Appendix A
Some Further Developments

Due to the character of these notes with the objective to present and explain the basic principles of conformal field theory on a mathematical basis in a rather detailed manner there has been nearly no room to mention further developments.

In this appendix we concentrate on boundary conformal field theory (BCFT) and on stochastic Loewner evolution (SLE) as two developments which lead to new structures not being part of conformal field theory (CFT) as described in these notes but strongly connected with CFT.

We only give a brief description and some references.

Boundary Conformal Field Theory. Boundary conformal field theory is essentially conformal field theory on domains with a boundary. As an example, let us consider strings moving in a background Minkowski space M as in Chap. 7. For a closed string, that is a closed loop moving in M, one gets a closed surface. After quantization one obtains the corresponding CFT on this surface as developed in Chap. 7. In case of an open string, that is a connected part of a closed loop (which is the image of an interval under an injective embedding) with two endpoints, the string weeps out an open surface or better a surface with boundary. The boundary is given by the movement of the two endpoints of the string. We obtain the corresponding CFT in the interior of the surface, the bulk CFT, together with compatibility conditions on the boundary of the surface.

BCFT has important applications in string theory, in particular, in the physics of open strings and D-branes (cf. [FFFS00b*], for instance), and in condensed matter physics in boundary critical behavior.

BCFT is in some respect simpler than CFT. For instance, in the case of the upper half plane H with the real axis as its boundary one possible boundary condition is that the energy–momentum tensor T satisfies $T(z) = \overline{T}(\bar{z})$. This implies that the correlation functions of $\overline{T}$ are the same as those of T, analytically continued to the lower halfplane. This simplifies among other things the conformal Ward identities. Moreover, there is only one Virasoro algebra.

For general reviews on BCFT we refer to [Zub02*] and [Car04*]. See also [Car89*] and [FFFS00a*].

Stochastic Loewner Evolution. There is a deep connection between BCFT and conformally invariant measures on spaces of curves in a simply connected domain

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H in $\mathbb{C}$ which start at the boundary of the domain. This has been indicated in both the survey articles of Cardy [Car04*] on BCFT and [Car05*] on SLE and in a certain sense already in [LPSA94]. Such measures arise naturally in the continuum limit of certain statistical mechanics models.

For instance, in the case of the upper half plane H a measure of this type can be constructed using a family of conformal mappings $g_t,\ t \geq 0$. In such a construction one uses the stochastic Loewner evolution (SLE) first described by [Schr00*]. More precisely, for a constant $\kappa \in \mathbb{R},\ \kappa \geq 0$, the so-called SLE$_\kappa$ curve $\gamma : [0,\infty[\to \mathbb{C}$ in the upper half plane H is generated as follows: $\gamma : [0,\infty[\to \mathbb{C}$ is continuous with $\gamma(0) = 0$ and $\gamma(]0,\infty[) \subset H$. γ is furthermore determined by the unique conformal diffeomorphism

$$g_t : H \setminus \gamma(]0,t]) \to H,\ t \geq 0,$$

satisfying the Loewner equation

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa} b_t},\ g_0(z) = z,$$

normalized by the condition $g_t(z) = z + o(1)$ for $z \to \infty$. Here, $b_t,\ t \geq 0$, is an ordinary brownian motion starting at $b_0 = 0$. Hence, $\gamma(t) = \gamma_t$ is precisely the point satisfying $g_t^{\sim}(\gamma_t) = \sqrt{\kappa} b_t$ for the continuous extension $g_t^{\sim}$ of g_t to $H \setminus \gamma(]0,t[)$ that is into the boundary point $\gamma(t)$ of $H \setminus \gamma(]0,t[)$.

A comprehensive introduction to SLE is given in Lawler's book [Law05*]. A first exact application to the critical behavior of statistical mechanics models can be found in [Smi01*].

The relation of SLE to CFT is not easy to detect. It has been uncovered in the articles [BB03*] and [FW03*].

Modularity. Modularity properties have been studied in the articles on vertex algebras and CFT from the very beginning, in particular with respect to the examples of large finite simple groups (see [Bor86*] and [FLM88*], for instance). A comprehensive survey can be found in [Gan06*].

References

[BB03*] M. Bauer and D. Bernard. Conformal field theories of stochastic Loewner evolutions. *Comm. Math. Phys.* **239** (2003) 493–521.

[Bor86*] R. E. Borcherds. Vertex algebras, Kac-Moody algebra and the monster. *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068–3071.

[Car89*] J.L. Cardy. Boundary conditions, fusion rules and the Verlinde formula. *Nucl. Phys.* **B324** (1989), 581–596.

[Car04*] J.L. Cardy. *Boundary Conformal Field Theory*. [arXiv:hepth/ 0411189v2] (2004) (To appear in *Encyclopedia of Mathematical Physics*, Elsevier).

[Car05*] J.L. Cardy. SLE for theoretical physicists. *Ann. Phys.* **318** (2005), 81–118.

[FFFS00a*] G. Felder, J. Fröhlich, J. Fuchs, and C. Schweigert. Conformal boundary conditions and three-dimensional topological field theory. *Phys. Rev. Lett.* **84** (2000), 1659–1662.

[FFFS00b*] G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert. The geometry of WZW branes. *J. Geom. Phys*. **34** (2000), 162–190.

[FLM88*] I. Frenkel, J. Lepowsky, and A. Meurman. *Vertex Operator Algebras and the Monster*. Academic Press, New York, 1988.

[FW03*] R. Friedrich and W. Werner. Conformal restriction, highest weight representations and SLE. *Comm. Math. Phys*. **243** (2003), 105–122.

[Gan06*] T. Gannon. *Moonshine Beyond the Monster. The Bridge Connecting Algebra, Modular Forms and Physics*. Cambridge University Press, Cambridge, 2006.

[LPSA94] R. Langlands, P. Pouliot, and Y. Saint-Aubin. Conformal invariance in two-dimensional percolation. *Bull. Am. Math*. Soc. **30** (1994), 1–61.

[Law05*] G.F. Lawler. *Conformally Invariant Processes in the Plane. Mathematical Surveys and Monographs* **114**. AMS, Providence, RI, 2005.

[Schr00*] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math*. **118** (2000), 221–288.

[Smi01*] S. Smirnov. Critical percolation in the plane: Conformal invariance, Cardy's formula, scaling limits. *C. R. Acad*. Sci. Paris **333** (2001), 239–244.

[Zub02*] J.B. Zuber. CFT, BCFT, ADE and all that. In: *Quantum Symmetries in Theoretical Physics and Mathematics*, Coquereaux et alii (Eds.), Contemporary Mathematics **294**, 233–271, AMS, Providence, RI, 2002.