17. Vector Bundles

Version 1.1

We have chosen to present the geometry of vector bundles before we study the general case of principal fibre bundles because the vector bundle case is slightly more elementary. In particular, we do not need the notion of a Lie group and its Lie algebra.

Vector bundles are fibrations with a (finite dimensional) vector space as its typical fibre such that transition functions are linear:

(17.1) Definition: A vector bundle of rank r ($r \in \mathbb{N}$, $r \geq 0$) over $\mathbb{K}$ ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$) is given by a smooth map $\pi : E \to B$ such that

1° For every $a \in B$ the fibre $E_a := \pi^{-1}(a)$ has the structure of an r-dim. $\mathbb{K}$ vector space.

2° For every $a \in B$ there exists an open nbhd U of a and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{K}^r$ with $pr_1 \circ \psi = \pi|_{\pi^{-1}(U)}$ (i.e. a local trivialization) such that $\psi_b := pr_2 \circ \psi|_{E_b} : E_b \to \mathbb{K}^r$ is a linear isomorphism for all $b \in U$.

Evidently, for a vector bundle $\pi : E \to B$ of rank r over $\mathbb{K}$ the quadruple $(E, \pi, B, \mathbb{K}^r)$ is a fibration for which the transition functions $\psi_{ij} : U_{ij} \to \mathrm{Diff}(\mathbb{K}^r)$ with respect to a suitable open cover are linear in the sense that each

$$\psi_{ij}(a) : \mathbb{K}^r \to \mathbb{K}^r \quad , a \in U_{ij},$$

is a $\mathbb{K}$-linear isomorphism, i.e. $\psi_{ij}(a) \in GL(r, \mathbb{K})$
Conversely, suppose we have a fibration $(E, \pi, B, \mathbb{K}^r)$ ($\mathbb{K}^r$ with its usual norm topology) such that there exists an open cover (U_j) with trivializations ψ_j for which the transition functions are $\mathbb{K}$-linear. Then the trivializations $\psi_{j,a} : E_a \to \{a\} \times \mathbb{K}^r$ define on E_a a unique structure of an r-dimensional $\mathbb{K}$ vector space, and with these structures, the fibration is a vector bundle of rank r.

A morphism of vector bundles (or a vector bundle homomorphism) from $\xi = (E, \pi, B, \mathbb{K}^r)$ to $\xi' = (E', \pi', B', \mathbb{K}^{r'})$ is - by definition - a morphism of the fibration

i.e. $h : E \to E'$ with $\pi' \circ h = h_B \circ \pi$

such that all $h_a = h|_{E_a} : E_a \to E_{h_B a}$ are linear over $\mathbb{K}$. In the following we restrict mostly to the case of $B = B'$ and $h_B = \mathrm{id}_B$. The corresponding category is (vect_B).

Notation: $\mathrm{Hom}_B(E, E')$ for the morphisms in (vect_B)

Note, that $\mathrm{Hom}_B(E, E')$ is an $\mathcal{E}(B)$ module: $f \in \mathcal{E}(B)$ and $h, g \in \mathrm{Hom}_B(E, E')$ induce

$$f \cdot h \in \mathrm{Hom}_B(E, E') \text{ by } f \cdot h(v) := f(\pi(v))\, h(v)$$
$$h + g \in \mathrm{Hom}_B(E, E') \text{ by } (h+g)(v) := h(v) + g(v)$$

(17.2) Fact: Any cocycle (ψ_{ij}) with values in $GL(r, \mathbb{K})$ defines a vector bundle of rank r (cf. 16.2). Note, that $GL(r, \mathbb{K}) \subset \mathrm{Diff}(\mathbb{K}^r)$, and that $GL(r, \mathbb{K}) \subset \mathbb{K}^{r \times r}$ is an open submanifold of $\mathbb{K}^{r \times r}$. The condition

1° $(a, y) \longmapsto \psi_{ij}(a) \cdot y$ is smooth

of 16.1 is equivalent to the more natural condition

1° $\psi_{ij} : U_{ij} \to GL(r, \mathbb{K})$ is smooth.

Hence, the isomorphism classes of vector bundles over B of rank r which are trivial over each U_j are in one-to-one correspondence to $H^1(\mathfrak{U}, GL(r,\mathbb{K}))$.

(17.3) Examples:

- product bundle $B \times \mathbb{K}^r$
- restriction to open subsets $B' \subset B$
- g^*E for $g : B' \to B$
- tangent and cotangent bdles $TM \to M$, $T^*M \to M$
- tensor bundles $T^r_s M \to M$
- Möbius band
- universal (tautological) line bundle $U \to \mathbb{P}_n(\mathbb{K})$
- tautological r-vector bundle $T \to G_{r,n}(\mathbb{K})$
- normal bundle of a k-dim. submanifold $M \subset \mathbb{R}^n$, $k < n$.

(17.4) Def - Proposition: For a vector bundle $E \to B$ the set $\Gamma(B,E)$ of smooth sections over B is an $\mathcal{E}(B)$ module which is finitely generated if B has countable topology.