Chapter 8
Axioms of Relativistic Quantum Field Theory

Although quantum field theories have been developed and used for more than 70 years a generally accepted and rigorous description of the structure of quantum field theories does not exist. In many instances quantum field theory is approached by quantizing classical field theories as for example elaborated in the last chapter on strings. A more systematic specification uses axioms. We present in Sect. 8.3 the system of axioms which has been formulated by Arthur Wightman in the early 1950s. This chapter follows partly the thorough exposition of the subject in [SW64*]. In addition, we have used [Simo74*], [BLT75*], [Haa93*], as well as [OS73] and [OS75].

The presentation of axiomatic quantum field theory in this chapter serves several purposes:

- It gives a general motivation for the axioms of two-dimensional conformal field theory in the Euclidean setting which we introduce in the next chapter.
- It explains in particular the transition from Minkowski spacetime to Euclidean spacetime (Wick rotation) and thereby the transition from relativistic quantum field theory to Euclidean quantum field theory (cf. Sect. 8.5).
- It explains the equivalence of the two descriptions of a quantum field theory using either the fields (as operator-valued distributions) or the correlation functions (resp. correlation distributions) as the main objects of the respective system (cf. Sect. 8.4).
- It motivates how the requirement of conformal invariance in addition to the Poincaré invariance leads to the concept of a vertex algebra.
- It points out important work which is known already for about 50 years and still leads to many basic open problems like one out of the seven millennium problems (cf. the article of Jaffe and Witten [JW06*]).
- It gives the opportunity to describe the general framework of quantum field theory and to introduce some concepts and results on distributions and functional analysis (cf. Sect. 8.1).

The results from functional analysis and distributions needed in this chapter can be found in most of the corresponding textbooks, e.g., in [Rud73*] or [RS80*].
First of all, we recall some aspects of distribution theory in order to present a precise concept of a quantum field.

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8.1 Distributions

A quantum field theory consists of quantum states and quantum fields with various properties. The quantum states are represented by the lines through 0 (resp. by the rays) of a separable complex Hilbert space $\mathbb{H}$, that is by points in the associated projective space $\mathbb{P} = \mathbb{P}(\mathbb{H})$ and the observables of the quantum theory are the self-adjoint operators in $\mathbb{H}$.

In a direct analogy to classical fields one is tempted to understand quantum fields as maps on the configuration space $\mathbb{R}^{1,3}$ or on more general spacetime manifolds M with values in the set of self-adjoint operators in $\mathbb{H}$. However, one needs more general objects, the quantum fields have to be operator-valued distributions. We therefore recall in this section the concept of a distribution with a couple of results in order to introduce the concept of a quantum field or field operator in the next section.

Distributions. Let $\mathscr{S}(\mathbb{R}^n)$ be the *Schwartz space* of *rapidly decreasing smooth functions*, that is the complex vector space of all functions $f : \mathbb{R}^n \to \mathbb{C}$ with continuous partial derivatives of any order for which

$$|f|_{p,k} := \sup_{|\alpha| \le p} \sup_{x \in \mathbb{R}^n} |\partial^\alpha f(x)|(1+|x|^2)^k < \infty, \tag{8.1}$$

for all $p,k \in \mathbb{N}$. (∂^α is the partial derivative for the multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ with respect to the usual cartesian coordinates $x = (x^1, x^2, \dots, x^n)$ in $\mathbb{R}^n$.)

The elements of $\mathscr{S} = \mathscr{S}(\mathbb{R}^n)$ are the *test functions* and the dual space contains the (tempered) distributions.

Observe that (8.1) defines seminorms $f \mapsto |f|_{p,k}$ on $\mathscr{S}$.

Definition 8.1. A *tempered distribution* T is a linear functional $T : \mathscr{S} \to \mathbb{C}$ which is continuous with respect to all the seminorms $|\ |_{p,k}$ defined in (8.1), $p,k \in \mathbb{N}$.

Consequently, a linear $T : \mathscr{S} \to \mathbb{C}$ is a tempered distribution if for each sequence (f_j) of test functions which converges to $f \in \mathscr{S}$ in the sense that

$$\lim_{j \to \infty} |f_j - f|_{p,k} = 0 \quad \text{for all } p,k \in \mathbb{N},$$

the corresponding sequence $(T(f_j))$ of complex numbers converges to $T(f)$. Equivalently, a linear $T : \mathscr{S} \to \mathbb{C}$ is continuous if it is bounded, that is there are $p,k \in \mathbb{N}$ and $C \in \mathbb{R}$ such that

$$|T(f)| \le C|f|_{p,k}$$

for all $f \in \mathscr{S}$.

The vector space of tempered distributions is denoted by $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^n)$. $\mathscr{S}'$ will be endowed with the topology of uniform convergence on all the compact subsets of $\mathscr{S}$. Since we only consider tempered distributions in these notes we often call a tempered distribution simply a distribution in the sequel.

Some distributions are represented by functions, for example for an arbitrary measurable and bounded function g on $\mathbb{R}^n$ the functional

$$T_g(f) := \int_{\mathbb{R}^n} g(x)f(x)dx, f \in \mathscr{S},$$

defines a distribution. A well-known distribution which cannot be represented as a distribution of the form T_g for a function g on $\mathbb{R}^n$ is the *delta distribution*

$$\delta_y : \mathscr{S} \to \mathbb{C}, f \mapsto f(y),$$

the evaluation at $y \in \mathbb{R}^n$. Nevertheless, δ_y is called frequently the delta function at y and one writes $\delta_y = \delta(x-y)$ in order to use the formal integral

$$\delta_y(f) = f(y) = \int_{\mathbb{R}^n} \delta(x-y)f(x)dx.$$

Here, the right-hand side of the equation is defined by the left-hand side.
Distributions T have derivatives. For example

$$\frac{\partial}{\partial q^j} T(f) := -T(\frac{\partial}{\partial q^j} f),$$

and $\partial^\alpha T$ is defined by

$$\partial^\alpha T(f) := (-1)^{|\alpha|} T(\partial^\alpha f), f \in \mathscr{S}.$$

By using partial integration one obtains $\partial^\alpha T_g = T_{\partial^\alpha g}$ if g is differentiable and suitably bounded.

An important example in the case of $n = 1$ is $T_H(f) := \int_0^\infty f(x)dx$, $f \in \mathscr{S}$, with

$$\frac{d}{dt} T(f) = -\int_0^\infty f'(x)dx = f(0) = \delta_0(f).$$

We observe that the delta distribution δ_0 has a representation as the derivative of a function (the *Heaviside function* $H(x) = \chi_{[0,\infty[}$) although δ_0 is not a true function. This fact has the following generalization:

Proposition 8.2. *Every tempered distribution $T \in \mathscr{S}'$ has a representation as a finite sum of derivatives of continuous functions of polynomial growth, that is there exist $g_\alpha : \mathbb{R}^n \to \mathbb{C}$ such that*

$$T = \sum_{0 \le |\alpha| \le k} \partial^\alpha T_{g_\alpha}.$$

Partial Differential Equations. Since a distribution possesses partial derivatives of arbitrary order it is possible to regard partial differential equations as equations for distributions and not only for differentiable functions. Distributional solutions in general lead to results for true functions. This idea works especially well in the case of partial differential equations with constant coefficients.

For a polynomial $P(X) = c_\alpha X^\alpha \in \mathbb{C}[X_1, \dots, X_n]$ in n variables with complex coefficients $c_\alpha \in \mathbb{C}$ one obtains the partial differential operator

$$P(-i\partial) = c_\alpha(-i\partial)^\alpha = \sum c_{(\alpha_1,\dots,\alpha_n)}\partial_1^{\alpha_1}\dots\partial_n^{\alpha_n},$$

and the corresponding inhomogeneous partial differential equation

$$P(-i\partial)u = v,$$

which is meaningful for functions as well as for distributions. As an example, the basic partial differential operator determined by the geometry of the Euclidean space $\mathbb{R}^n = \mathbb{R}^{n,0}$ is the Laplace operator

$$\Delta = \partial_1^2 + \dots + \partial_n^2,$$

with $\Delta = P(-i\partial)$ for $P = -(X_1^2 + \dots + X_n^2)$.

In the same way, the basic partial differential operator determined by the geometry of the Minkowski space $\mathbb{R}^{1,D-1}$ is the wave operator (the Laplace–Beltrami operator with respect to the Minkowski-metric, cf. 1.6)

$$\Box = \partial_0{}^2 - (\partial_1^2 + \dots + \partial_{D-1}^2) = \partial_0^2 - \Delta,$$

and $\Box = P(-i\partial)$ for $P = -X_0^2 + X_1^2 + \dots + X_{D-1}^2$.

A *fundamental solution* of the partial differential equation $P(-i\partial)u = v$ is any distribution G satisfying

$$P(-i\partial)G = \delta.$$

Proposition 8.3. *Such a fundamental solution provides solutions of the inhomogeneous partial differential equation $P(-i\partial)u = v$ by convolution of G with v:*

$$P(-i\partial)(G * v) = v.$$

Proof. Here, the *convolution* of two rapidly decreasing smooth functions $u, v \in \mathscr{S}$, is defined by

$$u * v(x) := \int_{\mathbb{R}^n} u(y)v(x-y)dy = \int_{\mathbb{R}^n} u(x-y)v(y)dy.$$

The identity $\partial_j(u*v) = (\partial_j u)*v = u*\partial_j v$ holds. The convolution is extended to the case of a distribution $T \in \mathscr{S}'$ by $T * v(u) := T(v*u)$. This extension again satisfies

$$\partial_j(T*v) = (\partial_j T)*v = T*\partial_j v.$$

Furthermore, we see that

$$\delta * v(u) = \delta(v*u) = \int_{\mathbb{R}^n} v(y)u(y)dy,$$

thus $\delta * v = v$. Now, the defining identity $P(-i\partial)G = \delta$ for the fundamental solution implies $P(-i\partial)(G*v) = \delta * v = v$. □

Fundamental solutions are not unique, the difference u of two fundamental solutions is evidently a solution of the homogeneous equation $P(-i\partial)u = 0$.

Fundamental solutions are not easy to obtain directly. They often can be derived using Fourier transform.

Fourier Transform. The Fourier transform of a suitably bounded measurable function $u : \mathbb{R}^n \to \mathbb{C}$ is

$$\widehat{u}(p) := \int_{\mathbb{R}^n} u(x) e^{ix\cdot p} dx$$

for $p = (p_1, \ldots, p_n) \in (\mathbb{R}^n)' \cong \mathbb{R}^n$ whenever this integral is well-defined. Here, $x \cdot p$ stands for a nondegenerate bilinear form appropriate for the problem one wants to consider. For example, it might be the Euclidean scalar product or the Minkowski scalar product in $\mathbb{R}^n = \mathbb{R}^{1,D-1}$ with $x \cdot p = x^\mu \eta_\mu^\nu p_\nu = x^0 p_0 - x^1 p_1 - \ldots - x^{D-1} p_{D-1}$.

The Fourier transform is, in particular, well-defined for a rapidly decreasing smooth function $u \in \mathscr{S}(\mathbb{R}^n) = \mathscr{S}$ and, moreover, the transformed function $\mathscr{F}(u) = \widehat{u}$ is again a rapidly decreasing smooth function $\mathscr{F}(u) \in \mathscr{S}$. The inverse Fourier transform of a function $v = v(p)$ is

$$\mathscr{F}^{-1} v(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} v(p) e^{-ix\cdot p} dp.$$

Proposition 8.4. *The Fourier transform is a linear continuous map*

$$\mathscr{F} : \mathscr{S} \to \mathscr{S}$$

whose inverse is $\mathscr{F}^{-1}$. As a consequence, $\mathscr{F}$ has an adjoint

$$\mathscr{F}' : \mathscr{S}' \to \mathscr{S}', T \mapsto T \circ \mathscr{F}.$$

On the basis of this result we can define the Fourier transform $\mathscr{F}(T)$ of a tempered distribution T as the adjoint

$$\mathscr{F}(T)(v) := T(\mathscr{F}(v)) = \mathscr{F}'(T)(v), v \in \mathscr{S},$$

and we obtain a map $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$ which is linear, continuous, and invertible. Note that for a function $g \in \mathscr{S}$ the Fourier transforms of the corresponding distribution T_g and that of g are the same:

$$\mathscr{F}(T_g)(v) = T_g(\widehat{v}) = \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n)'} g(x) v(p) e^{ix\cdot p} dp dx = T_{\mathscr{F}(g)}(v).$$

Typical examples of Fourier transforms of distributions are

$$\begin{aligned}
\mathscr{F}(H)(\omega) &= \int_0^\infty e^{it\omega} dt = \frac{i}{\omega + i0}, \\
\mathscr{F}(\delta_0) &= \int_{\mathbb{R}^D} \delta_0(x) e^{ix\cdot p} dx = 1, \\
\mathscr{F}^{-1}(e^{ip\cdot y}) &= (2\pi)^{-D} \int_{\mathbb{R}^D} e^{ip\cdot(y-x)} dp = \delta(x-y).
\end{aligned}$$

The fundamental importance of the Fourier transform is that it relates partial derivatives in the x^k with multiplication by the appropriate coordinate functions p_k after Fourier transformation:

$$\mathscr{F}(\partial_k u) = -ip_k \mathscr{F}(u)$$

by partial integration

$$\mathscr{F}(\partial_k u)(p) = \int \partial_k u(x) e^{ix\cdot p} dx = -\int u(x) ip_k e^{ix\cdot p} dx = -ip_k \mathscr{F}(u)(p),$$

and consequently,

$$\mathscr{F}(\partial^\alpha u) = (-ip)^\alpha \mathscr{F}(u).$$

This has direct applications to partial differential equations of the type

$$P(-i\partial)u = v.$$

The general differential equation $P(-i\partial)u = v$ will be transformed by $\mathscr{F}$ into the equation

$$P(p)\widehat{u} = \widehat{v}.$$

Now, trying to solve the original partial differential equation leads to a division problem for distributions. Of course, the multiplication of a polynomial $P = P(p)$ and a distribution $T \in \mathscr{S}'$ given by $PT(u) := T(Pu)$ is well-defined because $Pu(p) = P(p)u(p)$ is a function $Pu \in \mathscr{S}$ for each $u \in \mathscr{S}$. Solving the division problem, that is determining a distribution T with $PT = f$ for a given polynomial P and function f, is in general a difficult task.

For a polynomial P let us denote $G = G_P$ the inverse Fourier transform $\mathscr{F}^{-1}(T)$ of a solution of the division problem $PT = 1$, that is $P\widehat{G} = 1$. Then G is a *fundamental solution* of $P(-i\partial)u = v$, that is

$$P(-i\partial)G = \delta$$

since $\mathscr{F}(P(-i\partial)G) = P(p)\widehat{G} = 1$ and $\mathscr{F}^{-1}(1) = \delta$.

Klein–Gordon Equation. We study as an explicit example the fundamental solution of the Klein–Gordon equation. The results will be used later in the description of the free boson within the framework of Wightman's axioms, cf. p. 135, in order to construct a model satisfying all the axioms of quantum field theory.

The dynamics of a free bosonic classical particle is governed by the Klein–Gordon equation. The Klein–Gordon equation with mass $m > 0$ is

$$(\Box + m^2)u = v,$$

where $\Box$ is the wave operator for the Minkowski space $\mathbb{R}^{1,D-1}$ as before. A fundamental solution can be determined by solving the division problem

$$\left(-p^2 + m^2\right) T = 1:$$

A suitable

$$T \text{ is } \left(m^2 - p^2\right)^{-1}$$

as a distribution given by

$$T(v) = \int_{\mathbb{R}^{D-1}} \left(PV \int_{\mathbb{R}} \frac{v(p)}{\omega(\mathsf{p}) - p_0^2} dp_0 \right) d\mathsf{p},$$

where $PV \int$ is the principal value of the integral. The corresponding fundamental solution (the propagator) is

$$G(x) = (2\pi)^{-D} \int_{\mathbb{R}^D} (m^2 - p^2)^{-1} e^{-ix\cdot p} dp.$$

G can be expressed more concretely by Bessel, Hankel, etc., functions.

We restrict our considerations to the free fields which are the solutions of the homogeneous equation

$$(\Box - m^2)\phi = 0.$$

The Fourier transform $\widehat{\phi}$ satisfies

$$(p^2 - m^2)\widehat{\phi} = 0,$$

where $p^2 = \langle p, p \rangle = p_0^2 - (p_1^2 + \ldots + p_{D-1}^2)$. Therefore, $\widehat{\phi}$ has its support in the mass-shell $\{p \in (\mathbb{R}^{1,D-1})' : p^2 = m^2\}$. Consequently, $\widehat{\phi}$ is proportional to $\delta(p^2 - m^2)$ as a distribution, that is $\widehat{\phi} = g(p)\delta(p^2 - m^2)$, and we get ϕ by the inverse Fourier transform

$$\phi(x) = (2\pi)^{-D} \int_{\mathbb{R}^D} g(p)\delta(p^2 - m^2) e^{-ip\cdot x} dp.$$

Definition 8.5. The distribution

$$D_m(x) := 2\pi i \mathscr{F}^{-1}((\operatorname{sgn}(p_0)\delta(p^2 - m^2))(x)$$

is called the *Pauli–Jordan function*.

($\operatorname{sgn}(t)$ is the sign of t, $\operatorname{sgn}(t) = H(t) - H(-t)$.) D_m generates all solutions of the homogeneous Klein–Gordon equation. In order to describe D_m in detail and to use the integration

$$D_m(x) = 2\pi i (2\pi)^{-D} \int_{\mathbb{R}^D} \operatorname{sgn}(p_0)\delta(p^2 - m^2) e^{-ip\cdot x} dp$$

for further calculations we observe that for a general g the distribution

$$\widehat{\phi} = g(p)\delta(p^2 - m^2)$$

can also be written as

$$\widehat{\phi} = H(p_0)g_+(p)\delta(p^2 - m^2) - H(-p_0)g_-(p)\delta(p^2 - m^2)$$

taking into account the two components of the hyperboloid $\{p \in (\mathbb{R}^{1,D-1})' : p^2 = m^2\}$: the upper hyperboloid

$$\Gamma_m := \{p \in (\mathbb{R}^{1,D-1})' : p^2 = m^2, p_0 > 0\}$$

and the lower hyperboloid

$$-\Gamma_m = \{p \in ((\mathbb{R}^{1,D-1}))' : p^2 = m^2, p_0 < 0\}.$$

Here, the g_+, g_- are distributions on the upper resp. lower hyperboloid, which in our situation can be assumed to be functions which simply depend on $\mathsf{p} \in \mathbb{R}^{D-1}$ via the global charts

$$\xi_\pm : \mathbb{R}^{D-1} \to \pm\Gamma_m, \mathsf{p} \mapsto (\pm\omega(\mathsf{p}), \mathsf{p}),$$

where $\omega(\mathsf{p}) := \sqrt{\mathsf{p}^2 + m^2}$ and $\mathsf{p} = (p_1, \ldots, p_{D-1})$, hence $\mathsf{p}^2 = p_1^2 + \ldots + p_{D-1}^2$. Let λ_m be the invariant measure on Γ_m given by the integral

$$\int_{\Gamma_m} h(\xi)d\lambda_m(\xi) := \int_{\mathbb{R}^{D-1}} h(\xi_+(\mathsf{p}))(2\omega(\mathsf{p}))^{-1}d\mathsf{p}$$

for functions h defined on Γ_m and analogously on $-\Gamma_m$. Then for $v \in \mathscr{S}(\mathbb{R}^D)$ the value of $\delta(p^2 - m^2)$ is

$$\delta(p^2 - m^2)(v) = \int_{\Gamma_m} v(\omega(\mathsf{p}), \mathsf{p})d\lambda_m + \int_{-\Gamma_m} v(-\omega(\mathsf{p}), \mathsf{p})d\lambda_m.$$

Here, we use the identity $\delta(t^2 - b^2) = (2b)^{-1}(\delta(t-b) + \delta(t+b))$ in one variable t with respect to a constant $b > 0$.

These considerations lead to the following ansatz which is in close connection to the formulas in the physics literature. We separate the coordinates $x \in \mathbb{R}^{1,D-1}$ into $x = (t, \mathsf{x})$ with $t = x^0$ and $\mathsf{x} = (x^1, \ldots, x^{D-1})$. Let

$$\phi(t,\mathsf{x}) := (2\pi)^{-D} \int_{\mathbb{R}^{D-1}} (a(\mathsf{p})e^{i(\mathsf{p}\cdot\mathsf{x} - \omega(\mathsf{p})t)} + a^*(\mathsf{p})e^{-i(\mathsf{p}\cdot\mathsf{x} - \omega(\mathsf{p})t)})d\lambda_m(\mathsf{p})$$

for arbitrary functions $a, a^* \in \mathscr{S}(\mathbb{R}^{D-1})$ in $D-1$ variables. Then $\phi(t,\mathsf{x})$ satisfies $(\Box + m^2)\phi = 0$ which is clear from the above derivation (because of $a(\mathsf{p}) = g_+(\omega(\mathsf{p}), \mathsf{p}), a^*(\mathsf{p}) = g_-(-\omega(\mathsf{p}), \mathsf{p})$ up to a constant). That $\phi(t,\mathsf{x})$ satisfies $(\Box + m^2)\phi = 0$ is in fact very easy to show directly: With the abbreviation

$$k(t,\mathsf{x},\mathsf{p}) := (2\pi)^{-D}(a(\mathsf{p})e^{i(\mathsf{p}\cdot\mathsf{x} - \omega(\mathsf{p})t)} + a^*(\mathsf{p})e^{-i(\mathsf{p}\cdot\mathsf{x} - \omega(\mathsf{p})t)})$$

we have

$$\partial_0^2\phi(t,\mathsf{p}) = \int_{\gamma_m} i^2\omega(\mathsf{p})^2 k(t,\mathsf{x},\mathsf{p})d\lambda_m \text{ and}$$

$$\partial_j^2\phi(t,\mathsf{p}) = \int_{\gamma_m} i^2 p_j^2 k(t,\mathsf{x},\mathsf{p})d\lambda_m \text{ for } j > 0.$$

Hence,

$$\Box\phi(t,\mathsf{p}) = -\int_{\gamma_m} (\omega(\mathsf{p})^2 - \mathsf{p}^2) k(t,\mathsf{x},\mathsf{p}) d\lambda_m = -m^2\phi(t,\mathsf{x}).$$

We have shown the following result:

Proposition 8.6. *Each solution $\phi \in \mathscr{S}$ of $(\Box + m^2)\phi = 0$ can be represented uniquely as*

$$\phi(t,\mathsf{x}) := (2\pi)^D \int_{\mathbb{R}^{D-1}} (a(\mathsf{p}) e^{i(\mathsf{p}\cdot\mathsf{x}-\omega(\mathsf{p})t)} + a^*(\mathsf{p}) e^{-i(\mathsf{p}\cdot\mathsf{x}-\omega(\mathsf{p})t)}) d\lambda_m(\mathsf{p})$$

with $a, a^* \in \mathscr{S}((\mathbb{R}^{D-1})')$. The real solutions correspond to the case $a^* = \overline{a}$.

8.2 Field Operators

Operators and Self-Adjoint Operators. Let $\mathscr{S}\mathscr{O} = \mathscr{S}\mathscr{O}(\mathbb{H})$ denote the set of self-adjoint operators in $\mathbb{H}$ and $\mathscr{O} = \mathscr{O}(\mathbb{H})$ the set of all densely defined operators in $\mathbb{H}$. (A general reference for operator theory is [RS80*].) Here, an *operator* in $\mathbb{H}$ is a pair (A,D) consisting of a subspace $D = D_A \subset \mathbb{H}$ and a $\mathbb{C}$-linear mapping $A : D \to \mathbb{H}$, and A is densely defined whenever D_A is dense in $\mathbb{H}$. In the following we are interested only in densely defined operators. Recall that such an operator can be unbounded, that is $\sup\{\|Af\| : f \in D, \|f\| \leq 1\} = \infty$, and many relevant operators in quantum theory are in fact unbounded. As an example, the position and momentum operators mentioned in Sect. 7.2 in the context of quantization of the harmonic oscillator are unbounded.

If a densely defined operator A is bounded (that is $\sup\{\|Af\| : f \in D_A, \|f\| \leq 1\} < \infty$), then A is continuous and possesses a unique linear and continuous continuation to all of $\mathbb{H}$.

Let us also recall the notion of a self-adjoint operator. Every densely defined operator A in $\mathbb{H}$ has an adjoint operator A^* which is given by

$$D_{A^*} := \{f \in \mathbb{H} | \exists h \in \mathbb{H}\ \forall g \in D_A : \langle h,g\rangle = \langle f,Ag\rangle\},$$
$$\langle A^*f,g\rangle = \langle f,Ag\rangle, f \in D_{A^*}, g \in D_A.$$

A^*f for $f \in D_{A^*}$ is thus the uniquely determined $h = A^*f \in \mathbb{H}$ with $\langle h,g\rangle = \langle f,Ag\rangle$ for all $g \in D_A$.

It is easy to show that the adjoint A^* of a densely defined operator A is a *closed operator*. A closed operator B in $\mathbb{H}$ is defined by the property that the graph of B, that is the subspace

$$\Gamma(B) = \{(f,B(f)) : f \in D_B\} \subset \mathbb{H} \times \mathbb{H}$$

of $\mathbb{H} \times \mathbb{H}$, is closed, where the Hilbert space structure on $\mathbb{H} \times \mathbb{H} \cong \mathbb{H} \oplus \mathbb{H}$ is defined by the inner product

$$\langle (f,f'),(g,g')\rangle := \langle f,g\rangle + \langle f',g'\rangle.$$

Hence, an operator B is closed if for all sequences (f_n) in D_B such that $f_n \to f \in \mathbb{H}$ and $Bf_n \to g \in \mathbb{H}$ it follows that $f \in D_B$ and $Bf = g$. Of course, every continuous operator defined on all of $\mathbb{H}$ is closed. Conversely, every closed operator B defined on all of $\mathbb{H}$ is continuous by the closed graph theorem. Note that a closed densely defined operator which is continuous satisfies $D_B = \mathbb{H}$.

Self-adjoint operators are sometimes mixed up with symmetric operators. For operators with domain of definition $D_B = \mathbb{H}$ the two notions agree and this holds more generally for closed operators also. A *symmetric operator* is a densely defined operator A such that

$$\langle Af,g\rangle = \langle f,Ag\rangle, f,g \in D_A.$$

By definition, a *self-adjoint operator* A is an operator which agrees with its adjoint A^* in the sense of $D_A = D_{A^*}$ and $A^*f = Af$ for all $f \in D_A$. Clearly, a self-adjoint operator is symmetric and it is closed since adjoint operators are closed in general. Conversely, it can be shown that a symmetric operator is self-adjoint if it is closed. An operator B is called *essentially self-adjoint* when it has a unique continuation to a self-adjoint operator, that is there is a self-adjoint operator A with $D_B \subset D_A$ and $B = A|_{D_B}$.

For a closed operator A, the spectrum $\sigma(A) = \{\lambda \in \mathbb{C} : (A - \lambda \mathrm{id}_{\mathbb{H}})^{-1}$does not exist as a bounded operator$\}$ is a closed subset of $\mathbb{C}$. Whenever A is self-adjoint, the spectrum $\sigma(A)$ is completely contained in $\mathbb{R}$.

For a self-adjoint operator A there exists a unique representation $U : \mathbb{R} \to \mathrm{U}(\mathbb{H})$ satisfying

$$\lim_{t\to 0} \frac{U(t)f - f}{t} = -iAf$$

for each $f \in D_A$ according to the spectral theorem. U is denoted $U(t) = e^{-itA}$ and A (or sometimes $-iA$) is called the *infinitesimal generator* of $U(t)$. Conversely (cf. [RS80*]),

Theorem 8.7 (Theorem of Stone). *Let $U(t)$ be a one parameter group of unitary operators in the complex Hilbert space $\mathbb{H}$, that is U is a unitary representation of $\mathbb{R}$. Then the operator A, defined by*

$$Af := \lim_{t\to 0} i\frac{U(t)f - f}{t}$$

in the domain in which this limit exists with respect to the norm of $\mathbb{H}$, is self-adjoint and generates $U(t)$: $U(t) = e^{-itA}$, $t \in \mathbb{R}$.

With the aid of (tempered) distributions and (self-adjoint) operators we are now in the position to explain what quantum fields are.

Field Operators. The central objects of quantum field theory are the quantum fields or field operators. A field operator is the analogue of a classical field but now in the quantum model. Therefore, in a first attempt, one might try to consider a field operator Φ to be a map from M to $\mathscr{S}\mathscr{O}$ assigning to a point $x \in M = \mathbb{R}^{1,D-1}$ a self-adjoint operator $\Phi(x)$ in a suitable way. However, for various reasons such a map is not sufficient to describe quantum fields (see also Proposition 8.15). For example, in some classical field theories the Poisson bracket of a field ϕ at points $x, y \in M$ with $x^0 = y^0$ (at equal time) is of the form

$$\{\phi(x), \phi(y)\} = \delta(\mathsf{x} - \mathsf{y}),$$

where $\mathsf{x} := (x^1, \dots, x^{D-1})$, the space part of $x = (x^0, x^1, \dots, x^{D-1})$. This equation has a rigorous interpretation in the context of the theory of distributions.

As a consequence, a quantum field will be an operator-valued distribution.

Definition 8.8. A *field operator* or *quantum field* is now by definition an *operator-valued distribution* (on $\mathbb{R}^n$), that is a map

$$\Phi : \mathscr{S}(\mathbb{R}^n) \to \mathscr{O}$$

such that there exists a dense subspace $D \subset \mathbb{H}$ satisfying

1. For each $f \in \mathscr{S}$ the domain of definition $D_{\Phi(f)}$ contains D.
2. The induced map $\mathscr{S} \to \mathrm{End}(D), f \mapsto \Phi(f)|_D$, is linear.
3. For each $v \in D$ and $w \in \mathbb{H}$ the assignment $f \mapsto \langle w, \Phi(f)(v) \rangle$ is a tempered distribution.

The concept of a quantum field as an operator-valued distribution corresponds better to the actual physical situation than the more familiar notion of a field as a quantity defined at each point of spacetime. Indeed, in experiments the field strength is always measured not at a point x of spacetime but rather in some region of space and in a finite time interval. Therefore, such a measurement is naturally described by the expectation value of the field as a distribution applied to a test function with support in the given spacetime region. See also Proposition 8.15 below.

As a generalization of the Definition 8.8, it is necessary to consider *operator-valued tensor distributions* also. Here, the term *tensor* is used for a quantity which transforms according to a finite-dimensional representation of the Lorentz group L (resp. of its universal cover).

8.3 Wightman Axioms

In order to present the axiomatic quantum field theory according to Wightman we need the notion of a quantum field or field operator Φ as an operator-valued distribution which we have introduced in Definition 8.8 and some informations about properties on geometric invariance which we recall in the sequel.

Relativistic Invariance. As before, let $M = \mathbb{R}^{1,D-1}$ D-dimensional Minkowski space (in particular the usual four-dimensional Minkowski space $M = \mathbb{R}^{1,3}$ or the Minkowski plane $M = \mathbb{R}^{1,1}$) with the (Lorentz) metric

$$x^2 = \langle x,x\rangle = x^0x^0 - \sum_{j=1}^{D-1} x^jx^j, x = (x^0,\ldots,x^{D-1}) \in M.$$

Two subsets $X,Y \subset M$ are called to be *space-like separated* if for any $x \in X$ and any $y \in Y$ the condition $(x-y)^2 < 0$ is satisfied, that is

$$(x^0-y^0)^2 < \sum_{j=1}^{D-1}(x^j-y^j)^2.$$

The *forward cone* is $C_+ := \{x \in M : x^2 = <x,x> \geq 0, x^1 \geq 0\}$ and the *causal order* is given by $x \geq y \Longleftrightarrow x-y \in C_+$.

Relativistic invariance of classical point particles in $M = \mathbb{R}^{1,D-1}$ or of classical field theory on M is described by the *Poincaré group* $\mathrm{P} := \mathrm{P}(1,D-1)$, the identity component of the group of all transformations of M preserving the metric. P is generated by the Lorentz group L, the identity component $\mathrm{L} := \mathrm{SO}_0(1,D-1) \subset \mathrm{GL}(D,\mathbb{R})$ of the orthogonal group $\mathrm{O}(1,D-1)$ of all *linear* transformations of M preserving the metric. (L is sometimes written $\mathrm{SO}(1,D-1)$ by abuse of notation.) In fact, the Poincaré group P is the semidirect product (see Sect. 3.1) $\mathrm{L} \ltimes \mathbb{R}^n \cong \mathrm{P}$ of L and the translation group $M = \mathbb{R}^D$.

The Poincaré group P preserves the causal structure and the space-like separateness. Observe that the corresponding conformal group $\mathrm{SO}(2,D)$ (cf. Theorem 2.9) which contains the Poincaré transformations also preserves the causal structure, but not the space-like separateness.

The Poincaré group acts on $\mathscr{S} = \mathscr{S}(\mathbb{R}^D)$, the space of test functions, from the left by $h \cdot f(x) := f(h^{-1}x)$ with $g\cdot(h\cdot f) = (gh)\cdot f$ and this left action is continuous. It is mostly written in the form

$$(q,\Lambda)f(x) = f(\Lambda^{-1}(x-q)),$$

where the Poincaré transformations h are parameterized by $(q,\Lambda) \in \mathrm{L} \ltimes M, q \in M$, $\Lambda \in \mathrm{L}$.

The relativistic invariance of the quantum system with respect to Minkowski space $M = \mathbb{R}^{1,D-1}$ is in general given by a projective representation $\mathrm{P} \to \mathrm{U}(\mathbb{P}(\mathbb{H}))$ of the Poincaré group P, a representation in the space $\mathbb{P}(\mathbb{H})$ of states of the quantum system as we explain in Sect. 3.2. By Bargmann's Theorem 4.8 such a representation can be lifted to an essentially uniquely determined unitary representation of the 2-to-1 covering group of P, the simply connected universal cover of P. This group is isomorphic to the semidirect product $\mathrm{Spin}(1,D-1) \ltimes \mathbb{R}^D$ for $D > 2$ where $\mathrm{Spin}(1,D-1)$ is the corresponding spin group, the universal covering group of the Lorentz group L$= \mathrm{SO}(1,D-1)$. In the sequel we often call these covering groups the Poincaré group and Lorentz group, respectively, and denote them simply again by P and L.

Note that in the two-dimensional case, the Lorentz group L is isomorphic to the abelian group $\mathbb{R}$ of real numbers (cf. Remark 1.15) and therefore agrees with its universal covering group.

We thus suppose to have a unitary representation of the Poincaré group P which will be denoted by

$$U : \mathrm{P} \to \mathrm{U}(\mathbb{H}), (q,\Lambda) \mapsto U(q,\Lambda),$$

$(q,\Lambda) \in M \times \mathrm{L} = \mathrm{L} \ltimes M$.

Since the transformation group $M \subset \mathrm{P}$ is abelian one can apply Stone's Theorem 8.7 in order to obtain the restriction of the unitary representation U to M in the form

$$U(q,1) = \exp iqP = \exp i(q^0P_0 - q^1P_1 - \ldots - q^{D-1}P_{D-1}), \tag{8.2}$$

$q \in \mathbb{R}^{1,D-1}$, with self-adjoint commuting operators $P_0,\ldots,P_{D-1}$ on $\mathbb{H}$. P_0 is interpreted as the energy operator $P_0 = H$ and the $P_j, j > 0$, as the components of the momentum.

We are now in the position to formulate the axioms of quantum field theory.

Wightman Axioms. A *Wightman quantum field theory* (Wightman QFT) in dimension D consists of the following data:

- the space of states, which is the projective space $\mathbb{P}(\mathbb{H})$ of a separable complex Hilbert space $\mathbb{H}$,
- the vacuum vector $\Omega \in \mathbb{H}$ of norm 1,
- a unitary representation $U : \mathrm{P} \to \mathrm{U}(\mathbb{H})$ of P, the covering group of the Poincaré group,
- a collection of field operators $\Phi_a, a \in I$ (cf. Definition 8.8),

$$\Phi_a : \mathscr{S}(\mathbb{R}^D) \to \mathscr{O},$$

  with a dense subspace $D \subset \mathbb{H}$ as their common domain (that is the domain $D_a(f)$ of Φ_a contains D for all $a \in A, f \in \mathscr{S}$) such that Ω is in the domain D.

These data satisfy the following three axioms:

Axiom W1 (Covariance)

1. Ω *is* P-*invariant, that is* $U(q,\Lambda)\Omega = \Omega$ *for all* $(q,\Lambda) \in \mathrm{P}$*, and* D is P-invariant*, that is* $U(q,\Lambda)D \subset D$ *for all* $(q,\Lambda) \in \mathrm{P}$*,*
2. *the common domain* $D \subset \mathbb{H}$ *is* invariant *in the sense that* $\Phi_a(f)D \subset D$ *for all* $f \in \mathscr{S}$ *and* $a \in I$*,*
3. *the actions on* $\mathbb{H}$ *and* $\mathscr{S}$ *are equivariant where* P *acts on* $\mathrm{End}(D)$ *by conjugation. That is on D we have*

$$U(q,\Lambda)\Phi_a(f)U(q,\Lambda)^* = \Phi((q,\Lambda)f) \tag{8.3}$$

   for all $f \in \mathscr{S}$ *and for all* $(q,\Lambda) \in \mathrm{P}$.

Axiom W2 (Locality) *$\Phi_a(f)$ and $\Phi_b(g)$ commute on D if the supports of $f, g \in \mathscr{S}$ are space-like separated*, that is on D

$$\Phi_a(f)\Phi_b(g) - \Phi_b(g)\Phi_a(f) = [\Phi_a(f), \Phi_b(g)] = 0. \tag{8.4}$$

Axiom W3 (Spectrum Condition) *The joint spectrum of the operators P_j is contained in the forward cone C_+.*

Recall that the support of a function f is the closure of the points x with $f(x) \neq 0$. If one represents the operator-valued distribution Φ_a symbolically by a function $\Phi_a = \Phi_a(x) \in \mathscr{O}$ the equivariance (8.3) can be written in the following form:

$$U(q,\Lambda)\Phi_a(x)U(q,\Lambda)^* = \Phi_a(\Lambda x + q).$$

This form is frequently used even if Φ_a cannot be represented as a function, and the equality is only valid in a purely formal way.

Remark 8.9. The relevant fields, that is the operators $\Phi_a(f)$ for real-valued test functions $f \in \mathscr{S}$, should be essentially self-adjoint. In the above axioms this has not been required from the beginning because often one considers a larger set of field operators so that only certain combinations are self-adjoint. In that situation it is reasonable to require Φ_a^* to be in the set of quantum fields, that is $\Phi_a^* = \Phi_{a'}$ for a suitable $a' \in A$ (where $a = a'$ if $\Phi_a(f)$ is essentially self-adjoint).

Remark 8.10. Axiom W1 is formulated for scalar fields only which transform under the trivial representation of L. In general, if fields have to be considered which transform according to a nontrivial (finite-dimensional) complex or real representation $R : \mathrm{L} \to \mathrm{GL}(W)$ of the (double cover of the) Lorentz group (like spinor fields, for example) the equivariance in *(8.3)* has to be replaced by

$$U(q,\Lambda)\Phi_j(f)U(q,\Lambda)^* = \sum_{k=1}^{m} R_{jk}(\Lambda^{-1})\Phi_k((q,\Lambda)f). \tag{8.5}$$

Here, W is identified with $\mathbb{R}^m$ resp. $\mathbb{C}^m$, and the $R(\Lambda)$ are given by matrices $\big(R_{jk}(\Lambda)\big)$. Moreover, the fields Φ_a are merely components and have to be grouped together to vectors $(\Phi_1, \ldots, \Phi_m)$.

Remark 8.11. In the case of $D = 2$ there exist nontrivial one-dimensional representations $R : \mathrm{L} \to \mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^\times$ of the Lorentz group L, since the Lie algebra Lie L of L is $\mathbb{R}$ and therefore not semi-simple. In this situation the equivariance (8.3) has to be extended to

$$U(q,\Lambda)\Phi_a(f)U(q,\Lambda)^* = R(\Lambda^{-1})\Phi_a((q,\Lambda)f). \tag{8.6}$$

Remark 8.12. Another generalization of the axioms of a completely different nature concerns the locality. In the above axioms only bosonic fields are considered. For the fermionic case one has to impose a grading into even and odd (see also Remark 10.19), and the commutator of odd fields in Axiom W2 has to be replaced with the anticommutator.

Remark 8.13. The spectrum condition (Axiom W3) implies that for eigenvalues p_μ of P_μ the vector $p = (p_0, \dots, p_{D-1})$ satisfies $p \in C_+$. In particular, with the interpretation of $P_0 = H$ as the energy operator the system has no negative energy states: $p_0 \geq 0$. Moreover, $P^2 = P_0^2 - P_1^2 - \dots - P_{D-1}^2$ has the interpretation of the mass-squared operator with the condition $p^2 \geq 0$ for each $D-$tuple of eigenvalues p_μ of P_μ in case Axiom W3 is satisfied.

Remark 8.14. In addition to the above axioms in many cases an irreducibility or completeness condition is required. For example, it is customary to require that the vacuum is cyclic in the sense that the subspace $D_0 \subset D$ spanned by all the vectors

$$\Phi_{a_1}(f_1)\Phi_{a_2}(f_2)\dots\Phi_{a_m}(f_m)\Omega^1$$

is dense in D and thus dense in $\mathbb{H}$.

Moreover, as an additional axiom one can require the vacuum Ω to be unique:

Axiom W4 (Uniqueness of the Vacuum) *The only vectors in $\mathbb{H}$ left invariant by the translations $U(q,1)$, $q \in M$, are the scalar multiples of the vacuum Ω.*

Although the above postulates appear to be quite evident and natural, it is by no means easy to give examples of Wightman quantum field theories even for the case of free theories. For the case of proper interaction no Wightman QFT is known so far in the relevant case of $D = 4$, and it is one of the millennium problems discussed in [JW06*] to construct such a theory. For $D = 2$, however, there are theories with interaction (cf. [Simo74*]), and many of the conformal field theories in two dimensions have nontrivial interaction.

Example: Free Bosonic QFT. In the following we sketch a Wightman QFT for a quantized boson of mass $m > 0$ in three-dimensional space (hence $D = 4$, the considerations work for arbitrary $D \geq 2$ without alterations). The basic differential operator, the Klein–Gordon operator $\square + m^2$ with mass m, has already been studied in Sect. 8.1. We look for a field operator

$$\Phi : \mathscr{S} = \mathscr{S}(\mathbb{R}^4) \longrightarrow \mathscr{S}\mathscr{O}(\mathbb{H})$$

on a Hilbert space $\mathbb{H}$ such that for all test function $f, g \in \mathscr{S}$:

1. Φ satisfies the Klein–Gordon equation in the following sense:

$$\Phi(\square f + m^2 f) = 0 \text{ for all } f \in \mathscr{S}.$$

2. Φ obeys the commutation relation

$$[\Phi(f), \Phi(g)] = -i \int_{\mathbb{R}^4 \times \mathbb{R}^4} f(x) D_m(x-y) g(y) dxdy.$$

[1] As before, we write the composition $B \circ C$ of operators as multiplication BC and similarly the value $B(v)$ as multiplication Bv.

Here, D_m is the Pauli–Jordan function (cf. Definition 8.5)

$$D_m(x) := i(2\pi)^{-3} \int_{\mathbb{R}^D} \operatorname{sgn}(p_0)\delta(p^2 - m^2)e^{-ip\cdot x}dp.$$

The construction of such a field and the corresponding Hilbert space is a Fock space construction. Let $H_1 = \mathscr{S}(\Gamma_m) \cong \mathscr{S}(\mathbb{R}^3)$. The isomorphism is induced by the global chart

$$\xi : \mathbb{R}^3 \to \Gamma_m, \mathsf{p} \mapsto (\omega(\mathsf{p}), \mathsf{p}),$$

where $\omega(\mathsf{p}) = \sqrt{\mathsf{p}^2 + m^2}$. We denote the points in Γ_m by ξ or ξ_j in the following:

H_1 is dense in $\mathbb{H}_1 := \mathrm{L}^2(\Gamma_m, d\lambda_m)$, the complex Hilbert space of square-integrable functions on the upper hyperboloid Γ_m. Furthermore, let H_N denote the space of rapidly decreasing functions on the N-fold product of the upper hyperboloid Γ_m which are symmetric in the variables $(\mathsf{p}_1, \dots, \mathsf{p}_N) \in \Gamma_m^N$. H_N has the inner product

$$\langle u, v\rangle := \int_{\Gamma_m^N} \overline{u}(\xi_1, \dots, \xi_N)v(\xi_1, \dots, \xi_N)d\lambda_m(\xi_1)\dots d\lambda_m(\xi_N).$$

The Hilbert space completion of H_N will be denoted by $\mathbb{H}_N$. H_N contains the N-fold symmetric product of H_1 and this space is dense in H_N and thus also in $\mathbb{H}_N$. Now, the direct sum

$$D := \bigoplus_{N=0}^{\infty} H_N$$

($H_0 = \mathbb{C}$ with the vacuum $\Omega := 1 \in H_0$) has a natural inner product given by

$$\langle f, g\rangle := \overline{f_0}g_0 + \sum_{N\geq 1} \frac{1}{N!}\langle f_N, g_N\rangle,$$

where $f = (f_0, f_1, \dots), g = (g_0, g_1, \dots) \in D$. The completion of D with respect to this inner product is the Fock space $\mathbb{H}$. $\mathbb{H}$ can also be viewed as a suitable completion of the symmetric algebra

$$S(H_1) = \bigoplus H_1^{\odot N},$$

where $H_1^{\odot N}$ is the N-fold symmetric product

$$H_1^{\odot N} = H_1 \odot \dots \odot H_1.$$

The operators $\Phi(f)$, $f \in \mathscr{S}$, will be defined on $g = (g_0, g_1, \dots) \in D$ by

$$\begin{aligned}(\Phi(f)g)_N(\xi_1, \dots, \xi_N) := &\int_{\Gamma_m} \widehat{f}(\xi)g_{N+1}(\xi, \xi_1, \dots, \xi_N)d\lambda_m(\xi)\\ &+ \sum_{j=1}^{N} \widehat{f}(-\xi_j)g_{N-1}(\xi_1, \dots \hat{\xi}_j \dots, \xi_N),\end{aligned}$$

where $\hat{\xi}_j$ means that this variable has to be omitted. This completes the construction of the Wightman QFT for the free boson.

The various requirements and axioms are not too difficult to verify. For example, we obtain $\Phi(\Box f - m^2 f) = 0$ since

$$\mathscr{F}(\Box f - m^2 f) = (-p^2 + m^2)\widehat{f}$$

vanishes on Γ_m, and similarly we obtain the second requirement on the commutators the formula

$$[\Phi(f), \Phi(g)] = -i \int_{\mathbb{R}^4 \times \mathbb{R}^4} f(x) D_m(x-y) g(y) dx dy.$$

Furthermore, we observe that the natural action of the Poincaré group on $\mathbb{R}^{1,3}$ and on $\mathscr{S}(\mathbb{R}^{1,3})$ induces a unitary representation U in the Fock space $\mathbb{H}$ leaving invariant the vacuum and the domain of definition D. Of course, Φ is a field operator in the sense of our Definition 8.8 with $\Phi(f)D \subset D$ and, moreover, it can be checked that Φ is covariant in the sense of Axiom W1 and that the joint spectrum of the operators P_j is supported in Γ_m hence in the forward light cone (Axiom W3). Finally, the construction yields locality (Axiom W2) according to the above formula for $[\Phi(f), \Phi(g)]$.

We conclude this section with the following result of Wightman which demonstrates that in QFT it is necessary to consider operator-valued distributions instead of operator-valued mappings:

Proposition 8.15. *Let Φ be a field in a Wightman QFT which can be realized as a map $\Phi : M \to \mathscr{O}$ and where Φ^* belongs to the fields. Moreover, assume that Ω is the only translation-invariant vector (up to scalars). Then $\Phi(x) = c\Omega$ is the constant operator for a suitable constant $c \in \mathbb{C}$.*

In fact, it is enough to require equivariance with respect to the transformation group only and the property that $\Phi(x)$ and $\Phi(y)^*$ commute if $x - y$ is spacelike.

8.4 Wightman Distributions and Reconstruction

Let $\Phi = \Phi_a$ be a field operator in a Wightman QFT acting on the space $\mathscr{S} = \mathscr{S}(\mathbb{R}^{1,D-1})$ of test functions

$$\Phi : \mathscr{S} \longrightarrow \mathscr{O}(\mathbb{H}).$$

We assume $\Phi(f)$ to be self-adjoint for real-valued $f \in \mathscr{S}$ (cf. 8.9), hence $\Phi(f)^* = \Phi(\overline{f})$ in general. Then for $f_1, \dots, f_N \in \mathscr{S}$ one can define

$$W_N(f_1, \dots, f_N) := \langle \Omega, \Phi(f_1) \dots \Phi(f_N) \Omega \rangle$$

according to Axiom W1 part 2. Since Φ is a field operator the mapping

$$W_N : \mathscr{S} \times \mathscr{S} \ldots \times \mathscr{S} \longrightarrow \mathbb{C}$$

is multilinear and separately continuous. It is therefore jointly continuous and one can apply the nuclear theorem of Schwartz to obtain a uniquely defined distribution on the space in DN variables, that is a distribution in $\mathscr{S}'((\mathbb{R}^D)^N) = \mathscr{S}'(\mathbb{R}^{DN})$. This continuation of W_N will be denoted again by W_N.

The sequence (W_N) of distributions generated by Φ is called the sequence of *Wightman distributions*. The $W_N \in \mathscr{S}'(\mathbb{R}^{DN})$ are also called *vacuum expectation values* or *correlation functions*.

Theorem 8.16. *The Wightman distributions associated to a Wightman QFT satisfy the following conditions: Each $W_N, N \in \mathbb{N}$, is a tempered distribution*

$$W_N \in \mathscr{S}'(\mathbb{R}^{DN})$$

with

WD1 (Covariance) *W_N is Poincaré invariant in the following sense:*

$$W_N(f) = W_N((q,\Lambda)f) \; for \; all \; (q,\Lambda)) \in P.$$

WD2 (Locality) *For all $N \in \mathbb{N}$ and $j, 1 \le j < N$,*

$$W_N(x_1,\ldots,x_j,x_{j+1},\ldots,x_N) = W_N(x_1,\ldots,x_{j+1},x_j,\ldots,x_N),$$

if $(x_j$ - $x_{j+1})^2 < 0$.)

WD3 (Spectrum Condition) *For each $N > 0$ there exists a distribution $M_N \in \mathscr{S}'(\mathbb{R}^{D(N-1)})$ supported in the product $(C_+)^{N-1} \subset \mathbb{R}^{D(N-1)}$ of forward cones such that*

$$W_N(x_1,\ldots,x_N) = \int_{\mathbb{R}^{D(N-1)}} M_N(p) e^{i\sum p_j \cdot (x_{j+1} - x_j)} dp,$$

where $p = (p_1,\ldots,p_{n-1}) \in (R^D)^{N-1}$ and $dp = dp_1 \ldots dp_{N-1}$.

WD4 (Positive Definiteness) *For any sequence $f_N \in \mathscr{S}(\mathbb{R}^{DN}) N \in \mathbb{N}$ one has for all $m \in \mathbb{N}$:*

$$\sum_{M,N=0}^{k} W_{M+N}(\overline{f}_M \otimes f_N) \ge 0.$$

$f \otimes g$ for $f \in \mathscr{S}(\mathbb{R}^{DM}), g \in \mathscr{S}(\mathbb{R}^{DN})$ is defined by

$$f \otimes g(x_1,\ldots,x_{M+N}) = f(x_1,\ldots,x_M) g(x_{M+1},\ldots,x_{M+N}).$$

Proof. WD1 follows directly from W1. Observe that the unitary representation of the Poincaré group is no longer visible. And WD2 is a direct consequence of W2. WD4 is essentially the property that a vector of the form

$$\sum_{M=1}^{k} \Phi(f_M)\Omega \in \mathbb{H}$$

has a non-negative norm where $\Phi(f_M)\Omega$ is defined as follows: The map

$$(f_1,\dots,f_M) \mapsto \Phi(f_1)\dots\Phi(f_M)\Omega, (f_1,\dots,f_M) \in \mathscr{S}(\mathbb{R}^D)^M,$$

is continuous and multilinear by the general assumptions on the field operator Φ and therefore induces by the nuclear theorem a vector-valued distribution $\Phi_M : \mathscr{S}(\mathbb{R}^{DM}) \to \mathbb{H}$ which is symbolically written as $\Phi_M(x_1,\dots,x_M)$. Now, $\Phi(f_M)\Omega := \Phi_M(f_M)\Omega$ and

$$0 \leq \left\| \sum_{M=1}^{k} \Phi(f_M)\Omega \right\|^2 \leq \left\langle \sum_{M=1}^{k} \Phi(f_M)\Omega, \sum_{N=1}^{k} \Phi(f_N)\Omega \right\rangle$$
$$\leq \sum_{M,N} \langle \Omega, \Phi(f_M)^*\Phi(f_N)\Omega\rangle = \sum_{M,N} W_{M+N}(\overline{f}_M \otimes f_N).$$

WD3 will be proven in the next proposition. □

In the sequel we write the distributions Φ and W_N symbolically as functions $\Phi(x)$ and $W_N(x_1,\dots,x_N)$ in order to simplify the notation and to work more easily with the supports of the distributions in consideration.

The covariance of the field operator Φ implies the covariance

$$W_N(x_1,\dots,x_n) = W_N(\Lambda x_1 + q,\dots,\Lambda x_N + q)$$

for every $(q,\Lambda) \in \mathrm{P}$. In particular, the Wightman distributions are translation-invariant:

$$W_N(x_1,\dots,x_n) = W_N(x_1 + q,\dots,x_N + q).$$

Consequently, W_N depends only on the differences

$$\xi_1 = x_1 - x_2,\dots,\xi_{N-1} = x_{N-1} - x_N.$$

We define

$$w_N(\xi_1,\dots,\xi_{N-1}) := W_N(x_1,\dots,x_N).$$

Proposition 8.17. *The Fourier transform $\widehat{w}_N$ has its support in the product $(C_+)^{N-1}$ of the forward cone $C_+ \in \mathbb{R}^D$. Hence*

$$W_N(x) = (2\pi)^{-D(N-1)} \int_{\mathbb{R}^{D(N-1)}} \widehat{w}_N(p) e^{-i\sum p_j\cdot(x_j - x_{j+1})} dp.$$

Proof. Because of $U(x,1)^* = U(-x,1) = e^{-ix\cdot P}$ for $x \in \mathbb{R}^D$ (cf. 8.2) the spectrum condition W2 implies

$$\int_{\mathbb{R}^D} e^{ix\cdot p} U(x,1)^* v dx = 0$$

for every $v \in \mathbb{H}$ if $p \notin C_+$. Since $w_N(\xi_1,\dots,\xi_j+x,\xi_{j+1},\dots,\xi_{N-1}) = W_N(x_1,\dots,x_j, x_{j+1}-x,\dots,x_N-x)$ the Fourier transform of w_N with respect to ξ_j gives

$$\begin{aligned}
&\int_{\mathbb{R}^D} w_N(\xi_1,\dots,\xi_j+x,\xi_{j+1},\dots,\xi_{N-1})e^{ip_j\cdot x}dx \\
&= \left\langle \Omega, \Phi(x_1)\dots\Phi(x_j)\int_{\mathbb{R}^D}\Phi(x_{j+1}-x)\dots\Phi(x_N-x)e^{ip_j\cdot x}\Omega dx\right\rangle \\
&= \left\langle \Omega, \Phi(x_1)\dots\Phi(x_j)\int_{\mathbb{R}^D} e^{ix\cdot p_j}U^*(x,1)\Phi(x_{j+1})\dots\Phi(x_N)\Omega dx\right\rangle = 0,
\end{aligned}$$

where the last identity is a result of applying the above formula to $v = \Phi(x_{j+1})\dots \Phi(x_N)\Omega$ whenever $p_j \notin C_+$. Hence,

$$\widehat{w}_N(p_1,\dots,p_{N-1}) = 0$$

if $p_j \notin C_+$ for at least one index j. □

Having established the basic properties of the Wightman functions we now explain how a sequence of distributions with the properties WD 1–4 induce a Wightman QFT by the following:

Theorem 8.18. (Wightman Reconstruction Theorem) *Given any sequence (W_N), $W_N \in \mathscr{S}'(\mathbb{R}^{DN})$, of tempered distributions obeying the conditions WD1–WD4, there exists a Wightman QFT for which the W_N are the Wightman distributions.*

Proof. We first construct the Hilbert space for the Wightman QFT. Let

$$\underline{\mathscr{S}} := \bigoplus_{N=0}^{\infty} \mathscr{S}(\mathbb{R}^{DN})$$

denote the vector space of finite sequences $\underline{f} = (f_N)$ with $f_N \in \mathscr{S}(\mathbb{R}^{DN}) =: \mathscr{S}_N$. On $\underline{\mathscr{S}}$ we define a multiplication

$$\underline{f}\times\underline{g} := (h_N), h_N := \sum_{k=0}^{N} f_k(x_1,\dots,x_k)g_{N-k}(x_{k+1},\dots,x_N).$$

The multiplication is associative and distributive but not commutative. Therefore, $\underline{\mathscr{S}}$ is an associative algebra with unit $1 = (1,0,0,\dots)$ and with a convolution $\gamma(\underline{f}) := (\overline{f}_N) = \underline{\overline{f}}$. γ is complex antilinear and satisfies $\gamma^2 = \mathrm{id}$.

Our basic algebra $\underline{\mathscr{S}}$ will be endowed with the direct limit topology and thus becomes a complete locally convex space which is separable. (The direct limit topology is the finest locally convex topology on $\underline{\mathscr{S}}$ such that the natural inclusions $\mathscr{S}(\mathbb{R}^{DN}) \to \underline{\mathscr{S}}$ are continuous.) The continuous linear functionals $\mu : \underline{\mathscr{S}} \to \mathbb{C}$ are represented by sequences (μ_N) of tempered distributions $\mu_N \in \mathscr{S}'_N$: $\mu((f_N)) = \sum \mu_N(f_N)$.

Such a functional is called positive semi-definite if $\mu(\underline{\overline{f}}\times\underline{f}) \geq 0$ for all $\underline{f} \in \underline{\mathscr{S}}$ because the associated bilinear form $\omega = \omega_\mu$ given by $\omega(\underline{f},\underline{g}) := \mu(\underline{\overline{f}}\times\underline{g})$ is positive semi-definite. For a positive semi-definite continuous linear functional μ the subspace

$$J = \{\underline{f} \in \underline{\mathscr{S}} : \mu(\overline{\underline{f}} \times \underline{f}) = 0\}$$

turns out to be an ideal in the algebra $\underline{\mathscr{S}}$.

It is not difficult to show that in the case of a positive semi-definite $\mu \in \underline{\mathscr{S}}'$ the form ω is hermitian and defines on the quotient $\underline{\mathscr{S}}/J$ a positive definite hermitian scalar product. Therefore, $\underline{\mathscr{S}}/J$ is a pre-Hilbert space and the completion of this space with respect to the scalar product is the Hilbert space $\mathbb{H}$ needed for the reconstruction. This construction is similar to the so-called GNS construction of Gelfand, Naimark, and Segal.

The vacuum $\Omega \in \mathbb{H}$ will be the class of the unit $1 \in \underline{\mathscr{S}}$ and the field operator Φ is defined by fixing $\Phi(f)$ for any test function $f \in \mathscr{S}$ on the subspace $D = \underline{\mathscr{S}}/J$ of classes $[\underline{g}]$ of elements of $\underline{\mathscr{S}}$ by

$$\Phi(f)([\underline{g}]) := [\underline{g} \times f],$$

where f stands for the sequence $(0, f, 0, \ldots,)$. Evidently, $\Phi(f)$ is an operator defined on D depending linearly on f. Moreover, for $\underline{h}, \underline{g} \in \underline{\mathscr{S}}$ the assignment

$$f \mapsto \langle [\underline{h}], \Phi(f)([\underline{g}]) \rangle = \mu(\underline{h} \times (\underline{g} \times f))$$

is a tempered distribution because μ is continuous. This means that Φ is a field operator in the sense of Definition 8.8. Obviously, $\Phi(f)D \subset D$ and $\Omega \in D$.

So far, the Wightman distributions W_N have not been used at all. We consider now the above construction for the continuous functional $\mu := (W_N)$. Because of property WD4 this functional is positive semi-definite and provides the Hilbert space $\mathbb{H}$ constructed above depending on (W_N) together with a vacuum Ω and a field operator Φ. The properties of the Wightman distributions which eventually ensure that the Wightman axioms for this construction are fulfilled are encoded in the ideal

$$J = \{\underline{f} = (f_N) \in \underline{\mathscr{S}} : \sum W_N(\underline{f} \times \overline{\underline{f}}) = 0\}.$$

To show covariance, we first have to specify a unitary representation of the Poincaré group P in $\mathbb{H}$. This representation is induced by the natural action $\underline{f} \mapsto (q,\Lambda)\underline{f}$ of P on $\underline{\mathscr{S}}$ given by

$$(q,\Lambda)f_n(x_1, \ldots, x_n) := f(\Lambda^{-1}(x_1 - q), \ldots, \Lambda^{-1}(x_n - q))$$

for $(q,\Lambda) \in \mathrm{L} \ltimes M \cong \mathrm{P}$. This action leads to a homomorphism $\mathrm{P} \to \mathrm{GL}(\underline{\mathscr{S}})$ and the action respects the multiplication. Now, because of the covariance of the Wightman distributions (property WD1) the ideal J is invariant, that is for $\underline{f} \in J$ and $(q,\Lambda) \in \mathrm{P}$ we have $(q,\Lambda)\underline{f} \in J$. As a consequence, $\mathrm{U}(q,\Lambda)([\underline{f}]) := [(q,\Lambda)\underline{f})]$ is well-defined on $D \subset \mathbb{H}$ with

$$\langle \mathrm{U}(q,\Lambda)([\underline{f}]), \mathrm{U}(q,\Lambda)([\underline{f}]) \rangle = \langle [\underline{f}], [\underline{f}] \rangle.$$

Altogether, this defines a unitary representation of P in $\mathbb{H}$ leaving Ω invariant such that the field operator is equivariant. We have shown that the covariance axiom W1 is satisfied.

In a similar way, one can show that property WD2 implies W2 and property WD3 implies W3. Locality (property WD2) implies that J includes the ideal J_{lc} generated by the linear combinations of the form

$$f_N(x_1,\ldots,x_N) = g(x_1,\ldots,x_j,x_{j+1},\ldots,x_N) - g(x_1,\ldots,x_{j+1},x_j,\ldots,x_N)$$

with $g(x_1,\ldots,x_N) = 0$ for $(x_{j+1} - x_j)^2 \geq 0$. And property WD3 (spectrum condition) implies that the ideal

$$J_{sp} := \{(f_N) : f_0 = 0, \widehat{f}(p_1,\ldots,p_N) = 0 \text{ in a neighborhood of } C_N\},$$

where $C_N = \{p : p_1 + \ldots + p_j \in C_+, j = 1,\ldots,N\}$, is also contained in J. □

As a result of this section, in an axiomatic approach to quantum field theory the Wightman axioms W1–W3 on the field operators can be replaced by the equivalent properties or axioms WD1–WD4 on the corresponding correlation functions W_N, the Wightman distributions. This second approach is formulated without explicit reference to the Hilbert space.

In the next section we come to a different but again equivalent description of the axiomatics which is formulated completely in the framework of Euclidean geometry.

8.5 Analytic Continuation and Wick Rotation

In this section we explain how the Wightman axioms induce a Euclidean field theory through analytic continuation of the Wightman distributions.

We first collect some results and examples on analytic continuation of holomorphic functions. Recall that a complex-valued function $F : U \to \mathbb{C}$ on an open subset $U \subset \mathbb{C}^n$ is *holomorphic* or *analytic* if it has complex partial derivatives $\frac{\partial}{\partial z^j} F = \partial_j F$ on U with respect to each of its variables z^j or, equivalently, if F can be expanded in each point $a \in U$ into a convergent power series $\sum c_\alpha z^\alpha$ such that

$$F(a+z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$$

for z in a suitable open neighborhood of 0. The partial derivatives of F in a of any order exist and appear in the power series expansions in the form $\partial^\alpha F(a) = \alpha! c_\alpha$.

A holomorphic function F on a connected domain $U \subset \mathbb{C}^n$ is completely determined by the restriction $F|_W$ to any nonempty open subset $W \subset U$ or by any of its germs (that is power series expansion) at a point $a \in U$. This property leads to the phenomenon of analytic continuation, namely that a holomorphic function g on an open subset $W \subset \mathbb{C}^n$ may have a so-called *analytic continuation* to a holomorphic $F : U \to \mathbb{C}$, that is $F|_W = g$, which is uniquely determined by g.

A different type of analytic continuation occurs if a real analytic function $g : W \to \mathbb{C}$ on an open subset $W \subset \mathbb{R}^n$ is regarded as the restriction of a holomorphic $F : U \to \mathbb{C}$ where U is an open subset in $\mathbb{C}^n$ with $U \cap \mathbb{R}^n = W$. Such a holomorphic function F is obtained by simply exploiting the power series expansions of the real analytic function g: For each $a \in W$ there are $c_\alpha \in \mathbb{C}$ and $r^j(a) > 0, j = 1,\dots,n$, such that $g(a+x) = \sum_\alpha c_\alpha x^\alpha$ for all x with $|x^j| < r^j(a)$. By inserting $z \in \mathbb{C}, |z^j| < r^j(a)$, instead of x into the power series we get such an analytic continuation defined on the open neighborhood $U = \{a+z \in \mathbb{C}^n : a \in W, |z^j| < r^j(a)\} \subset \mathbb{C}^n$ of W.

Another kind of analytic continuation is given by the Laplace transform. As an example in one dimension let $u : \mathbb{R}_+ \to \mathbb{C}$ be a polynomially bounded continuous function on $\mathbb{R}_+ = \{t \in \mathbb{R} : t > 0\}$.

Then the integral ("Laplace transform")

$$\mathscr{L}(u)(z) = F(z) := \int_0^\infty u(t)e^{itz}dt, \ \mathrm{Im}\, z \in \mathbb{R}_+,$$

defines a holomorphic function F on the "tube" domain $U = \mathbb{R} \times \mathbb{R}_+ \subset \mathbb{C}$ such that,

$$\lim_{y \searrow 0} F(x+iy) = g(x) \text{ where } g(x) := \int_0^\infty u(t)e^{itx}dt.$$

In this situation the $g(x)$ are sometimes called the boundary values of $F(z)$. The analytic continuation is given by the Laplace transform.

Of course, the integral exists because of $|u(t)e^{itz}| = |u(t)e^{-ty}| \leq |u(x)|$ for $z = x+iy \in U$ and $t \in \mathbb{R}_+$. F is holomorphic since we can interchange integration and derivation to obtain

$$\frac{d}{dz}F(z) = F'(z) = i\int_0^\infty tu(t)e^{itz}dt.$$

We now present a result which shows how in a similar way even a distribution $T \in \mathscr{S}(\mathbb{R}^n)'$ can, in principle, be continued analytically from $\mathbb{R}^n$ into an open neighborhood $U \subset \mathbb{C}^n$ of $\mathbb{R}^n$ and in which sense T is a boundary value of this analytic continuation.

Let $C \subset \mathbb{R}^n$ be a convex cone with its dual $C' := \{p \in \mathbb{R}^n : p \cdot x \geq 0 \forall\, x \in C\}$ and assume that C' has a nonempty interior C°. Let $\mathscr{T} := \mathbb{R}^n \times (-C^\circ)$ be the induced open tube in $\mathbb{C}^n$. Here, the dot "$\cdot$" represents any scalar product on $\mathbb{R}^n$, that is any symmetric and nondegenerate bilinear form.

The particular case in which we are mainly interested is the case of the forward cone $C = C_+$ in $\mathbb{R}^D = \mathbb{R}^{1,D-1}$ with respect to the Minkowki scalar product. Here, the cone C is self-dual $C' = C$ and C° is the open forward cone

$$C^\circ = \left\{x \in \mathbb{R}^{1,D-1} : x^2 = <x,x> \, > 0, x^0 > 0\right\}$$

and $\mathscr{T} = \mathbb{R}^n \times (-C^\circ)$ is the *backward tube*.

Theorem 8.19. *For every distribution $T \in \mathscr{S}(\mathbb{R}^n)'$ whose Fourier transform has its support in the cone C there exists an analytic function F on the tube $\mathscr{T} \subset \mathbb{C}^n$ with*

- $|F(z)| \le c(1+|z|)^k(1+d^e(z,\partial\mathscr{T}))^{-m}$ *for suitable* $c \in \mathbb{R}$, $k,m \in \mathbb{N}$. *(Here,* d^e *is the Euclidean distance in* $\mathbb{C}^n = \mathbb{R}^{2n}$.*)*
- T *is the boundary value of the holomorphic function* F *in the following sense. For any* $f \in \mathscr{S}$ *and* $y \in -C^\circ \subset \mathbb{R}^n$*:*

$$\lim_{t \searrow 0} \int_{\mathbb{R}^n} f(x)F(x+ity)dx = T(f),$$

where the convergence is the convergence in $\mathscr{S}'$.

Proof. Let us first assume that $\widehat{T}$ is a polynomially bounded continuous function $g = g(p)$ with support in C. In that case the (Laplace transform) formula

$$F(z) := (2\pi)^{-n} \int_{\mathbb{R}^n} g(p)e^{-ip\cdot z}dp,\ z \in \mathscr{T},$$

defines a holomorphic function fulfilling the assertions of the theorem. Indeed, since the exponent $-ip\cdot z = -ip\cdot x + p\cdot y$ has a negative real part $p\cdot y < 0$ for all $z = x+iy \in \mathscr{T} = \mathbb{R}^n \times (-C^\circ)$ the integral is well-defined. F is holomorphic in z since one can take derivatives under the integral. To show the bounds is straightforward. Finally, for $y \in -C^\circ$ and $f \in \mathscr{S}(\mathbb{R}^n)$ the limit of

$$\int f(x)F(x+ity)dx = \int f(x)\left((2\pi)^{-n}\int g(p)e^{-ip\cdot x}e^{tp\cdot y}dp\right)dx$$

for $t \searrow 0$ is $\int f(x)\mathscr{F}^{-1}g(x)dx = T(f)$.

Suppose now that $\widehat{T}$ is of the form $P(-i\partial)g$ for a polynomial $P \in \mathbb{C}[X^1,\dots,X^n]$ and g a polynomially bounded continuous function with support in C. Then

$$F(z) = P(z)(2\pi)^{-n} \int_{\mathbb{R}^n} g(p)e^{-ip\cdot z}dp, z \in \mathscr{T},$$

satisfies all conditions since $\mathscr{F}(P(x)\mathscr{F}^{-1}g) = P(-i\partial)g = \widehat{T}$.

Now the theorem follows from a result of [BEG67*] which asserts that for any distribution $S \in \mathscr{S}'$ with support in a convex cone C there exists a polynomial P and a polynomially bounded continuous function g with support in C and with $S = P(-i\partial)g$. □

We now draw our attention to the Wightman distributions.

Analytic Continuation of Wightman Functions. Given a Wightman QFT with field operator $\Phi : \mathscr{S}(\mathbb{R}^{1,D-1}) \longrightarrow \mathscr{O}$ (cf. Sect. 8.3) we explain in which sense and to which extent the corresponding Wightman distributions (cf. Sect. 8.4)

$$W_N \in \mathscr{S}'(\mathbb{R}^{DN})$$

can be continued analytically to an open connected domain $U_N \subset \mathbb{C}^{DN}$ of the complexification

$$\mathbb{C}^{DN} \cong \mathbb{R}^{DN} \otimes \mathbb{C}$$

of $\mathbb{R}^{DN}$.

The Minkowski inner product will be continued to a complex-bilinear form on $\mathbb{C}^D$ by $\langle z,w\rangle = z\cdot w = z^0w^0 - \sum_{j=1}^{D-1} z^j w^j$.

An important and basic observation in this context is the possibility of identifying the Euclidean $\mathbb{R}^D$ with the real subspace

$$E := \{(it,x^1,\dots,x^{D-1}) \in \mathbb{C}^D : (t,x^1,\dots,x^{D-1}) \in \mathbb{R}^D\}$$

the "Euclidean points" of $\mathbb{C}^D$, since

$$\langle (it,x^1,\dots,x^{D-1}),(it,x^1,\dots,x^{D-1})\rangle = -t^2 - \sum_{j=1}^{D-1} x^j x^j.$$

The Wightman distributions W_N will be analytically continued in three steps into open subsets U_N containing a great portion of the Euclidean points E^N, so that the restrictions of the analytically continued Wightman functions W_N to $U_N \cap E^N$ define a Euclidean field theory.
We have already used the fact that W_N is translation-invariant and therefore depends only on the differences $\xi_j := x_j - x_{j+1}$, $j = 1,\dots,N-1$:

$$w_N(\xi_1,\dots,\xi_{N-1}) := W_N(x_1,\dots,x_N).$$

Each w_N is the inverse Fourier transform of its Fourier transform $\widehat{w}_N$, that is

$$w_N(\xi_1,\dots,\xi_{N-1}) = \\ (2\pi)^{-D(N-1)} \int_{\mathbb{R}^{D(N-1)}} \widehat{w}_N(\omega_1,\dots,\omega_{N-1}) e^{-i\sum_k \omega_k\cdot\xi_k} d\omega_1 \dots d\omega_{N-1} \tag{8.7}$$

with

$$\widehat{w}_N(\omega_1,\dots,\omega_{N-1}) = \int_{\mathbb{R}^{D(N-1)}} w(\xi_1,\dots,\xi_{N-1}) e^{i\sum_k \omega_k\cdot\xi_k} d\xi_1 \dots d\xi_{N-1}.$$

By the spectrum condition the Fourier transform $\widehat{w}_N(\omega_1,\dots,\omega_{N-1})$ vanishes if one of the $\omega_1,\dots,\omega_{N-1}$ lies outside the forward cone C_+ (cf. 8.17).

If we now take complex vectors $\zeta_k = \xi_k + i\eta_k \in \mathbb{C}^D$ instead of the ξ_k in the above formula for w_N, then the integrand in (8.7) has the form

$$\widehat{w}_N(\omega) e^{-i\sum_k \omega_k\cdot\xi_k} e^{\sum_k \omega_k\cdot\eta_k},$$

and the corresponding integral will converge if η_k fulfills $\sum_k \omega_k\cdot\eta_k < 0$ for all ω_k in the forward cone. With the N-fold backward tube $\mathscr{T}_N = (\mathbb{R}^D \times (-C^\circ))^N \subset (\mathbb{C}^D)^N$ this approach leads to the following result whose proof is similar to the proof of Theorem 8.19.

Proposition 8.20. *The formula*

$$w_N(\zeta) = (2\pi)^{-D(N-1)} \int \widehat{w}_N(\omega) e^{-i\sum_k \omega_k\cdot\zeta_k} d\omega, \zeta \in \mathscr{T}_{N-1},$$

provides a holomorphic function in $\mathcal{T}_{N-1}$ *with the property*

$$\lim_{t \searrow 0} w_N(\xi + it\eta) = w_N(\xi)$$

if $\xi + i\eta \in \mathcal{T}_{N-1}$ *and where the convergence is the convergence in* $\mathcal{S}'(\mathbb{R}^{D(N-1)})$.

As a consequence, the Wightman distributions have analytic continuations to $\{z \in (\mathbb{C}^D)^N : \mathrm{Im}(z_{j+1} - z_j) \in C^\circ\}$.

This first step of analytic continuation is based on the spectrum condition. In a second step the covariance is exploited.

The covariance implies that the identity

$$w_N(\zeta_1, \ldots, \zeta_{N-1}) = w_N(\Lambda\zeta_1, \ldots, \Lambda\zeta_{N-1}) \tag{8.8}$$

holds for $(\zeta_1, \ldots, \zeta_{N-1}) \in (R^D)^{N-1}$ and $\Lambda \in \mathrm{L}$. Since analytic continuation is unique the identity also holds for $(\zeta_1, \ldots, \zeta_{N-1}) \in \mathcal{T}_{N-1}$ for those $(\zeta_1, \ldots, \zeta_{N-1})$ satisfying $(\Lambda\zeta_1, \ldots, \Lambda\zeta_{N-1}) \in \mathcal{T}_{N-1}$.

Moreover, the identity (8.8) extends to transformations Λ in the (proper) complex Lorentz group $\mathrm{L}(\mathbb{C})$. This group $\mathrm{L}(\mathbb{C})$ is the component of the identity of the group of complex $D \times D$-matrices Λ satisfying $\Lambda z \cdot \Lambda w = z \cdot w$ with respect to the complex Minkowski scalar product. This follows from the covariance and the fact that

$$\Lambda \mapsto w_N(\Lambda\zeta_1, \ldots, \Lambda\zeta_{N-1})$$

is holomorphic in a neighborhood of $\mathrm{id}_{\mathbb{C}^D}$ in $\mathrm{L}(\mathbb{C})$. By the identity (8.8) one obtains an analytic continuation of w_N to $(\Lambda^{-1}(\mathcal{T}_{N-1}))^{N-1}$.
Let

$$\mathcal{T}_N^e = \bigcup \{\Lambda(\mathcal{T}_N) : \Lambda \in \mathrm{L}(\mathbb{C})\}$$

be the *extended tube* where $\Lambda(\mathcal{T}_N) = \{(\Lambda\zeta_1, \ldots, \Lambda\zeta_N) : (\zeta_1, \ldots, \zeta_N) \in \mathcal{T}_N\}$. We have shown

Proposition 8.21. w_N *has an analytic continuation to the extended tube* $\mathcal{T}_{N-1}^e$.

While the tube $\mathcal{T}_N$ has no real points (that is points with only real coordinates $z_j \in \mathbb{R}^D$) as is clear from the definition of the tube, the extended tube contains real points.

For example, in the case $N = 1$ let $x \in \mathbb{R}^D$ be a real point with $x \cdot x < 0$. We can assume $x_2 = x_3 = \ldots = x_{D-1} = 0$ with $|x^1| > |x^0|$ by rotating the coordinate system. The complex Lorentz transformation $w = \Lambda z$, $w^0 = iz^1, w^1 = iz^0$ produces $w = \Lambda x$ with $\mathrm{Im}\ w^0 = x^1, \mathrm{Im}\ w^1 = x^0$, thus $\mathrm{Im}\ w \cdot \mathrm{Im}\ w = (x^1)^2 - (x^0)^2 > 0$ and $\Lambda x \in C^\circ$ if $x^1 < 0$. In the case $x^1 > 0$ one takes the transformation $w = \Lambda' z, w^0 = -iz^1, w^1 = -iz^0$. These two transformations are indeed in $\mathrm{L}(\mathbb{C})$ since they can be connected with the identity by $\Lambda(\theta)$ acting on the first two variables by

$$\Lambda(\theta) = \begin{pmatrix} \cosh i\theta & \sinh i\theta \\ \sinh i\theta & \cosh i\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$$

and leaving the remaining coordinates invariant.

We have proven that any real x with $x \cdot x < 0$ is contained in the extended tube $\mathscr{T}_1^e$. Similarly, one can show the converse, namely that a real x point of $\mathscr{T}_1^e$ satisfies $x \cdot x < 0$. In particular, the subset $\mathbb{R}^D \cap \mathscr{T}_1^e$ is open and not empty.

For general N, we have the following theorem due to Jost:

Theorem 8.22. *A real point $(\zeta_1, \dots, \zeta_N)$ lies in the extended tube $\mathscr{T}_N^e$ if and only if all convex combinations*

$$\sum_{j=1}^{N} t_j \zeta_j, \ \sum_{j=1}^{N} t_j = 1, t_j \geq 0,$$

are space-like, that is $(\sum_{j=1}^{N} t_j \zeta_j)^2 < 0$.

In the third step of analytic continuation we exploit the locality. For a permutation $\sigma \in S_N$, that is a permutation of $\{1, \dots, N\}$, let W_N^σ denote the Wightman distribution where the coordinates are interchanged by σ:

$$W_N^\sigma(x_1, \dots, x_N) := W_N(x_{\sigma(1)}, \dots, x_{\sigma(N)}),$$

and denote $w_N^\sigma(\xi_1, \dots, \xi_{N-1}) = W_N(x_{\sigma(1)}, \dots, x_{\sigma(N)}), \xi_j = x_j - x_{j+1}$.

Proposition 8.23. *Let w_N and w_N^σ be the holomorphic functions defined on the extended tube $\mathscr{T}_{N-1}^e$ by analytic continuation of the Wightman distributions w_N and w_N^σ according to Proposition 8.21. Then these holomorphic functions w_N and w_N^σ agree on their common domain of definition, which is not empty, and therefore define a holomorphic continuation on the union of their domains of definition.*

This result will be obtained by regarding the permuted tube ${}^\sigma\mathscr{T}'_{N-1}$ which is defined in analogy to $\Lambda\mathscr{T}_{N-1}$. The two domains $\mathscr{T}_{N-1}^e$ and ${}^\sigma\mathscr{T}_{N-1}^e$ have a nonempty open subset V of real points ξ with $\xi^2 < 0$ in common according to Theorem 8.22. Since all $\xi_j = x_j - x_{j+1}$ are space-like, this implies that w_N and w_N^σ agree on this open subset V and therefore w_N and w_N^σ agree in the intersection of the domains of definition.

We eventually have the following result:

Theorem 8.24. *w_N has an analytic continuation to the* permuted extended tube *$\mathscr{T}_{N-1}^{pe} = \bigcup\{{}^\sigma\mathscr{T}_{N-1}^e : \sigma \in S_N\}$ and similarly W_N has a corresponding analytic continuation to the permuted extended tube $\mathscr{T}_N^{pe}$. Moreover this tube contains all non-coincident points of E^N.*

Here E is the space of Euclidean points, $E := \{(it, x^1, \dots, x^{D-1}) \in \mathbb{C}^D : (t, x^1, \dots, x^{D-1}) \in \mathbb{R}^D\}$, and the last statement asserts that $E^N \setminus \Delta$ is contained in $\mathscr{T}_N^{pe}$ where $\Delta = \{(x_1, \dots, x_N) \in E^N : x_j = x_k \text{ for some } j \neq k\}$.

As a consequence the W_N have an analytic continuation to $E^N \setminus \Delta$ and define the so-called Schwinger functions

$$S_N := W_N|_{E^N \setminus \Delta}.$$

8.6 Euclidean Formulation

In order to state the essential properties of the Schwinger functions S_N we use the Euclidean *time reflection*

$$\theta : E \to E, (it, x^1, \ldots, x^{D-1}) \mapsto (-it, x^1, \ldots, x^{D-1})$$

and its action Θ on

$$\mathscr{S}_+(\mathbb{R}^{DN}) = \{f : E^N \to \mathbb{C} : f \in \mathscr{S}(E^N) \text{ with support in } Q_+^N\},$$

where

$$Q_+^N = \{(x_1, \ldots, x_N) : x_j = (it_j, x_j^1, \ldots, x_j^{D-1}), 0 < t_1 < \ldots < t_N\} :$$
$$\Theta \;:\; \mathscr{S}_+(\mathbb{R}^{DN}) \to \mathscr{S}(\mathbb{R}^{DN}),\; \Theta f(x_1, \ldots, x_N) := \overline{f}(\theta x_1, \ldots, \theta x_N).$$

Theorem 8.25. *The Schwinger functions S_N are analytic functions $S_N : E^N \setminus \Delta \to \mathbb{C}$ satisfying the following axioms:*

S1 (Covariance) $S_N(gx_1, \ldots, gx_N) = S_N(x_1, \ldots, x_N)$ *for Euclidean motions* $g = (q, R), q \in \mathbb{R}^D, R \in \mathrm{SO}(D)$ *(or* $R \in \mathrm{Spin}(D)$*).*

S2 (Locality) $S_N(x_1, \ldots, x_N) = S_N(x_{\sigma(1)}, \ldots, x_{\sigma(N)})$ *for any permutation* σ.

S3 (Reflection Positivity)

$$\sum_{M,N} S_{M+N}(\Theta f_M \otimes f_N) \geq 0$$

for finite sequences $(f_N), f_N \in \mathscr{S}_+(\mathbb{R}^{DN})$*, where, as before,*

$$g_M \otimes f_N(x_1, \ldots, x_{M+N}) = g_M(x_1, \ldots, x_M) f_N(x_{M+1}, \ldots, x_{M+N}).$$

These properties of correlation functions are called the Osterwalder–Schrader axioms.

Reconstruction. Several slightly different concepts are called *reconstruction* in the context of axiomatic quantum field theory when Wightman's axioms are involved and also the Euclidean formulation (Osterwalder–Schrader axioms) is considered.

For example from the axioms S1–S3 one can deduce the Wightman distributions satisfying WD1–WD4 and this procedure can be called reconstruction. Moreover, after this step one can reconstruct the Hilbert space (cf. Theorem 8.18) with the relativistic fields Φ as in W1–W3. Altogether, on the basis of Schwinger functions and its properties we thus have reconstructed the relativistic fields and the corresponding Hilbert space of states. This procedure is also called reconstruction.

But starting with S1–S3 one could, as well, build a Euclidean field theory by constructing a Hilbert space directly with the aid of S3 and then define the Euclidean fields as operator-valued distributions similar to the reconstruction of the relativistic fields as described in Sect. 8.4, in particular in the proof of the Wightman Reconstruction Theorem 8.18. Of course, this procedure is also called reconstruction. In the next chapter this kind of reconstruction is described with some additional details in Sects. 9.2 and 9.3 in the two-dimensional case.

8.7 Conformal Covariance

The theories described in this chapter do not incorporate conformal symmetry, so far. Let us describe how the covariance with respect to conformal mappings can be formulated within the framework of the axioms. Recall (cf. Theorem 1.9) that the conformal mappings not already included in the Poincaré group resp. the Euclidean group of motions are the special conformal transformations

$$q \mapsto q^b = \frac{q - \langle q,q\rangle b}{1 - 2\langle q,b\rangle + \langle q,q\rangle\langle b,b\rangle},\ q \in \mathbb{R}^n,$$

where $b \in \mathbb{R}^n$, and the dilatations

$$q \mapsto q^\lambda = e^\lambda q, q \in \mathbb{R}^n,$$

where $\lambda \in \mathbb{R}$.

The Wightman Axioms 8.3 are now extended in such a way that one requires U to be a unitary representation $U = U(q,\Lambda,b)$ of the conformal group $\mathrm{SO}(n,2)$ or $\mathrm{SO}(n,2)/\{\pm 1\}$ (cf. Sect. 2.2), resp. of its universal covering, such that in addition to the Poincaré covariance

$$U(q,\Lambda)\Phi_a(x)U(q,\Lambda)^* = \Phi_a(\Lambda x),$$

the following has to be satisfied:

$$U(0,E,b)\Phi_a(x)U(0,E,b)^* = N(q,b)^{-h_a}\Phi_a(x^b),$$

where $N(x,b) = 1 - 2\langle q,b\rangle + \langle q,q\rangle\langle b,b\rangle$ is the corresponding denominator and where $h_a \in \mathbb{R}$ is a so-called *conformal weight* of the field Φ_a. Moreover, the conformal covariance for the dilatations is

$$U(\lambda)\Phi_a(x)U(\lambda)^* = e^{\lambda d_a}\Phi_a(x^\lambda),$$

with a similar weight d_a. Observe that $N(x,b)^{-n}$ resp. $e^{n\lambda}$ is the Jacobian of the transformation x^b resp. x^λ.

We now turn our attention to the two-dimensional case. Since the Lorentz group of the Minkowski plane is isomorphic to the abelian group $\mathbb{R}$ (cf. Remark 1.15) and the rotation group of the Euclidean plane is isomorphic to $\mathbb{S}$, the one-dimensional representations of the isometry groups are no longer trivial (as in the higher-dimensional case). Consequently, in the covariance condition, in principle, these one-dimensional representations could occur, see also Remark 8.11. As an example, one can expect that the Lorentz boosts

$$\Lambda = \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix}, \ \chi \in \mathbb{R},$$

in the two-dimensional case satisfy the following covariance condition:

$$U(\Lambda)\Phi_a(x)U(\Lambda)^* = e^{\chi s_a}\Phi_a(\Lambda x),$$

where s_a would represent a spin of the field. Similarly, in the Euclidean case

$$U(\Lambda)\Phi_a(x)U(\Lambda)^* = e^{i\alpha s_a}\Phi_a(\Lambda x),$$

if α is the angle of the rotation Λ.

It turns out that in two-dimensional conformal field theory this picture is even refined further when formulating the covariance condition for the other conformal transformations. The light cone coordinates are regarded separately in the Minkowski case and similarly in the Euclidean case the coordinates are split into the complex coordinate and its conjugate.

With respect to the Minkowski plane one first considers the restricted conformal group (cf. Remark 2.16) only which is isomorphic to $\mathrm{SO}(2,2)/\{\pm 1\}$ and not the full infinite dimensional group of conformal transformations. With respect to the light cone coordinates the restricted conformal group $\mathrm{SO}(2,2)/\{\pm 1\}$ acts in the form of two copies of $\mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$ (cf. Proposition 2.17). For a conformal transformation $g = (A_+, A_-), A_\pm \in \mathrm{SL}(2,\mathbb{R})$,

$$A_+ = \begin{pmatrix} a_+ & b_+ \\ c_+ & d_+ \end{pmatrix}, \quad A_- = \begin{pmatrix} a_- & b_- \\ c_- & d_- \end{pmatrix},$$

with the action

$$(A_+, A_-)(x_+, x_-) = \left(\frac{a_+x_+ + b_+}{c_+x_+ + d_+}, \frac{a_-x_- + b_-}{c_-x_- + d_-} \right),$$

the covariance condition now reads

$$U(g)\Phi_a(x)U(g)^* = \left(\frac{1}{(c_+x_+ + d_+)^2}\right)^{h_a^+} \left(\frac{1}{(c_-x_- + d_-)^2}\right)^{h_a^-} \Phi_a(gx),$$

where the conformal weights h_a^+, h_a^- are in general independent of each other. Note that the factor

$$\frac{1}{(c_+x_+ + d_+)^2}$$

is the derivative of

$$x_+ \mapsto A_+(x_+) = \frac{a_+x_+ + b_+}{c_+x_+ + d_+},$$

and therefore essentially the conformal factor.

The boost described above is given by $g = (A_+, A_-)$ with $a_+ = \exp\frac{1}{2}\chi = d_-, d_+ = \exp -\frac{1}{2}\chi = a_-$, the bs and cs being zero. By comparison we obtain

$$s_a = h_a^+ - h_a^-,$$

for the spins s_a and, similarly, for the weights d_a related to the dilatations:

$$d_a = h_a^+ + h_a^-.$$

In the Euclidean case one writes the general point in the Euclidean plane as $z = x + iy$ or $t + iy$ and $\overline{z} = x - iy$. The conformal covariance for the rotation $w(z) = e^{i\alpha}z$ will correspondingly be formulated by

$$U(\Lambda)\Phi_a(z)U(\Lambda)^* = \left(\frac{dw}{dz}\right)^{h_a} \left(\overline{\frac{dw}{dz}}\right)^{\overline{h}_a} \Phi_a(w),$$

where again $h_a, \overline{h}_a$ are independent. Equivalently, one writes

$$U(\Lambda)\Phi_a(z,\overline{z})U(\Lambda)^* = \left(\frac{dw}{dz}\right)^{h_a} \left(\overline{\frac{dw}{dz}}\right)^{\overline{h}_a} \Phi_a(w,\overline{w}),$$

emphasizing the two components of z resp. w (cf. the Axiom 2 in the following chapter). This is the formulation of covariance for other conformal transformations as well.

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