Chapter 3
Central Extensions of Groups

The notion of a central extension of a group or of a Lie algebra is of particular importance in the quantization of symmetries. We give a detailed introduction to the subject with many examples, first for groups in this chapter and then for Lie algebras in the next chapter.

3.1 Central Extensions

In this section let A be an abelian group and let G be an arbitrary group. The trivial group consisting only of the neutral element is denoted by 1.

Definition 3.1. An *extension* of G *by the group* A is given by an *exact sequence* of group homomorphisms

$$1 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1.$$

Exactness of the sequence means that the kernel of every map in the sequence equals the image of the previous map. Hence the sequence is exact if and only if ι is injective, π is surjective, the image im ι is a normal subgroup, and

$$\ker \pi = \mathrm{im}\ \iota (\cong A).$$

The extension is called *central* if A is abelian and its image im ι is in the center of E, that is

$$a \in A, b \in E \Rightarrow \iota(a)b = b\iota(a).$$

Note that A is written multiplicatively and 1 is the neutral element although A is supposed to be abelian.

Examples:

- A *trivial* extension has the form

$$1 \longrightarrow A \xrightarrow{i} A \times G \xrightarrow{pr_2} G \longrightarrow 1,$$

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where $A \times G$ denotes the product group and where $i : A \to G$ is given by $a \mapsto (a,1)$. This extension is central.

- An example for a nontrivial central extension is the exact sequence

$$1 \longrightarrow \mathbb{Z}/k\mathbb{Z} \longrightarrow E = \mathrm{U}(1) \xrightarrow{\pi} \mathrm{U}(1) \longrightarrow 1$$

with $\pi(z) := z^k$ for $k \in \mathbb{N}$, $k \geq 2$. This extension cannot be trivial, since $E = \mathrm{U}(1)$ and $\mathbb{Z}/k\mathbb{Z} \times \mathrm{U}(1)$ are not isomorphic. Another argument for this uses the fact – known for example from function theory – that a homomorphism $\tau : \mathrm{U}(1) \to E$ with $\pi \circ \tau = id_{\mathrm{U}(1)}$ does not exist, since there is no global kth root.

- A special class of group extensions is given by semidirect products. For a group G acting on another group H by a homomorphism $\tau : G \to \mathrm{Aut}(H)$ the *semidirect product* group $G \ltimes H$ is the set $H \times G$ with the multiplication given by the formula

$$(x,g).(x',g') := (x\tau(g)(x'), gg')$$

for $(g,x),(g',x') \in G \times H$. With $\pi(g,x) = x$ and $\iota(x) = (a,x)$, one obtains the group extension

$$1 \longrightarrow H \xrightarrow{\iota} G \ltimes H \xrightarrow{\pi} G \longrightarrow 1.$$

For example, for a vector space V the general linear group $\mathrm{GL}(V)$ of invertible linear mappings acts naturally on the additive group V, $\tau(g)(x) = g(x)$, and the resulting semidirect group $\mathrm{GL}(V) \ltimes V$ is (isomorphic to) the group of affine transformations.

With the same action $\tau : \mathrm{GL}(V) \to \mathrm{Aut}(V)$ the group of motions of $\mathbb{R}^{p,q}, n = p+q > 2$, as a semi-Riemannian space can be described as a semidirect product $\mathrm{O}(p,q) \ltimes \mathbb{R}^n$ (see the example in Sect. 1.4). As a particular case, we obtain the Poincaré group as the semidirect group $\mathrm{SO}(1,3) \ltimes \mathbb{R}^4$ (cf. Sect. 8.1).

Observe that these examples of group extensions are not central, although the additive group V (resp. $\mathbb{R}^n$) of translations is abelian.

- The universal covering group of the Lorentz group $\mathrm{SO}(1,3)$ (that is the identity component of the group $\mathrm{O}(1,3)$ of all metric-preserving linear maps $\mathbb{R}^{1,3} \to \mathbb{R}^{1,3}$) is (isomorphic to) a central extension of $\mathrm{SO}(1,3)$ by the group $\{+1,-1\}$. In fact, there is the exact sequence of Lie groups

$$1 \longrightarrow \{+1,-1\} \longrightarrow \mathrm{SL}(2,\mathbb{C}) \xrightarrow{\pi} \mathrm{SO}(1,3) \longrightarrow 1,$$

where π is the 2-to-1 covering.

This is a special case of the general fact that for a given connected Lie group G the universal covering group E of G is an extension of G by the group of deck transformations which in turn is isomorphic to the fundamental group $\pi(G)$ of G.

- Let V be a vector space over a field K. Then

$$1 \longrightarrow K^\times \xrightarrow{i} \mathrm{GL}(V) \xrightarrow{\pi} \mathrm{PGL}(V) \longrightarrow 1$$

with $i : K^\times \to \mathrm{GL}(V), \lambda \mapsto \lambda \mathrm{id}_V$, is a central extension by the (commutative) multiplicative group $K^\times = K \setminus \{0\}$ of units in K. Here, the *projective linear group* $\mathrm{PGL}(V)$ is simply the factor group $\mathrm{PGL}(V) = \mathrm{GL}(V)/K^\times$.

- The main example in the context of quantization of symmetries is the following: Let $\mathbb{H}$ be a Hilbert space and let $\mathbb{P} = \mathbb{P}(\mathbb{H})$ be the projective space of one-dimensional linear subspaces of $\mathbb{H}$, that is

$$\mathbb{P}(\mathbb{H}) := (\mathbb{H} \setminus \{0\})/\sim,$$

with the equivalence relation

$$f \sim g :\Leftrightarrow \exists \lambda \in \mathbb{C}^\times : f = \lambda g \quad \text{for} \quad f, g \in \mathbb{H}.$$

$\mathbb{P}$ is the space of states in quantum physics, that is the quantum mechanical phase space. In Lemma 3.4 it is shown that the group $\mathrm{U}(\mathbb{H})$ of unitary operators on $\mathbb{H}$ is in a natural way a nontrivial central extension of the group $\mathrm{U}(\mathbb{P})$ of (unitary) projective transformations on $\mathbb{P}$ by $\mathrm{U}(1)$

$$1 \longrightarrow \mathrm{U}(1) \xrightarrow{\iota} \mathrm{U}(\mathbb{H}) \xrightarrow{\widehat{\gamma}} \mathrm{U}(\mathbb{P}) \longrightarrow 1.$$

To explain this last example and for later purposes we recall some basic notions concerning Hilbert spaces. A *pre-Hilbert* space $\mathbb{H}$ is a complex vector space with a positive definite hermitian form, called an inner product or scalar product. A *hermitian form* is an $\mathbb{R}$-bilinear map

$$\langle\ ,\ \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{C},$$

which is complex antilinear in the first variable (another convention is to have the form complex linear in the first variable) and satisfies

$$\langle f, g\rangle = \overline{\langle g, f\rangle}$$

for all $f, g \in \mathbb{H}$. A hermitian form is an *inner product* if, in addition,

$$\langle f, f\rangle > 0 \text{ for all } f \in \mathbb{H} \setminus \{0\}.$$

The inner product induces a norm on $\mathbb{H}$ by $\|f\| := \sqrt{\langle f, f\rangle}$ and hence a topology. $\mathbb{H}$ with the inner product is called a *Hilbert space* if $\mathbb{H}$ is complete as a normed space with respect to this norm.

Typical finite-dimensional examples of Hilbert spaces are the $\mathbb{C}^m$ with the standard inner product

$$\langle z, w\rangle := \sum_{j=1}^{m} \bar{z}_j w_j.$$

In quantum theory important Hilbert spaces are the spaces $\mathrm{L}^2(X, \lambda)$ of square-integrable complex functions $f : X \to \mathbb{C}$ on various measure spaces X with a measure λ on X, where the inner product is

$$\langle f,g\rangle := \int_X \overline{f}(x)g(x)d\lambda(x).$$

In the case of $X = \mathbb{R}^n$ with the Lebesgue measure, this space is separable, that is there exists a countable dense subset in $\mathbb{H}$. A separable Hilbert space has a countable (Schauder) basis, that is a sequence (e_n), $e_n \in \mathbb{H}$, which is mutually orthonormal, $\langle e_n, e_m\rangle = \delta_{n,m}$, and such that every $f \in \mathbb{H}$ has a unique representation as a convergent series

$$f = \sum_n \alpha_n e_n$$

with coefficients $\alpha_n \in \mathbb{C}$. These coefficients are $\alpha_n = \langle e_n, f\rangle$.

In quantum theory the Hilbert spaces describing the states of the quantum system are required to be separable. Therefore, in the sequel the Hilbert spaces are assumed to be separable.

A *unitary operator* U on $\mathbb{H}$ is a $\mathbb{C}$-linear bijective map $U : \mathbb{H} \to \mathbb{H}$ leaving the inner product invariant:

$$f,g \in \mathbb{H} \quad \Longrightarrow \quad \langle Uf, Ug\rangle = \langle f,g\rangle.$$

It is easy to see that the inverse $U^{-1} : \mathbb{H} \to \mathbb{H}$ of a unitary operator $U : \mathbb{H} \to \mathbb{H}$ is unitary as well and that the composition $U \circ V$ of two unitary operators U,V is always unitary. Hence, the composition of operators defines the structure of a group on the set of all unitary operators on $\mathbb{H}$. This group is denoted by $\mathrm{U}(\mathbb{H})$ and called the *unitary group* of $\mathbb{H}$.

In the finite-dimensional situation ($m = \dim \mathbb{H}$) the unitary group $\mathrm{U}(\mathbb{H})$ is isomorphic to the matrix group $\mathrm{U}(m)$ of all complex $m \times m$-matrices B with $B^{-1} = B^*$. For example, $\mathrm{U}(1)$ is isomorphic to $\mathbb{S}^1$. The special unitary groups are the

$$\mathrm{SU}(m) = \{B \in \mathrm{U}(m) : \det B = 1\}.$$

$\mathrm{SU}(2)$ is isomorphic to the group of unit quaternions and can be identified with the unit sphere $\mathbb{S}^3$ and thus provides a 2-to-1 covering of the rotation group $\mathrm{SO}(3)$ (which in turn is the three-dimensional real projective space $\mathbb{P}(\mathbb{R}^4)$).

Let $\gamma : \mathbb{H} \setminus \{0\} \to \mathbb{P}$ be the canonical map into the quotient space $\mathbb{P}(\mathbb{H}) = (\mathbb{H} \setminus \{0\})/\sim$ with respect to the equivalence relation which identifies all points on a complex line through 0 (see above). Let $\varphi = \gamma(f)$ and $\psi = \gamma(g)$ be points in the projective space $\mathbb{P}$ with $f,g \in \mathbb{H}$. We then define the "*transition probability*" as

$$\delta(\varphi,\psi) := \frac{|\langle f,g\rangle|^2}{\|f\|^2\|g\|^2}.$$

δ is not quite the same as a metric but it defines in the same way as a metric a topology on $\mathbb{P}$ which is the natural topology on $\mathbb{P}$. This topology is generated by the open subsets $\{\varphi \in \mathbb{P} : \delta(\varphi,\psi) < r\}$, $r \in \mathbb{R}$, $\psi \in \mathbb{P}$. It is also the quotient topology on $\mathbb{P}$ with respect to the quotient map γ, that is a subset $W \subset \mathbb{P}$ is open if and only if $\gamma^{-1}(W) \subset \mathbb{H}$ is open in the Hilbert space topology.

Definition 3.2. A bijective map $T : \mathbb{P} \to \mathbb{P}$ with the property

$$\delta(T\varphi, T\psi) = \delta(\varphi, \psi) \ \text{ for } \ \varphi, \psi \in \mathbb{P},$$

is called a *projective transformation* or *projective automorphism.*

Furthermore, we define the group $\mathrm{Aut}(\mathbb{P})$ of projective transformations to be the set of all projective transformations where the group structure is again given by composition. Hence, $\mathrm{Aut}(\mathbb{P})$ is the group of bijections of $\mathbb{P}$, the quantum mechanical phase space, preserving the transition probability. This means that $\mathrm{Aut}(\mathbb{P})$ is the full symmetry group of the quantum mechanical state space.

For every $U \in \mathrm{U}(\mathbb{H})$ we define a map $\widehat{\gamma}(U) : \mathbb{P} \to \mathbb{P}$ by

$$\widehat{\gamma}(U)(\varphi) := \gamma(U(f))$$

for all $\varphi = \gamma(f) \in \mathbb{P}$ with $f \in \mathbb{H}$. It is easy to show that $\widehat{\gamma}(U) : \mathbb{P} \to \mathbb{P}$ is well defined and belongs to $\mathrm{Aut}(\mathbb{P})$. This is true not only for unitary operators, but also for the so-called anti-unitary operators V, that is for the $\mathbb{R}$-linear bijective maps $V : \mathbb{H} \to \mathbb{H}$ with

$$\langle Vf, Vg\rangle = \overline{\langle f, g\rangle}, V(if) = -iV(f)$$

for all $f, g \in \mathbb{H}$.

Note that $\widehat{\gamma} : \mathrm{U}(\mathbb{H}) \to \mathrm{Aut}(\mathbb{P})$ is a homomorphism of groups.

The following theorem is a complete characterization of the projective automorphisms:

Theorem 3.3. (Wigner [Wig31], Chap. 20, Appendix) *For every projective transformation $T \in \mathrm{Aut}(\mathbb{P})$ there exists a unitary or an anti-unitary operator U with $T = \widehat{\gamma}(U)$.*

The elementary proof of Wigner has been simplified by Bargmann [Bar64].

Let

$$\mathrm{U}(\mathbb{P}) := \widehat{\gamma}(\mathrm{U}(\mathbb{H})) \subset \mathrm{Aut}(\mathbb{P}).$$

Then $\mathrm{U}(\mathbb{P})$ is a subgroup of $\mathrm{Aut}(\mathbb{P})$, called the group of unitary projective transformations. The following result is easy to show:

Lemma 3.4. *The sequence*

$$1 \longrightarrow \mathrm{U}(1) \xrightarrow{\iota} \mathrm{U}(\mathbb{H}) \xrightarrow{\widehat{\gamma}} \mathrm{U}(\mathbb{P}) \longrightarrow 1$$

with $\iota(\lambda) := \lambda\, \mathrm{id}_{\mathbb{H}}$, $\lambda \in \mathrm{U}(1)$, is an exact sequence of homomorphism and hence defines a central extension of $\mathrm{U}(\mathbb{P})$ *by* $\mathrm{U}(1)$.

Proof. In order to prove this statement one only has to check that $\ker \widehat{\gamma} = \mathrm{U}(1)\mathrm{id}_{\mathbb{H}}$.

Let $U \in \ker \widehat{\gamma}$, that is $\widehat{\gamma}(U) = \mathrm{id}_{\mathbb{P}}$. Then for all $f \in \mathbb{H}$, $\varphi := \gamma(f)$,

$$\widehat{\gamma}(U)(\varphi) = \varphi = \gamma(f) \quad \text{and} \quad \widehat{\gamma}(U)(\varphi) = \gamma(Uf),$$

hence $\gamma(Uf) = \gamma(f)$. Consequently, there exists $\lambda \in \mathbb{C}$ with $\lambda f = Uf$. Since U is unitary, it follows that $\lambda \in \mathrm{U}(1)$. By linearity of U, λ is independent of f, that is U has the form $U = \lambda \mathrm{id}_{\mathbb{H}}$. Therefore, $U \in \mathrm{U}(1)\mathrm{id}_{\mathbb{H}}$.

Conversely, let $\lambda \in \mathrm{U}(1)$. Then for all $f \in \mathbb{H}$, $\varphi := \gamma(f)$, we have

$$\widehat{\gamma}(\lambda \mathrm{id}_{\mathbb{H}})(\varphi) = \gamma(\lambda f) = \gamma(f) = \varphi,$$

that is $\widehat{\gamma}(\lambda \mathrm{id}_{\mathbb{H}}) = \mathrm{id}_{\mathbb{P}}$ and hence, $\lambda \mathrm{id}_{\mathbb{H}} \in \ker \widehat{\gamma}$. □

Note that this basic central extension is nontrivial, cf. Example 3.21.

The significance of Wigner's Theorem in quantum theory is the following: The states of a quantum system are represented by points in $\mathbb{P} = \mathbb{P}(\mathbb{H})$ for a suitable separable Hilbert space. A *symmetry* of such a quantum system or an *invariance principle* is a bijective transformation leaving invariant the transition probability δ, hence it is an element of the automorphism group $\mathrm{Aut}(\mathbb{P})$, that is a projective transformation. Now Wigner's Theorem 3.3 asserts that such a symmetry is always induced by either a unitary or an anti-unitary operator on the Hilbert space $\mathbb{H}$. In physical terms, "Every symmetry transformation between coherent states is implementable by a one-to-one complex-linear or antilinear isometry of $\mathbb{H}$."

In the next section we consider the same question not for a single symmetry given by only one transformation but for a group of symmetries. Note that this means that the notion of symmetry is extended from a single invariance principle to a group of symmetry operations.

3.2 Quantization of Symmetries

Examples for classical systems with a symmetry group G are

- $G = \mathrm{SO}(3)$ for systems with rotational symmetry;
- $G =$ Galilei group, for free particles in classical nonrelativistic mechanics;
- $G =$ Poincaré group $\mathrm{SO}(1,3) \ltimes \mathbb{R}^4$, for free particles in the special theory of relativity;
- $G = \mathrm{Diff}_+(\mathbb{S}) \times \mathrm{Diff}_+(\mathbb{S})$ in string theory and in conformal field theory on $\mathbb{R}^{1,1}$;
- $G =$ gauge group $= \mathrm{Aut}(P)$, where P is a principal fiber bundle, for gauge theories;
- $G =$ unitary group $\mathrm{U}(\mathbb{H})$ as a symmetry of the Hilbert space $\mathbb{H}$ (resp. $\mathrm{U}(\mathbb{P})$ as a symmetry of $\mathrm{P} = \mathrm{P}\,(\mathbb{H})$) when $\mathbb{H}$ (resp. $\mathbb{P}$) is considered as a classical phase space, for instance in the context of quantum electrodynamics (see below p. 51).

In these examples and in other classical situations the symmetry in question is manifested by a group homomorphism

$$\tau : G \to \mathrm{Aut}(Y)$$

with respect to the classical phase space Y (often represented by a manifold Y equipped with a symplectic form) and a suitable group $\mathrm{Aut}(Y)$ of transformations leaving invariant the physics of the classical system. (In case of a manifold with a symplectic form at least the symplectic form is left invariant so that the automorphisms have to be canonical transformations.) In addition, in most cases τ is supposed to be continuous for natural topologies on G and $\mathrm{Aut}(Y)$. The symmetry can also be described by the corresponding (continuous) action of the symmetry group G on Y:

$$G \times Y \to Y, (g,y) \mapsto \tau(g)(y).$$

Example: Rotationally invariant classical system with phase space $Y = \mathbb{R}^3 \times \mathbb{R}^3$ and action $\mathrm{SO}(3) \times Y \to Y, (g,(q,p)) \mapsto (g^{-1}q, g^{-1}p)$.

In general, such a group homomorphism is called a *representation* of G in Y. In case of a vector space Y and $\mathrm{Aut}(Y) = \mathrm{GL}(Y)$, the group of invertible linear maps $Y \to Y$ the representation space Y sometimes is called a *G-module*. Whether or not the representation is assumed to be continuous or more (e.g., differentiable) depends on the context.

Note, however, that the symmetry groups in the above six examples are topological groups in a natural way.

Definition 3.5. A *topological group* is a group G equipped with a topology, such that the group operation $G \times G \to G$, $(g,h) \mapsto gh$, and the inversion map $G \to G$, $g \mapsto g^{-1}$, are continuous.

The above examples of symmetry groups are even Lie groups, that is they are manifolds and the composition and inversion are differentiable maps. The first three examples are finite-dimensional Lie groups, while the last three examples are, in general, infinite dimensional Lie groups (modeled on Fréchet spaces). (The topology of $\mathrm{Diff}_+(\mathbb{S})$ will be discussed briefly at the beginning of Chap. 5, and the unitary group $\mathrm{U}(\mathbb{H})$ has a Lie group structure given by the operator norm (cf. p. 46), but it also carries another important topology, the strong topology which will be investigated below after Definition 3.6.)

Now, the quantization of a classical system Y means to find a Hilbert space $\mathbb{H}$ on which the classical observables (that is functions on Y) in which one is interested now act as (mostly self-adjoint) operators on $\mathbb{H}$ in such a way that the commutators of these operators correspond to the Poisson bracket of the classical variables, see Sect. 7.2 for further details on canonical quantization.

After quantization of a classical system with the classical symmetry $\tau : G \to \mathrm{Aut}(Y)$ a homomorphism

$$T : G \to \mathrm{U}(\mathbb{P})$$

will be induced, which in most cases is continuous for the strong topology on $\mathrm{U}(\mathbb{P})$ (see below for the definition of the strong topology).

This property cannot be proven – it is, in fact, an **assumption** concerning the quantization procedure. The reasons for making this assumption are the following. It seems to be evident from the physical point of view that each classical symmetry $g \in G$ acting on the classical phase space should induce after quantization a transformation of the quantum phase space $\mathbb{P}$. This requirement implies the existence of a map

$$T(g) : \mathbb{P} \to \mathbb{P}$$

for each $g \in G$. Again by physical arguments, $T(g)$ should preserve the transition probability, since δ is – at least in the case of classical mechanics – the quantum analogue of the symplectic form which is preserved by g. Hence, by these considerations, one obtains a map

$$T : G \to \mathrm{Aut}(\mathbb{P}).$$

In addition to these requirements it is simply reasonable and convenient to assume that T has to respect the natural additional structures on G and $\mathrm{Aut}(\mathbb{P})$, that is that T has to be a homomorphism since τ is a homomorphism, and that it is a continuous homomorphism when τ is continuous.

This (continuous) homomorphism $T : G \to \mathrm{U}(\mathbb{P})$ is sometimes called the *quantization of the symmetry* τ. See, however, Theorem 3.10 and Corollary 3.12 which yield a (continuous) homomorphism $S : E \to \mathrm{U}(\mathbb{H})$ of a central extension of G which is also called the quantization of the classical symmetry τ.

Definition 3.6. *Strong (operator) topology on* $\mathrm{U}(\mathbb{H})$: Typical open neighborhoods of $U_0 \in \mathrm{U}(\mathbb{H})$ are the sets

$$\mathscr{V}_f(U_0, r) := \{U \in \mathrm{U}(\mathbb{H}) : \|U_0(f) - U(f)\| < r\}$$

with $f \in \mathbb{H}$ and $r > 0$. These neighborhoods form a subbasis of the strong topology: A subset $\mathscr{W} \subset \mathrm{U}(\mathbb{H})$ is by definition open if for each $U_0 \in \mathscr{W}$ there exist finitely many such $\mathscr{V}_{f_j}(U_0, r_j), j = 1, \ldots, k$, so that the intersection is contained in $\mathscr{W}$, that is

$$U_0 \subset \bigcap_{j=1}^{k} \mathscr{V}_{f_j}(U_0, r_j) \subset \mathscr{W}.$$

On $\mathrm{U}(\mathbb{P}) = \widehat{\gamma}(\mathrm{U}(\mathbb{H}))$ a topology (the quotient topology) is defined using the map $\widehat{\gamma} : \mathrm{U}(\mathbb{H}) \to \mathrm{U}(\mathbb{P})$:

$$\mathscr{V} \subset \mathrm{U}(\mathbb{P})\text{open} \quad :\Longleftrightarrow \quad \widehat{\gamma}^{-1}(\mathscr{V}) \subset \mathrm{U}(\mathbb{H})\text{open}.$$

We see that the strong topology is the topology of pointwise convergence in both cases. The strong topology can be defined on any subset

$$M \subset \mathscr{B}_{\mathbb{R}}(\mathbb{H}) := \{A : \mathbb{H} \to \mathbb{H} | A \text{ is } \mathbb{R}\text{-linear and bounded}\}$$

of the space of $\mathbb{R}$-linear continuous endomorphisms, hence in particular on

$$M_u = \{U : \mathbb{H} \to \mathbb{H} | U \text{unitary or anti-unitary}\}.$$

Note that a linear map $A : \mathbb{H} \to \mathbb{H}$ is continuous if and only if it is *bounded*, that is if its *operator norm*

$$\|A\| := \sup\{\|Af\| : f \in \mathscr{B}_{\mathbb{R}}, \|f\| \leq 1\}$$

is finite. And with the operator norm the space $\mathscr{B}_{\mathbb{R}}(\mathbb{H})$ is a Banach space, that is a complete normed space. Evidently, a unitary or anti-unitary operator is bounded with operator norm equal to 1.

In the same way as above the strong topology on $\mathrm{Aut}(\mathbb{P})$ is defined using δ replacing the norm.

Observe that the strong topology on $\mathrm{U}(\mathbb{H})$ and $\mathrm{U}(\mathbb{P})$ as well as on M_u and $\mathrm{Aut}(\mathbb{P})$ is the topology of pointwise convergence. So, in contrast to its name, the strong topology is rather a weak topology.

Since all these sets of mappings are uniformly bounded they are equicontinuous by the theorem of Banach–Steinhaus and hence the strong topology also agrees with the compact open topology, that is the topology of uniform convergence on the compact subsets of $\mathbb{H}$ (resp. of $\mathbb{P}$). We also conclude that in the case of a separable Hilbert space (which we always assume), the strong topology on $\mathrm{U}(\mathbb{H})$ as well as on $\mathrm{U}(\mathbb{P})$ is metrizable.

On subsets M of $\mathscr{B}_{\mathbb{R}}(\mathbb{H})$ we also have the natural norm topology induced by the operator norm. This topology is much stronger than the strong topology in the infinite dimensional case, since it is the topology of uniform convergence on the unit ball of $\mathbb{H}$.

Definition 3.7. For a topological group G a *unitary representation* R of G in the Hilbert space $\mathbb{H}$ is a continuous homomorphism

$$R : G \to \mathrm{U}(\mathbb{H})$$

with respect to the strong topology on $\mathrm{U}(\mathbb{H})$. A *projective representation* R of G is, in general, a continuous homomorphism

$$R : G \to \mathrm{U}(\mathbb{P})$$

with respect to the strong topology on $\mathrm{U}(\mathbb{P})$ ($\mathbb{P} = \mathbb{P}(\mathbb{H})$).

Note that $\mathrm{U}(\mathbb{H})$ and $\mathrm{U}(\mathbb{P})$ are topological groups with respect to the strong topology (cf. 3.11). Moreover, both these groups are connected and metrizable (see below).

The reason that in the context of representation theory one prefers the strong topology over the norm topology is that only few homomorphisms $G \to \mathrm{U}(\mathbb{H})$ turn out to be continuous with respect to the norm topology. In particular, for a compact Lie group G and its Hilbert space $\mathbb{H} = \mathrm{L}^2(G)$ of square-integrable measurable functions with respect to Haar measure the regular representation

$$R : G \to \mathrm{U}(\mathrm{L}^2(G)), g \mapsto (R_g : f(x) \mapsto f(xg)),$$

is not continuous in the norm topology, in general. But R is continuous in the strong topology, since all the maps $g \mapsto R_g(f)$ are continuous for fixed $f \in \mathrm{L}^2(G)$. This last property is equivalent to the action

$$G \times \mathrm{L}^2(G) \to \mathrm{L}^2(G), (g, f) \mapsto R_g(f),$$

of G on $\mathrm{L}^2(G)$ being continuous.

Another reason to use the strong topology is the fact that various related actions, e.g., the natural action of $U(\mathbb{H})$ on the space of Fredholm operators on $\mathbb{H}$ or on the Hilbert space of Hilbert–Schmidt operators, are continuous in the strong topology. Hence, the strong topology is weak enough to allow many important representations to be continuous and strong enough to ensure that natural actions of $U(\mathbb{H})$ are continuous.

Lifting Projective Representations. When quantizing a classical symmetry group G the following question arises naturally: Given a projective representation T, that is a continuous homomorphism $T : G \to U(\mathbb{P})$ with $\mathbb{P} = \mathbb{P}(\mathbb{H})$, does there exist a unitary representation $S : G \to U(\mathbb{H})$, such that the following diagram commutes?

$$\begin{tikzcd}
 & & & G \arrow[dl, "S"'] \arrow[d, "T"] & \\
1 \arrow[r] & U(1) \arrow[r] & U(\mathbb{H}) \arrow[r, "\widehat{\gamma}"'] & U(\mathbb{P}) \arrow[r] & 1
\end{tikzcd}$$

In other words, can a projective representation T always be induced by a proper unitary representation S on $\mathbb{H}$ so that $T = \widehat{\gamma} \circ S$?

The answer is no; such a lifting does not exist in general. Therefore, it is, in general, not possible to take G as the quantum symmetry group in the sense of a unitary representation $S : G \to U(\mathbb{H})$ in the Hilbert space $\mathbb{H}$. However, a lifting exists with respect to the central extension of the universal covering group of the classical symmetry group. (Here and in the following, the *universal covering group* of a connected Lie group G is the (up to isomorphism) uniquely determined connected and simply connected universal covering $\widetilde{G}$ of G with its Lie group structure.) This is well known for the rotation group $SO(3)$ where the transition from $SO(3)$ to the simply connected 2-to-1 covering group $SU(2)$ can be described in the following way:

Example 3.8. To every projective representation $T' : SO(3) \to U(\mathbb{P})$ there corresponds a unitary representation $S : SU(2) \to U(\mathbb{H})$ such that $\widehat{\gamma} \circ S = T' \circ P =: T$. The following diagram is commutative:

$$\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r] & U(1) \times SU(2) \arrow[r, "pr_2"] \arrow[d] & SU(2) \arrow[r] \arrow[dl, "S"'] \arrow[d, "T"] \arrow[dr] & 1 \\
 & & & & SO(3) \arrow[dl] \\
1 \arrow[r] & U(1) \arrow[r] & U(\mathbb{H}) \arrow[r, "\widehat{\gamma}"'] & U(\mathbb{P}) \arrow[r] & 1
\end{tikzcd}$$

S is unique up to a scalar multiple of norm 1.

$SU(2)$ is the universal covering group of $SO(3)$ with covering map (and group homomorphism) $P : SU(2) \to SO(3)$. From a general point of view the lifting $S : SU(2) \to U(\mathbb{H})$ of $T := T' \circ P$ (that is $T = \widehat{\gamma} \circ S$) in the diagram is obtained via the lifting of T to a central extension of $SU(2)$ which always exists according to the subsequent Theorem 3.10. Since each central extension of $SU(2)$ is trivial (cf. Remark 4.10), this lifting factorizes and yields the lifting T (cf. Bargmann's Theorem 4.8).

Remark 3.9. In a similar matter one can lift every projective representation T' : $\mathrm{SO}(1,3) \to \mathrm{U}(\mathbb{P})$ of the Lorentz group $\mathrm{SO}(1,3)$ to a proper unitary representation $S : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{U}(\mathbb{H})$ in $\mathbb{H}$ of the group $\mathrm{SL}(2,\mathbb{C})$: $T' \circ P = \widehat{\gamma} \circ S$.

Here, $P : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(1,3)$ is the 2-to-1 covering map and homomorphism.

Because of these facts – the lifting up to the covering maps – the group $\mathrm{SL}(2,\mathbb{C})$ is sometimes called the quantum Lorentz group and, correspondingly, $\mathrm{SU}(2)$ is called the quantum mechanical rotation group.

Theorem 3.10. *Let G be a group and $T : G \to \mathrm{U}(\mathbb{P})$ be a homomorphism. Then there is a central extension E of G by $\mathrm{U}(1)$ and a homomorphism $S : E \to \mathrm{U}(\mathbb{H})$, so that the following diagram commutes:*

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{U}(1) \arrow[r] \arrow[d, "\mathrm{id}"'] & E \arrow[r, "\pi"] \arrow[d, "S"'] & G \arrow[r] \arrow[d, "T"] & 1 \\
1 \arrow[r] & \mathrm{U}(1) \arrow[r] & \mathrm{U}(\mathbb{H}) \arrow[r, "\widehat{\gamma}"'] & \mathrm{U}(\mathbb{P}) \arrow[r] & 1
\end{tikzcd}$$

Proof. We define

$$E := \{(U,g) \in \mathrm{U}(\mathbb{H}) \times G \mid \widehat{\gamma}(U) = Tg\}.$$

E is a subgroup of the product group $\mathrm{U}(\mathbb{H}) \times G$, because $\widehat{\gamma}$ and T are homomorphisms. Obviously, the inclusion

$$\iota : \mathrm{U}(1) \to E, \lambda \mapsto (\lambda \mathrm{id}_{\mathbb{H}}, 1_G)$$

and the projection $\pi := \mathrm{pr}_2 : E \to G$ are homomorphisms such that the upper row is a central extension. Moreover, the projection $S := \mathrm{pr}_1 : E \to \mathrm{U}(\mathbb{H})$ onto the first component is a homomorphism satisfying $T \circ \pi = \widehat{\gamma} \circ S$. □

Proposition 3.11. $\mathrm{U}(\mathbb{H})$ *is a topological group with respect to the strong topology.*

This property simplifies the proof of Bargmann's Theorem (4.8) significantly. The proposition is in sharp contrast to claims in the corresponding literature on quantization of symmetries (e.g., [Sim68]) and in other publications. Since even in the latest publications it is repeated that $\mathrm{U}(\mathbb{H})$ is not a topological group, we provide the simple proof (cf. [Scho95, p. 174]):

Proof. In order to show the continuity of the group operation $(U,U') \mapsto UU' = U \circ U'$ it suffices to show that to any pair $(U,U') \in \mathrm{U}(\mathbb{H}) \times \mathrm{U}(\mathbb{H})$ and to arbitrary $f \in \mathbb{H}, r > 0$, there exist open subsets $\mathscr{V}, \mathscr{V}'$ of $\mathrm{U}(\mathbb{H})$ satisfying

$$\{VV' | V \in \mathscr{V}, V' \in \mathscr{V}'\} \subset \mathscr{V}_f(UU', r).$$

Because of

$$\begin{aligned}\|UU'(f) - VV'(f)\| \\ &= \|UU'(f) - VU'(f) + VU'(f) - VV'(f)\| \\ &\leq \|UU'(f) - VU'(f)\| + \|VU'(f) - VV'(f)\| \\ &= \|UU'(f) - VU'(f)\| + \|U'(f) - V'(f)\| \\ &= \|U(g) - V(g)\| + \|U'(f) - V'(f)\|,\end{aligned}$$

where $g = U'(f)$, the condition is satisfied for $\mathscr{V} = \mathscr{V}_g(U, \frac{1}{2}r)$ and $\mathscr{V}' = \mathscr{V}_f(U', \frac{1}{2}r)$. To show the continuity of $U \mapsto U^{-1}$ let $g = U^{-1}(f)$ hence $f = U(g)$. Then

$$\|U^{-1}(f) - V^{-1}(f)\| = \|g - V^{-1}U(g)\| = \|V(g) - U(g)\|,$$

and the condition $\|V(g) - U(g)\| < r$ directly implies

$$\|U^{-1}(f) - V^{-1}(f)\| < r.$$

□

Note that the topological group $\mathrm{U}(\mathbb{H})$ is metrizable and complete in the strong topology and the same is true for $\mathrm{U}(\mathbb{P})$.

Because of Proposition 3.11, it makes sense to carry out the respective investigations in the topological setting from the beginning, that is for topological groups and continuous homomorphisms. Among others we have the following properties:

1. $\mathrm{U}(\mathbb{H})$ is connected, since $\mathrm{U}(\mathbb{H})$ is pathwise connected with respect to the norm topology. Every unitary operator is in the orbit of a suitable one-parameter group $\exp(iAt)$.
2. $\mathrm{U}(\mathbb{P})$ and $\mathrm{Aut}(\mathbb{P})$ are also topological groups with respect to the strong topology.
3. $\widehat{\gamma} : \mathrm{U}(\mathbb{H}) \to \mathrm{U}(\mathbb{P})$ is a continuous homomorphism (with local continuous sections, cf. Lemma 4.9).
4. $\mathrm{U}(\mathbb{P})$ is a connected metrizable group. $\mathrm{U}(\mathbb{P})$ is the connected component containing the identity in $\mathrm{Aut}(\mathbb{P})$.
5. Every continuous homomorphism $T : G \to \mathrm{Aut}(\mathbb{P})$ on a connected topological group G has its image in $\mathrm{U}(\mathbb{P})$, that is it is already a continuous homomorphism $T : G \to \mathrm{U}(\mathbb{P})$. This is the reason why – in the context of quantization of symmetries for connected groups G – it is in most cases enough to study continuous homomorphism $T : G \to \mathrm{U}(\mathbb{P})$ into $\mathrm{U}(\mathbb{P})$ instead of $T : G \to \mathrm{Aut}(\mathbb{P})$

Corollary 3.12. *If, in the situation of Theorem 3.10, G is a topological group and $T : G \to \mathrm{U}(\mathbb{P})$ is a projective representation of G, that is T is a continuous homomorphism, then the central extension E of G by $\mathrm{U}(1)$ has a natural structure of a topological group such that the inclusion $\iota : \mathrm{U}(1) \to E$, the projection $\pi : E \to G$ and the lift $S : E \to \mathrm{U}(\mathbb{H})$ are continuous. In particular, S is a unitary representation in $\mathbb{H}$.*

To show this statement one only has to observe that the product group $G \times \mathrm{U}(\mathbb{H})$ is a topological group with respect to the product topology and thus E is a topological group with respect to the induced topology.

Remark 3.13. In view of these results a quantization of a classical symmetry group G can in general be regarded as a central extension E of the universal covering group of G by the group $\mathrm{U}(1)$ of phases.

Quantum Electrodynamics. We conclude this section with an interesting example of a central extension of groups which occurs naturally in the context of second quantization in quantum electrodynamics. The first quantization leads to a separable Hilbert space $\mathbb{H}$ of infinite dimension, sometimes called the one-particle space, which decomposes according to the positive and negative energy states: We have two closed subspaces $\mathbb{H}_+, \mathbb{H}_- \subset \mathbb{H}$ such that $\mathbb{H}$ is the orthogonal sum of $\mathbb{H}_\pm$, that is $\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_-$. For example, $\mathbb{H}_\pm$ is given by the positive resp. negative or zero eigenspaces of the Dirac hamiltonian on $\mathbb{H} = \mathrm{L}^2(\mathbb{R}^3, \mathbb{C}^4)$.

An orthogonal decomposition $\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_-$ with infinite dimensional components $\mathbb{H}_\pm$ is called a *polarization*.

Now, the Hilbert space $\mathbb{H}$ (or its projective space $\mathbb{P} = \mathbb{P}(\mathbb{H})$) can be viewed as a classical phase space with the imaginary part of the scalar product as the symplectic form and with the unitary group $\mathrm{U}(\mathbb{H})$ (or $\mathrm{U}(\mathbb{P})$) as symmetry group. In this context the observables one is interested in are the elements of the CAR algebra $\mathscr{A}(\mathbb{H})$ of $\mathbb{H}$. Second quantization is the quantization of these observables.

The CAR (Canonical Anticommutation Relation) algebra $\mathscr{A}(\mathbb{H}) = \mathscr{A}$ of a Hilbert space $\mathbb{H}$ is the universal unital C^*-algebra generated by the *annihilation operators* $a(f)$ and the *creation operators* $a^*(f), f \in \mathbb{H}$, with the following commutation relations:

$$a(f)a^*(g) + a^*(g)a(f) = \langle f, g\rangle \mathbf{1},$$

$$a^*(f)a^*(g) + a^*(g)a^*(f) = 0 = a(f)a(g) + a(g)a(f).$$

Here, $a^* : \mathbb{H} \to \mathscr{A}$ is a complex-linear map and $a : \mathbb{H} \to \mathscr{A}$ is complex antilinear (other conventions are often used in the literature). The CAR algebra $\mathscr{A}(\mathbb{H})$ can be described as a Clifford algebra using the tensor algebra of $\mathbb{H}$.

Recall that a *Banach algebra* is an associative algebra B over $\mathbb{C}$ which is a complex Banach space such that the multiplication satisfies $\|ab\| \le \|a\|\|b\|$ for all $a, b \in B$. A *unital* Banach algebra B is a Banach algebra with a unit of norm 1. Finally, a C^*-algebra is a Banach algebra B with an antilinear involution $* : B \to B, b \mapsto b^*$ satisfying $(ab)^* = b^*a^*$ and $\|aa^*\| = \|a\|^2$ for all $a, b \in B$.

Let us now assume to have a polarization $\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_-$ induced by a (first) quantization (for example the quantization of the Dirac hamiltonian). For a general complex Hilbert space $\mathbb{W}$ the complex conjugate $\overline{\mathbb{W}}$ is $\mathbb{W}$ as an abelian group endowed with the "conjugate" scalar multiplication $(\lambda, w) \mapsto \overline{\lambda} w$ and with the conjugate scalar product.

The second quantization is obtained by representing the CAR algebra $\mathscr{A}$ in the *fermionic Fock space* (which also could be called *spinor space*) $\mathsf{S}(\mathbb{H}_+) = \mathsf{S}$ depending on the polarization. S is the Hilbert space completion of

$$\bigwedge \mathbb{H}_+ \otimes \bigwedge \overline{\mathbb{H}_-},$$

with the induced scalar product on $\bigwedge \mathbb{H}_+ \otimes \bigwedge \overline{\mathbb{H}_-}$, where

$$\bigwedge \mathbb{W} = \bigoplus \bigwedge^p \mathbb{W}$$

is the exterior algebra of the Hilbert space $\mathbb{W}$ equipped with the induced scalar product on $\bigwedge \mathbb{W}$.

In order to define the representation π of $\mathscr{A}$ in S, the actions of $a^*(f), a(f)$ on S are given in the following using $a^*(f) = a^*(f_+) + a^*(f_-), a(f) = a(f_+) + a(f_-)$ with respect to the decomposition $f = f_+ + f_- \in \mathbb{H}_+ \oplus \mathbb{H}_-$.

For $f_1, f_2, \dots f_n \in \mathbb{H}_+, g_1, g_2, \dots, g_m \in \mathbb{H}_-$, and $\xi \in \bigwedge^k \mathbb{H}_+, \eta \in \bigwedge \overline{\mathbb{H}_-}$, one defines

$$\begin{aligned}
\pi(a^*)(f_+)\xi \otimes \eta &:= (f_+ \wedge \xi) \otimes \eta, \\
\pi(a^*)(f_-)(\xi \otimes g_1 \wedge \dots g_m) &:= \sum_{j=1}^{j=n} (-1)^{k+j+1} \xi \otimes \langle g_j, f_- \rangle g_1 \wedge \dots \widehat{g_j} \wedge \dots g_m, \\
\pi(a)(f_+)(f_1 \wedge \dots \wedge f_n \otimes \eta) &:= \sum_{j=1}^{j=n} (-1)^{j+1} \langle f_+, f_j \rangle f_1 \wedge \dots \widehat{f_j} \wedge \dots f_n \otimes \eta, \\
\pi(a)(f_-)(\xi \otimes \eta) &:= (-1)^k \xi \otimes f_- \wedge \eta.
\end{aligned}$$

Lemma 3.14. *This definition yields a representation*

$$\pi : \mathscr{A} \to \mathscr{B}(\mathsf{S})$$

of C^*-algebras satisfying the anticommutation relations.

Here, $\mathscr{B}(\mathbb{H}) \subset \operatorname{End} \mathbb{H}$ is the C^*-algebra of bounded $\mathbb{C}$-linear endomorphisms of $\mathbb{H}$.

The representation induces the field operators $\Phi : \mathbb{H} \to \mathscr{B}(\mathsf{S})$ by $\Phi(f) = \pi(a(f))$ and its adjoint $\Phi^*, \Phi^* = \pi \circ a^*$.

One is interested to know which unitary operators $U \in \mathrm{U}(\mathbb{H})$ can be carried over to unitary operators in S in order to have the dynamics of the first quantization implemented in the Fock space (or spinor space) S, that is in the second quantized theory. To "carry over" means for a unitary $U \in \mathrm{U}(\mathbb{H})$ to find a unitary operator $U^\sim \in \mathrm{U}(\mathsf{S})$ in the Fock space S such that

$$U^\sim \circ \Phi(f) = \Phi(Uf) \circ U^\sim, f \in \mathbb{H},$$

with the same condition for Φ^*. In this situation $U^\sim$ is called an *implementation* of U.

A result of Shale and Stinespring [ST65*] yields the condition under which U is implementable.

Theorem 3.15. *Each unitary operator $U \in \mathrm{U}(\mathbb{H})$ has an implementation $U^{\sim} \in \mathrm{U}(\mathsf{S})$ if and only if in the block matrix representation of U*

$$U = \begin{pmatrix} U_{++} & U_{-+} \\ U_{+-} & U_{--} \end{pmatrix} : \mathbb{H}_+ \oplus \mathbb{H}_- \longrightarrow \mathbb{H}_+ \oplus \mathbb{H}_-$$

the off-diagonal components

$$U_{+-} : \mathbb{H}_+ \to \mathbb{H}_-, U_{-+} : \mathbb{H}_- \to \mathbb{H}_+$$

are Hilbert–Schmidt operators. Moreover, any two implementations $U^{\sim}$, $'U^{\sim}$ of such an operator U are the same up to a phase factor $\lambda \in \mathrm{U}(1)$: $'U^{\sim} = \lambda U^{\sim}$.

Recall that a bounded operator $T : \mathbb{H} \to \mathbb{W}$ between separable Hilbert spaces is *Hilbert–Schmidt* if with respect to a Schauder basis (e_n) of $\mathbb{H}$ the condition $\sum \|Te_n\|^2 < \infty$ holds.

$$\|T\|_{\mathrm{HS}} = \sqrt{\sum \|Te_n\|^2}$$

is the Hilbert–Schmidt norm.

Definition 3.16. The group $\mathrm{U}_{\mathrm{res}} = \mathrm{U}_{\mathrm{res}}(\mathbb{H}_+)$ of all implementable unitary operators on $\mathbb{H}$ is called the *restricted unitary group*.

The set of implemented operators

$$\mathrm{U}^{\sim}_{\mathrm{res}} = \mathrm{U}^{\sim}_{\mathrm{res}}(\mathbb{H}_+) = \{V \in \mathrm{U}(\mathsf{S}) | \exists U : U^{\sim} = V\}$$

is a subgroup of the unitary group $\mathrm{U}(\mathsf{S})$, and the natural “restriction” map

$$\pi : \mathrm{U}^{\sim}_{\mathrm{res}} \to \mathrm{U}_{\mathrm{res}}$$

is a homomorphism with kernel $\{\lambda\, \mathrm{id}_{\mathsf{S}} : \lambda \in \mathrm{U}(1)\} \cong \mathrm{U}(1)$.

As a result, with $\iota(\lambda) := \lambda\, \mathrm{id}_{\mathsf{S}}, \lambda \in \mathrm{U}(1)$, we obtain an exact sequence of groups

$$1 \longrightarrow \mathrm{U}(1) \xrightarrow{\iota} \mathrm{U}^{\sim}_{\mathrm{res}} \xrightarrow{\pi} \mathrm{U}_{\mathrm{res}} \longrightarrow 1, \tag{3.1}$$

and therefore another example of a central extension of groups appearing naturally in the context of quantization. This is the example we intended to present, and we want briefly to report about some properties of this remarkable central extension in the following.

We cannot expect to represent $\mathrm{U}_{\mathrm{res}}$ in the Fock space S, that is to have a homomorphism $\rho : \mathrm{U}_{\mathrm{res}} \to \mathrm{U}(\mathsf{S})$ with $\pi \circ \rho = \mathrm{id}_{\mathrm{U}_{\mathrm{res}}}$, because this would imply that the extension is trivial: such a ρ is a splitting, and the existence of a splitting implies triviality (see below in the next section). One knows, however, that the extension is not trivial (cf. [PS86*] or [Wur01*], for example).

As a compensation we obtain a homomorphism $\rho : \mathrm{U}_{\mathrm{res}} \to \mathrm{U}(\mathbb{P}(\mathsf{S}))$. The existence of ρ follows directly from the properties of the central extension (3.1).

In what sense can we expect $\rho : \mathrm{U}_{\mathrm{res}} \to \mathrm{U}(\mathbb{P}(\mathsf{S}))$ to be continuous? In other words, for which topology on $\mathrm{U}_{\mathrm{res}}$ is ρ a representation? The strong topology on $\mathrm{U}_{\mathrm{res}}$ is not enough. But on $\mathrm{U}_{\mathrm{res}}$ there is the natural topology induced by the norm

$$\|U_{++}\| + \|U_{--}\| + \|U_{+-}\|_{HS} + \|U_{-+}\|_{HS},$$

where $\| \; \|_{HS}$ is the Hilbert–Schmidt norm. With respect to this topology the group $\mathrm{U}_{\mathrm{res}}$ becomes a real Banach Lie group and ρ is continuous.

Moreover, on $\mathrm{U}^{\sim}_{\mathrm{res}}$ one obtains a topology such that this group is a Banach Lie group as well, and the natural projection is a Lie group homomorphism (cf. [PS86*], [Wur01*]). Altogether, the exact sequence (3.1) turns out to be an exact sequence of Lie group homomorphisms and hence a central extension of infinite dimensional Banach Lie groups.

According to Theorem 3.15 the phase of an implemented operator $U^{\sim}$ for $U \in \mathrm{U}_{\mathrm{res}}$ is not determined, and the possible variations are described by our exact sequence (3.1). In the search of a physically relevant phase of the second quantized theory, it is therefore natural to ask whether or not there exists a continuous map

$$s : \mathrm{U}_{\mathrm{res}} \to \mathrm{U}^{\sim}_{\mathrm{res}} \quad \text{with} \quad \pi \circ s = \mathrm{id}_{\mathrm{U}_{\mathrm{res}}}.$$

We know already that there is no such homomorphism since the central extension is not trivial. And it turns out that there also does not exist such a continuous section s.

The arguments which prove this result are rather involved and do not have their place in these notes. Nevertheless, we give some indications.

First of all, we observe that the restriction map

$$\pi : \mathrm{U}^{\sim}_{\mathrm{res}} \to \mathrm{U}_{\mathrm{res}}$$

in the exact sequence (3.1) is a principal fiber bundle with structure group $\mathrm{U}(1)$ (cf. [Diec91*] or [HJJS08*] for general properties of principal fiber bundles). This observation is in close connection with the investigation leading to Bargmann's Theorem, cf. Lemma 4.9. Note that a principal fiber bundle $\pi : P \to X$ is (isomorphic to) the trivial bundle if and only if there exists a global continuous section $s : X \to P$ satisfying $\pi \circ s = \mathrm{id}_X$.

The existence of a continuous section $s : \mathrm{U}_{\mathrm{res}} \to \mathrm{U}^{\sim}_{\mathrm{res}}$ in our situation, that is $\pi \circ s = \mathrm{id}_{\mathrm{U}_{\mathrm{res}}}$, would imply that the principal bundle is a trivial bundle and thus homeomorphic to $\mathrm{U}_{\mathrm{res}} \times \mathrm{U}(1)$. Although we know already that $\mathrm{U}^{\sim}_{\mathrm{res}}$ cannot be isomorphic to the product group $\mathrm{U}_{\mathrm{res}} \times \mathrm{U}(1)$ as a group, it is in principle not excluded that these spaces are homeomorphic, that is isomorphic as topological spaces.

But the principal bundle π cannot be trivial in the topological sense. To see this, one can use some interesting universal properties of another principal fiber bundle

$$\tau : \mathscr{E} \longrightarrow \mathrm{GL}^0_{\mathrm{res}}(\mathbb{H}_+),$$

which is in close connection to $\pi : \mathrm{U}^{\sim}_{\mathrm{res}} \to \mathrm{U}_{\mathrm{res}}$.

Here $\mathrm{GL}_{\mathrm{res}}(\mathbb{H}_+)$ is the group of all bounded invertible operators $\mathbb{H} \to \mathbb{H}$ whose off-diagonal components are Hilbert–Schmidt operators, so that $\mathrm{U}_{\mathrm{res}} = \mathrm{U}(\mathbb{H}) \cap \mathrm{GL}_{\mathrm{res}}(\mathbb{H}_+)$. $\mathrm{GL}_{\mathrm{res}}(\mathbb{H}_+)$ will be equipped with the topology analogous to the topology on $\mathrm{U}_{\mathrm{res}}$ respecting the Hilbert–Schmidt norms, and $\mathrm{GL}^0_{\mathrm{res}}(\mathbb{H}_+)$ is the connected component of $\mathrm{GL}_{\mathrm{res}}(\mathbb{H}_+)$ containing the identity. The group $\mathscr{E}$ is in a similar relation to $\mathrm{U}^{\sim}_{\mathrm{res}}$ as $\mathrm{GL}_{\mathrm{res}}(\mathbb{H}_+)$ to $\mathrm{U}_{\mathrm{res}}$. In concrete terms

$$\mathscr{E} := \{(T,P) \in \mathrm{GL}^0_{\mathrm{res}}(\mathbb{H}_+) \times \mathrm{GL}(\mathbb{H}_+) : T - P \in \mathscr{I}_1\},$$

where $\mathscr{I}_1$ is the class of operators having a trace, that is being a trace class operator. (We refer to [RS80*] for concepts and results about operators on a Hilbert space.) $\mathscr{E}$ obtains its topology from the embedding into $\mathrm{GL}^0_{\mathrm{res}}(\mathbb{H}_+) \times \mathscr{I}_1(\mathbb{H}_+)$. The structure group of the principal bundle $\tau : \mathscr{E} \longrightarrow \mathrm{GL}^0_{\mathrm{res}}(\mathbb{H}_+)$ is the Banach Lie group $\mathscr{D}$ of invertible bounded operators having a determinant, that is of operators of the form $1 + T$ with T having a trace.

τ is simply the projection into the first component and we obtain another exact sequence of infinite dimensional Banach Lie groups as well as a principal fiber bundle

$$1 \longrightarrow \mathscr{D} \xrightarrow{\iota} \mathscr{E} \xrightarrow{\tau} \mathrm{GL}^0_{\mathrm{res}}(\mathbb{H}_+) \longrightarrow 1. \tag{3.2}$$

$\mathscr{E}$ is studied in the book of Pressley and Segal [PS86*] where, in particular, it is shown that $\mathscr{E}$ is contractible. This crucial property is investigated by Wurzbacher [Wur06*] in greater detail. The main ingredient of the proof is Kuiper's result on the homotopy type of the unitary group $\mathrm{U}(\mathbb{H})$ of a separable and infinite dimensional Hilbert space $\mathbb{H}$: $\mathrm{U}(\mathbb{H})$ with the norm topology is contractible and this also holds for the general linear group $\mathrm{GL}(\mathbb{H})$ with the norm topology (cf. [Kui65*]).

By general properties of classifying spaces the contractibility of the group $\mathscr{E}$ implies that τ is a universal fiber bundle for $\mathscr{D}$ (see [Diec91*], for example). This means that every principal fiber bundle $P \to X$ with structure group $\mathscr{D}$ can be obtained as the pullback of τ with respect to a suitable continuous map $X \to \mathrm{GL}^0_{\mathrm{res}}(\mathbb{H})$. Since there exist nontrivial principal fiber bundles with structure group $\mathscr{D}$ the bundle $\tau : \mathscr{E} \to \mathrm{GL}^0_{\mathrm{res}}$ cannot be trivial, and thus there cannot exist a continuous section $\mathrm{GL}^0_{\mathrm{res}}(\mathbb{H}_+) \to \mathscr{E}$.

One can construct directly a nontrivial principal fiber bundle with structure group $\mathscr{D}$. Or one uses another interesting result, namely that the group $\mathscr{D}$ is homotopy equivalent to $\mathrm{U}(\infty)$ according to a result of Palais [Pal65*]. $\mathrm{U}(\infty)$ is the limit of the unitary groups $\mathrm{U}(n) \subset \mathrm{U}(n+1)$ and the above exact sequence (3.2) realizes the universal sequence

$$1 \longrightarrow \mathrm{U}(\infty) \longrightarrow \mathrm{EU}(\infty) \longrightarrow \mathrm{BU}(\infty) \longrightarrow 1.$$

Since there exist nontrivial fiber bundles with structure group $\mathrm{U}(n)$ it follows that there exist nontrivial principal fiber bundles with structure group $\mathrm{U}(\infty)$ as well, and hence the same holds for $\mathscr{D}$ as structure group.

The closed subgroup $\mathscr{D}_1 := \{P \in \mathscr{D} : \det P = 1\}$ of $\mathscr{D}$ induces the exact sequence

$$1 \longrightarrow \mathscr{D}_1 \stackrel{\iota}{\longrightarrow} \mathscr{D} \stackrel{\det}{\longrightarrow} \mathbb{C}^\times \longrightarrow 1.$$

With the quotient $\mathrm{GL}^{0\sim}_{\mathrm{res}}(\mathbb{H}_+) := \mathscr{E}/\mathscr{D}_1$ one obtains another universal bundle

$$\mathrm{GL}^{0\sim}_{\mathrm{res}}(\mathbb{H}_+) \to \mathrm{GL}^{0}_{\mathrm{res}}(\mathbb{H}_+),$$

now with the multiplicative group $\mathbb{C}^\times$ as structure group. We have the exact sequence

$$1 \longrightarrow \mathbb{C}^\times \stackrel{\iota}{\longrightarrow} \mathrm{GL}^{0\sim}_{\mathrm{res}}(\mathbb{H}_+) \stackrel{\pi}{\longrightarrow} \mathrm{GL}_{\mathrm{res}}(\mathbb{H}_+) \longrightarrow 1,$$

which is another example of a central extension. Using the universality of this sequence one concludes that $\mathrm{GL}^{0\sim}_{\mathrm{res}}(\mathbb{H}_+) \to \mathrm{GL}^{0}_{\mathrm{res}}(\mathbb{H}_+)$ again has no continuous section. It follows in the same way that eventually our original bundle $\pi : \mathrm{U}^{\sim}_{\mathrm{res}} \to \mathrm{U}_{\mathrm{res}}$ (3.1) cannot have a continuous section. In summary we have

Proposition 3.17. *The exact sequence of Banach Lie groups*

$$1 \longrightarrow \mathrm{U}(1) \stackrel{\iota}{\longrightarrow} \mathrm{U}^{\sim}_{\mathrm{res}} \stackrel{\pi}{\longrightarrow} \mathrm{U}_{\mathrm{res}} \longrightarrow 1$$

is a central extension of the restricted unitary group $\mathrm{U}_{\mathrm{res}}$ *and a principal fiber bundle which does not admit a continuous section.*

In the same manner the basic central extension

$$1 \longrightarrow \mathrm{U}(1) \stackrel{\iota}{\longrightarrow} \mathrm{U}(\mathbb{H}) \stackrel{\widehat{\gamma}}{\longrightarrow} \mathrm{U}(\mathbb{P}) \longrightarrow 1$$

introduced in Lemma 3.4 has no continuous section when endowed with the norm topology. Since $\mathrm{U}(\mathbb{H})$ is contractible [Kui65*] the bundle is universal. But we know that there exist nontrivial $\mathrm{U}(1)$-bundles, for instance the central extensions

$$1 \longrightarrow \mathrm{U}(1) \stackrel{\iota}{\longrightarrow} \mathrm{U}(n) \stackrel{\widehat{\gamma}}{\longrightarrow} \mathrm{U}(\mathbb{P}(\mathbb{C}^n)) \longrightarrow 1$$

are nontrivial fiber bundles for $n > 1$ (cf. Example 3.21 below).

As will be seen in the next section the basic central extension also has no sections which are group homomorphisms (that is there exists no splitting map, cf. Example 3.21).

3.3 Equivalence of Central Extensions

We now come to general properties of central extensions beginning the discussion without taking topological questions into account.

Definition 3.18. Two central extensions

$$1 \longrightarrow A \stackrel{\iota}{\longrightarrow} E \stackrel{\pi}{\longrightarrow} G \longrightarrow 1\ ,\ 1 \longrightarrow A \stackrel{\iota}{\longrightarrow} E' \stackrel{\pi}{\longrightarrow} G \longrightarrow 1$$

of a group G by A are equivalent, if there exists an isomorphism $\psi : E \to E'$ of groups such that the diagram

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"'] & A \times G \arrow[r] \arrow[d, "\psi"'] & G \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\ 1 \arrow[r] & A \arrow[r] & E \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

commutes.

Definition 3.19. An exact sequence of group homomorphisms

$$1 \longrightarrow A \stackrel{\iota}{\longrightarrow} E \stackrel{\pi}{\longrightarrow} G \longrightarrow 1$$

splits if there is a homomorphism $\sigma : G \to E$ such that $\pi \circ \sigma = \mathrm{id}_G$.

Of course, by the surjectivity of π one can always find a map $\tau : G \to E$ with $\pi \circ \tau = \mathrm{id}_G$. But this map will not be a group homomorphism, in general.

If the sequence splits with splitting map $\sigma : G \to E$, then

$$\psi : A \times G \to E, \quad (a,g) \mapsto \iota(a)\sigma(g),$$

is a group isomorphism leading to the trivial extension

$$1 \longrightarrow A \longrightarrow A \times G \longrightarrow G \longrightarrow 1,$$

which is equivalent to the original sequence: the diagram

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, "\mathrm{id}"'] & A \times G \arrow[r] \arrow[d, "\psi"'] & G \arrow[r] \arrow[d, "\mathrm{id}"] & 1 \\ 1 \arrow[r] & A \arrow[r] & E \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

commutes. Conversely, if such a commutative diagram with a group isomorphism ψ exists, the sequence

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1$$

splits with splitting map $\sigma(g) := \psi(1_A, g)$. We have shown that

Lemma 3.20. *A central extension splits if and only if it is equivalent to a trivial central extension.*

Example 3.21. There exist many nontrivial central extensions by $\mathrm{U}(1)$. A general example of special importance in the context of quantization is given by the exact sequence (Lemma 3.4)

$$1 \longrightarrow \mathrm{U}(1) \xrightarrow{\iota} \mathrm{U}(n) \xrightarrow{\widehat{\gamma}} \mathrm{U}(\mathbb{P}(\mathbb{C}^n)) \longrightarrow 1$$

for each $n \in \mathbb{N}, n > 1$, and

$$1 \longrightarrow \mathrm{U}(1) \xrightarrow{\iota} \mathrm{U}(\mathbb{H}) \xrightarrow{\widehat{\gamma}} \mathrm{U}(\mathbb{P}) \longrightarrow 1$$

for infinite dimensional Hilbert spaces $\mathbb{H}$. These extensions are not equivalent to the trivial extension. They are also nontrivial as fiber bundles (with respect to both topologies on $\mathrm{U}(\mathbb{H})$, the norm topology or the strong topology).

Proof. All these extensions are nontrivial if this holds for $n = 2$ since this extension is contained in the others induced by the natural embeddings $\mathbb{C}^2 \hookrightarrow \mathbb{C}^n$ resp. $\mathbb{C}^2 \hookrightarrow \mathbb{H}$. The nonequivalence to a trivial extension in the case $n = 2$ follows from well-known facts.
In particular, we have the following natural isomorphisms:

$$\mathrm{U}(2) \cong \mathrm{U}(1) \times \mathrm{SU}(2) \text{ and } \mathrm{PU}(2) = \mathrm{U}(\mathbb{P}(\mathbb{C}^2)) \cong \mathrm{SO}(3)$$

as groups (and as topological spaces). If the central extension

$$1 \longrightarrow \mathrm{U}(1) \xrightarrow{\iota} \mathrm{U}(2) \xrightarrow{\widehat{\gamma}} \mathrm{PU}(2) \longrightarrow 1$$

would be equivalent to the trivial extension then there would exist a splitting homomorphism

$$\sigma : \mathrm{SO}(3) \cong \mathrm{PU}(2) \to \mathrm{U}(2) \cong \mathrm{U}(1) \times \mathrm{SU}(2).$$

The two components of σ are homomorphisms as well, so that the second component $\sigma_2 : \mathrm{SO}(3) \to \mathrm{SU}(2)$ would be a splitting map of the natural central extension

$$1 \longrightarrow \{+1, -1\} \xrightarrow{\iota} \mathrm{SU}(2) \xrightarrow{\pi} \mathrm{SO}(3) \longrightarrow 1,$$

which also is the universal covering. This is a contradiction. For instance, the standard representation $\rho : \mathrm{SU}(2) \hookrightarrow \mathrm{GL}(\mathbb{C}^2)$ cannot be obtained as a lift of a representation of $\mathrm{SO}(3)$ because of $\pi(\pm 1) = 1$.

In the same way one concludes that there is no continuous section. □

Note that the nonexistence of a continuous section has the elementary proof just presented above without reference to the universal properties which have been considered at the end of the preceding section. One can give an elementary proof for Proposition 3.17 as well, with a similar ansatz using the fact that the projection $\mathrm{U}_{\mathrm{res}}^{\sim} \to \mathrm{U}_{\mathrm{res}}$ corresponds to the natural projection $\widehat{\gamma} : \mathrm{U}(\mathsf{S}) \to \mathrm{U}(\mathbb{P}(\mathsf{S}))$.

On the other hand, the basic exact sequence

$$1 \longrightarrow \mathrm{U}(1) \xrightarrow{\iota} \mathrm{U}(\mathbb{H}) \xrightarrow{\widehat{\gamma}} \mathrm{U}(\mathbb{P}) \longrightarrow 1$$

is universal also for the strong topology (not only for the norm topology as mentioned in the preceding section), since the unitary group $\mathrm{U}(\mathbb{H})$ is contractible in the strong topology as well whenever $\mathbb{H}$ is an infinite dimensional Hilbert space.

In the following remark we present a tool which helps to check which central extensions are equivalent to the trivial extension.

Remark 3.22. Let

$$1 \longrightarrow A \xrightarrow{\iota} E \xrightarrow{\pi} G \longrightarrow 1$$

be a central extension and let $\tau : G \to E$ be a map (not necessarily a homomorphism) with $\pi \circ \tau = \mathrm{id}_G$ and $\tau(1) = 1$. We set $\tau_x := \tau(x)$ for $x \in G$ and define a map

$$\begin{aligned} \omega : G \times G &\longrightarrow A \cong \iota(A) \subset E, \\ (x,y) &\longmapsto \tau_x \tau_y \tau_{xy}^{-1}. \end{aligned}$$

(Here, $\tau_{xy}^{-1} = (\tau_{xy})^{-1} = (\tau(xy))^{-1}$ denotes the inverse element of τ_{xy} in the group E.) This map ω is well-defined since $\tau_x \tau_y \tau_{xy}^{-1} \in \ker \pi$, and it satisfies

$$\omega(1,1) = 1 \quad \text{and} \quad \omega(x,y)\omega(xy,z) = \omega(x,yz)\omega(y,z) \tag{3.3}$$

for $x,y,z \in G$.

Proof. By definition of ω we have

$$\begin{aligned} \omega(x,y)\omega(xy,z) &= \tau_x \tau_y \tau_{xy}^{-1} \tau_{xy} \tau_z \tau_{xyz}^{-1} \\ &= \tau_x \tau_y \tau_z \tau_{xyz}^{-1} \\ &= \tau_x \tau_y \tau_z \tau_{yz}^{-1} \tau_{yz} \tau_{xyz}^{-1} \\ &= \tau_x \omega(y,z) \tau_{yz} \tau_{xyz}^{-1} \\ &= \tau_x \tau_{yz} \tau_{xyz}^{-1} \omega(y,z) \ (A \text{ is central}) \\ &= \omega(x,yz)\omega(y,z). \end{aligned}$$

□

Definition 3.23. Any map $\omega : G \times G \longrightarrow A$ having the property (3.3) is called a 2-*cocycle*, or simply a *cocycle* (*on G with values in A*).

The central extension of G by A *associated with a cocycle* ω is given by the exact sequence

$$\begin{aligned} 1 \longrightarrow A \xrightarrow{\iota} A \times_\omega G &\xrightarrow{pr_2} G \longrightarrow 1, \\ a \longmapsto (a,1). & \end{aligned}$$

Here, $A \times_\omega G$ denotes the product $A \times G$ endowed with the multiplication defined by

$$(a,x)(b,y) := (\omega(x,y)ab, xy)$$

for $(a,x),(b,y) \in A \times G$.

It has to be shown that this multiplication defines a group structure on $A \times_\omega G$ for which ι and pr_2 are homomorphisms. The crucial property is the associativity of the multiplication, which is guaranteed by the condition (3.3):

$$\begin{aligned} ((a,x)(b,y))(c,z) &= (\omega(x,y)ab,xy)(c,z) \\ &= (\omega(xy,z)\omega(x,y)abc,xyz) \\ &= (\omega(x,yz)\omega(y,z)abc,xyz) \\ &= (a,x)(\omega(y,z)bc,yz) \\ &= (a,x)((b,y)(c,z)). \end{aligned}$$

The other properties are easy to check.

Remark 3.24. This yields a correspondence between the set of cocycles on G with values in A and the set of central extensions of G by A.

The extension E in Theorem 3.10

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{E} \xrightarrow{\pi} G \longrightarrow 1$$

is of the type $\mathrm{U}(1) \times_\omega G$. How do we get a suitable map $\omega : G \times G \to \mathrm{U}(1)$ in this situation? For every $g \in G$ by Wigner's Theorem 3.3 there is an element $U_g \in \mathrm{U}(\mathbb{H})$ with $\widehat{\gamma}(U_g) = Tg$. Thus we have a map $\tau_g := (U_g, g)$, $g \in G$, which defines a map $\omega : G \times G \to \mathrm{U}(1)$ satisfying (3.3) given by

$$\omega(g,h) := \tau_g \tau_h \tau_{gh}^{-1} = (U_g U_h U_{gh}^{-1}, 1_G).$$

Note that $g \mapsto U_g$ is not, in general, a homomorphism and also not continuous (if G is a topological group and T is continuous); however, in particular cases which turn out to be quite important ones, the U_gs can be chosen to yield a continuous homomorphism (cf. Bargmann's Theorem (4.8)).

If G and A are topological groups then for a cocycle $\omega : G \times G \to A$ which is continuous the extension $A \times_\omega G$ is a topological group and the inclusion and projection in the exact sequence are continuous homomorphisms. The reverse implication does not hold, since continuous maps $\tau : G \to E$ with $\pi \circ \tau = \mathrm{id}_G$ need not exist, in general. The central extension $p : z \longmapsto z^2$

$$1 \longrightarrow \{+1,-1\} \longrightarrow \mathrm{U}(1) \xrightarrow{p} \mathrm{U}(1) \longrightarrow 1$$

provides a simple counterexample. A more involved counterexample is (cf. Proposition 3.17)

$$1 \longrightarrow \mathrm{U}(1) \xrightarrow{\iota} \mathrm{U}_{\mathrm{res}}^{\sim} \xrightarrow{\pi} \mathrm{U}_{\mathrm{res}} \longrightarrow 1.$$

Lemma 3.25. *Let $\omega : G \times G \longrightarrow A$ be a cocycle. Then the central extension $A \times_\omega G$ associated with ω splits if and only if there is a map $\lambda : G \to A$ with*

$$\lambda(xy) = \omega(x,y)\lambda(x)\lambda(y).$$

Proof. The central extension splits if and only if there is a map $\sigma : G \to A \times_\omega G$ with $pr_2 \circ \sigma = \mathrm{id}_G$ which is a homomorphism. Such a map σ is of the form $\sigma(x) := (\lambda(x), x)$ for $x \in G$ with a map $\lambda : G \to A$. Now, σ is a homomorphism if and only if for all $x, y \in G$:

$$\begin{aligned}
\sigma(xy) &= \sigma(x)\sigma(y) \\
\Longleftrightarrow (\lambda(xy), xy) &= (\lambda(x), x)(\lambda(y), y) \\
\Longleftrightarrow (\lambda(xy), xy) &= ((\omega(x,y)\lambda(x)\lambda(y)), xy) \\
\Longleftrightarrow \lambda(xy) \qquad &= \omega(x,y)\lambda(x)\lambda(y).
\end{aligned}$$

□

Definition 3.26.

$$H^2(G,A) := \{\omega : G \times G \to A \,|\, \omega \text{ is a cocycle}\}/\sim,$$

where the equivalence relation $\omega \sim \omega'$ holds by definition if and only if there is a $\lambda : G \to A$ with

$$\lambda(xy) = \omega(x,y)\omega'(x,y)^{-1}\lambda(x)\lambda(y).$$

$H^2(G,A)$ is called the *second cohomology group* of the group G with coefficients in A.

$H^2(G,A)$ is an abelian group with the multiplication induced by the pointwise multiplication of the maps $\omega : G \times G \to A$.

Remark 3.27. The above discussion shows that the second cohomology group $H^2(G,A)$ is in one-to-one correspondence with the equivalence classes of central extensions of G by A.

This is the reason why in the context of quantization of classical field theories with conformal symmetry $\mathrm{Diff}_+(\mathbb{S}) \times \mathrm{Diff}_+(\mathbb{S})$ one is interested in the cohomology group $H^2(\mathrm{Diff}_+(\mathbb{S}), \mathrm{U}(1))$.

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