section horizontal?

We restrict to the case $n = 2$ in the following



In the parameter space $I_1 \times I_2$:
The section $s = \beta_2 : I_1 \times I_2 \to E$
is horizontal in the vertical
directions $\updownarrow$, but in the
horizontal direction $\longleftrightarrow$ only for $q^2 = 0$: $s(q^1, 0) = \beta_1(q^1)$.
In other words:

$$\nabla_2 s = 0 \quad \text{always},$$
$$\nabla_1 s = 0 \quad \text{only along } q^2 = 0.$$

Assume now that ∇_1 & ∇_2 commute: $\nabla_1 \nabla_2 = \nabla_2 \nabla_1$. Then

$$\nabla_2(\nabla_1 s) = \nabla_1(\nabla_2 s) = 0 \qquad \text{since } \nabla_2 s = 0$$

Consequently, $\nabla_1 s(q^1, t)$ is horizontal lift of $\gamma_2(q^1, t)$
through $\nabla_1 s(q^1, 0)$. However, $\nabla_1 s(q^1, 0) = 0$, since
$s(q^1, 0) = \beta_1(q^1)$ is horizontal lift of γ_1.
As a result,

$$\nabla_1 s(q^1, t) = \mathbb{P}^{\gamma_2} \nabla_1 s(q^1, 0) = 0$$

(parallel transport is linear) and $\nabla_1 s = \nabla_2 s = 0$, i.e. s
is horizontal: $X = X^1 \partial_1 + X^2 \partial_2$: $\nabla_X s = X^1 \nabla_2 s + X^2 \nabla_2 s = 0$.

The case $n \geq 3$ can be treated in the same way

(26.2) DEFINITION: The curvature operator of a connection D on a vector bundle $E \to M$ is

$$F(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \qquad X,Y \in \mathcal{V}(W),\ W \subset M \text{ open.}$$

(26.3) PROPOSITION: For a connection D on a vector bundle $E \to M$ the following assertions are equivalent:

1° $F(X,Y)\,s = 0$ for $s \in \Gamma(W,E)$ and $X,Y \in \mathcal{V}(W)$

2° For every $a \in M$ there exists an open neighbourhood U of a such the to every $\xi \in E_a$ there is a horizontal section $s \in \Gamma(U,E)$ with $s(a) = \xi$.

3° Parallel transport is locally path independent: $\forall a \in M\ \exists U \subset M$ open: $a \in U$ & $\forall \gamma, \alpha : [t_0,t_1] \to U$: $\gamma(t_0) = \alpha(t_0) = a$, $\gamma(t_1) = \alpha(t_1) = b \implies \mathbb{P}^{\alpha}_{t_0,t_1} = \mathbb{P}^{\gamma}_{t_0,t_1}$

Proof. "1° ⇒ 2°": $F(\partial_i, \partial_j) = 0 \implies D_i D_j = D_j D_i$ since $[\partial_i, \partial_j] = 0$.
"2° <=> 3°" in (26.1)
"2° ⇒ 1°: There is basis $s_1, \dots s_r$ of horizontal sections. Let s by an arbitrary section $s \in \Gamma(U,E)$. $s = f^1 s_1 + \dots + f^r s_r$,

$f^\rho \in \mathcal{E}(U)$. Now, for $X, Y \in \mathfrak{X}(U)$:

$$\nabla_X \nabla_Y s = \nabla_X \nabla_Y (f^\rho s_\rho) = \nabla_X (L_Y f^\rho s_\rho + \underbrace{f^\rho \nabla_Y s_\rho}_{=0})$$

$$= L_X L_Y f^\rho s_\rho + L_Y f^\rho \nabla_X s_\rho = (L_X L_Y f^\rho) s_\rho$$

Hence,

$$[\nabla_X, \nabla_Y] s = (L_{[X,Y]} f^\rho) s_\rho .$$

Finally, $\nabla_{[X,Y]} s = (L_{[X,Y]} f^\rho) s_\rho + f^\rho \nabla_{[X,Y]} s = (L_{[X,Y]} f^\rho) s_\rho$

$$\Rightarrow \quad F(X,Y) s = 0$$

<u>Observation:</u> $TS^2 \to S^2$ with Levi-Civita connection has no path independent parallel transport. In particular: $F \neq 0$.