11. Tensor Fields as Sections

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In section 10 the tensor fields on an open subset $W \subset M$ of a smooth manifold M have been defined to be multilinear mappings on a product

$$\underbrace{\mathcal{V}(W)^* \times \dots \times \mathcal{V}(W)^*}_{r \text{ times}} \times \underbrace{\mathcal{V}(W) \times \dots \times \mathcal{V}(W)}_{s\text{-times}} \quad ;$$

An $\binom{r}{s}$-tensor field is accordingly a map

$$t \in T^r_s(\mathcal{V}(W)) = \mathrm{Hom}_E\left(\mathcal{V}(W)^{*\,r} \times \mathcal{V}(W)^s; \mathcal{E}(W)\right)$$

From another point of view these tensor fields are smooth sections in a suitable vector bundle in a similar manner as this holds for vector fields, the $\binom{1}{0}$ tensor fields, cf. 7.1, and differential forms, the $\binom{0}{1}$ tensor fields, cf. 9.3.

(11.1) Definition - Proposition: The set

$$T^r_s M := \bigcup_{a \in M} T^r_s(T_a M)$$

has a natural structure of a smooth manifold given by bundle charts such that the projection

$$\tau : T^r_s M \to M, \ \tau(T^r_a M) = \{a\},$$

is smooth and the whole object is a vector bundle *
with typical fibre $T^r_s \mathbb{R}^n$ ($n = \dim M$).

<u>The bundle charts:</u> For the vector space $V = \mathbb{R}^n$ we choose a basis $e_1, \dots e_n \in V$ which operate on V^* as $\mu \longmapsto \mu(e_j)$ and which induce a dual basis $\check{e}^j$, $j = 1, \dots n$, of $V^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$ by $\check{e}^j(e_k) = \delta^j_k$ (Kronecker).
Let

$$\varphi : U \longrightarrow Q \subset \mathbb{R}^n$$

be a chart of the defining atlas of the smooth structure on M. We want to define

$$\hat{\varphi} : T^r_s U \longrightarrow U \times T^r_s \mathbb{R}^n.$$

Each $t \in T^r_s U$ has a representation (cf. 10.5)

$$t = t^{j_1 \dots j_s}_{k_1 \dots k_r} \, \partial_{j_1} \otimes \dots \otimes \partial_{j_s} \otimes dq^{k_1} \otimes \dots \otimes dq^{k_r}$$

and we set

$$\hat{\varphi}(t) := t^{j_1 \dots j_s}_{k_1 \dots k_r} \, e_{j_1} \otimes \dots \otimes e_{j_s} \otimes \check{e}^{k_1} \otimes \dots \otimes \check{e}^{k_r}$$

for such a t.

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* cf. section 17

Clearly, the bundle charts $\hat{\varphi}$ are bijective. Therefore, they induce a unique topology on $T^r_s M$ by requiring all bundle charts to be topological maps. One has to check that this topology is again Hausdorff.

The bundle charts define a smooth structure on $T^r_s M$ such that the $\hat{\varphi}$ are diffeomorphisms (local trivializations, cf. section 15) which are linear in the fibres (vector bundle condition, cf. section 17) if the bundle charts are compatible with respect to the change of coordinates.

If now $\bar{\varphi} : \bar{U} \to \bar{Q} \subset \mathbb{R}^n$ is another chart with $U \cap \bar{U} \neq \emptyset$ we know that

$$\bar{\varphi} \circ \varphi^{-1} : Q' \to \bar{Q}' \ , \ Q' = \varphi(U \cap \bar{U}) \subset Q, \ \bar{Q}' = \bar{\varphi}(U \cap \bar{U}) \subset Q$$

is a diffeomorphism. The change of coordinates induces a change of the related bases in $T_a M$ and $T^*_a M$:

$$\frac{\partial}{\partial \bar{q}^j} = \frac{\partial (\varphi \circ \bar{\varphi}^{-1})^k}{\partial \bar{q}^j} \frac{\partial}{\partial q^k} = \frac{\partial q^k}{\partial \bar{q}^j} \frac{\partial}{\partial q^k} = c^k_j \frac{\partial}{\partial q^k}$$

$$d\bar{q}^k = \frac{\partial \bar{q}^k}{\partial q^j} dq^j = \check{c}^k_j dq^j \quad \text{with } \left(c^k_j\right)^{-1} = \left(\check{c}^l_m\right).$$

The transformation rules induced by the matrix (c^k_j) hold in greater generality (cf. next proposition) and show that $\hat{\bar{\varphi}} \circ \hat{\varphi}^{-1} : Q' \times T^r_s\mathbb{R}^n \to \bar{Q}' \times T^r_s\mathbb{R}^n$ is diffeomorphic since the coefficients c^k_j, $\check{c}^k_j$ are smooth.

<u>(11.2) Transformation Rules:</u> Let V have a basis $\partial_1, \dots \partial_n$ (over E) with the <u>dual</u> basis $d^1, \dots d^n \in V^*$ $(d^j(\partial_k) = \delta^j_k)$. Then $t \in T^r_s V$ has the unique description

$$t = t^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_r} \otimes d^{\nu_1} \otimes \dots \otimes d^{\nu_s} \quad (\text{cf. } 10.5).$$

If. $\bar{\partial}_1, \dots, \bar{\partial}_n$ is another basis of V with dual basis $\bar{d}_1, \dots \bar{d}_n \in V^*$
Let

$$\bar{\partial}_k = c^l_k \partial_l \quad \text{with } c^l_k \in E$$

(e.g. the change of coordinates from $q^1, \dots q^n$ to $\bar{q}^1, \dots \bar{q}^n$ with $c^l_k = \frac{\partial q^l}{\partial \bar{q}^k}$).
Denote $(\check{c}^\mu_\nu)$ the inverse matrix, i.e. $\check{c}^\mu_\nu \in E$ with

$c_k^{\ell} \check{c}^{\nu k}_{\mu} = \delta^{\ell}_{\mu}$, in particular $\partial_\nu = \check{c}^{\lambda}_{\nu} \bar{\partial}_\lambda$.

Then $\bar{d}^{\mu} = \check{c}^{\mu}_{k} d^k$ and for $t \in T^r_s V$ with

$$t = \bar{t}^{k_1 \dots k_r}_{\lambda_1 \dots \lambda_s} \bar{\partial}_{k_1} \otimes \dots \otimes \bar{\partial}_{k_r} \otimes \bar{d}^{\lambda_1} \otimes \dots \otimes \bar{d}^{\lambda_s}$$

we obtain

$$\boxed{t^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} = c^{\mu_1}_{k_1} c^{\mu_2}_{k_2} \dots c^{\mu_r}_{k_r} \check{c}^{\lambda_1}_{\nu_1} \dots \check{c}^{\lambda_s}_{\nu_s} \bar{t}^{k_1 \dots k_r}_{\lambda_1 \dots \lambda_s}.}$$

As a result: If V is free of rank n then $T^r_s V$ is free of rank n^{r+s}. Moreover, TV is free of countable rank.

<u>(11.3) Proposition</u>: For an open subset $W \subset M$ of a manifold M the set $\Gamma(W, T^r_s M)$ of sections $s: W \to T^r_s M$ (i.e. s smooth and $\tau \circ s = \mathrm{id}_W$) is a module over $\mathcal{E}(W)$. There is a natural isomorphism over $\mathcal{E}(W)$

$$\mathcal{T}^r_s W = T^r_s \mathfrak{V}(W) \cong \Gamma(W, T^r_s M).$$

<u>Proof</u>: The module structure on $\Gamma(W, T^r_s M)$ is given by pointwise addition and multiplication.

The isomorphism is given by $\sigma \in \Gamma(W, T^r_s M)$, $\sigma \mapsto t_\sigma$, where

$$t_\sigma(\eta^1, \dots, \eta^r, X_1, \dots, X_s)(a) := \sigma(a)\,(\eta^1(a), \dots \eta^r(a), X_1(a), \dots X_s(a))$$

is well-defined and smooth since $a \mapsto \sigma(a)$ is smooth and $\sigma(a) \in T^r_s T_a M$.

(11.4) Remark: The behaviour under transformation 11.2 can be used to define the concept of a tensor field! See 7.4, 9.3, and 12.11.

(11.5) Remark: Vector-valued versions:

$$\mathcal{T}^r_s(M, \mathbb{K}^l) := T^r_s(\mathcal{V}(M), \mathcal{E}(M)^l), \quad l \in \mathbb{N},$$

and – for a vector bundle $F \to M$

$$\mathcal{T}^r_s(M, F) := T^r_s(\mathcal{V}(M), \Gamma(M, F)).$$

The isomorphism: $\mathcal{T}^r_s(M, F) \cong \Gamma(M, T^r_s M \otimes F)$
(cf. section 18 for the tensor product of vector bundles).