Lecture Notes on Geometric Quantization

Based on the Course Given in 2021/22

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(Under Construction)

Objective

These Lecture Notes were intended to support the course "Geometric Quantization" held in the winter term 2021/2022 at the LMU München. Its content has been written down step by step on the basis of the actual development of the course in the lectures and exercises, and, in particular, in interaction with the comments and questions of the participating students. Some of the contributions of the participants are included in the Lecture Notes.

The result of these efforts (as of February 2022, end of the course) is roughly the content of the first 10 chapters (with about 130 pages at Febr. 22) and the 6 appendices (with about 90 pages at Febr. 22).

Now, after the course has been completed, while many questions remain open, the Lecture Notes will be developed further. Among others with the aim to present further approaches to Geometric Quantization and provide more examples. The main goal is to add several main achievements of Geometric Quantization such as half-density quantization, half-form quantization, pairing, metalinear structure, applications to field theory, etc. as well as to complement and improve the first 10 chapters and the appendices.

Contents

LEITFADEN

Geometric Quantization begins with a set of observables of a model of Classical Mechanics. This model is realized as a symplectic manifold (M, ω) and the observables form a subset $\mathfrak{o}$ of the space $\mathcal{E}(M, \mathbb{R})$ of real-valued functions on M. To describe this background, these notes start with a mathematical exposition of Classical Mechanics in chapter 1. They proceed with a first attempt to construct a quantum model on the basis of (M, ω) and $\mathfrak{o} \subset \mathcal{E}(M, \mathbb{R})$ in chapter 2.

This attempt shows that a Hermitian complex line bundle L with connection ∇ on M is required for the program of Geometric Quantization in such a way that the curvature $\mathrm{Curv}(L, \nabla)$ of the connection is the symplectic form ω. In order to formulate this requirement the basic notions of a complex line bundle (ch. 3), a connection (ch. 4), a curvature (ch. 5) and a Hermitian structure (ch. 6) are developed.

With these ingredients at hand the first step of Geometric Quantization – prequantization – is carried through in chapter 7: For a given symplectic manifold (M, ω) and and a Hermitian linde bundle (L, ∇, H) satisfying $\mathrm{Curv}(L, \nabla) = \omega$ – called a prequantum line bundle – a complex Hilbert space $\mathbb{H}$ (generated by a subspace of sections of L) and a map $q : \mathcal{E}(M, \mathbb{R}) \to \mathcal{S}(\mathbb{H})$ (using the connection ∇) is constructed, where $\mathcal{S}(\mathbb{H})$ is the set of linear operators on $\mathbb{H}$, such that the so called Dirac Conditions are satisfied:

(D1) $q(1) = \mathrm{id}_{\mathbb{H}}$,

(D2) $[q(F), q(G)] = \frac{i}{2\pi} q(\{F, G\})$, for all $F, G \in \mathcal{E}(M, \mathbb{R})$.

Here, "{ , }" is the Poisson bracket of the symplectic manifold. Note that prequantization works for arbitrary subsets $\mathfrak{o}$ of observables. But prequantization fails to provide good quantum models for several reasons. For instance, let M be the space$M = \mathbb{R}^{2n}$ with the standard symplectic form $\omega = dq^j \wedge dp_j$, and let $L = M \times \mathbb{C}$ be the trivial line bundle with connection ∇ given by the form $-p_j dq^j$. Then the Hilbert space $\mathbb{H}$ is $L^2(\mathbb{R}^{2n}, \mathbb{C})$, the space of square integrable functions $\phi : \mathbb{R}^{2n} \to \mathbb{C}$, and the observables q^i, p_j have as their prequantization operators

$$q\left(q^j\right) = \frac{i}{2\pi} \frac{\partial}{\partial p_j} + q^j =: Q^j \,,$$
$$q\left(p_j\right) = -\frac{i}{2\pi} \frac{\partial}{\partial q^j} =: P_j.$$

Although (D1), i.e. $q(1) = \mathrm{id}$, and (D2), i.e.

$$[Q^i, P_j] = \frac{i}{2\pi}\delta^i_j = \frac{i}{2\pi}\{q^i, p_j\} = \frac{i}{2\pi}q(\{q^i, p_j\})$$

are satisfied, this result is not in accordance with the usual quantum model. In particular, the prequantization q is not irreducible. However, if we restrict the operators Q^i, P_j to the smaller Hilbert space $\mathbb{H} = L^2(\mathbb{R}^n, \mathbb{C}) \subset L^2(\mathbb{R}^{2n}, \mathbb{C})$ of functions only depending on the variables q^i, the usual quantum model is achieved.

This procedure of cutting down the number of variables can be generalized by introducing polarizations on M (see ch. 9) along which the sections generating the Hilbert space have to be constant.

Before the presentation of polarizations in chapter 8 the question is discussed under which condition a symplectic manifold (M, ω) admits a Hermitian line bundle (L, ∇, H) with connection such that $\mathrm{Curv}(L, \nabla) = \omega$. It turns out that this condition is a purely topological condition on M and ω which can be expressed best by cohomology. We call symplectic manifolds quantizable if this condition is satisfied.

Chapter 10 is devoted to the construction of the first version of a full Geometric Quantization. The construction is based on the geometric data of a prequantum line bundle (L, ∇, H) on a symplectic manifold and a polarization P. The representation space $\mathbb{H}_P$ is then a suitable Hilbert completion of polarized sections. Here, a polarized section is a section s of L satisfying $\nabla_X s = 0$. Moreover, we need the notion of a directly quantizable observable F in order to confirm that for a polarized section s the derivative $\nabla_{X_F} s$ is polarized as well. Finally, the prequantum operator determines the quantum operator, which will be denoted as $q(F)$ as well, and which satisfies the Dirac conditions with respect to a smaller representation space $\mathbb{H}$ and the directly quantizable observables.

Several elementary examples are presented in detail in this chapter in order to illustrate the impact of Geometric Quantization. Among others, the geometric quantization of the harmonic oscillator is calculated, leading to a reasonable result. This result has, however, a shift in the eigenvalues in comparison to the known results from Quantum Mechanic. By a modification of Geometric Quantization this defect can be removed.

The content of the first 10 chapters of these notes covers the development of the course given in winter 21/22 which constitutes essentially half the notes. The second half deals with various improvements, modifications and generalizations.

The second part of the notes begins in ch. 11 with an analysis of the existence of enough polarized sections and quantizable observables. For instance, when some the leaves of the quotient M/P of M induced by the polarization P on M are not simply connected it can happen that there do not exist nontrivial polarized sections and holonomy comes into play (Bohr-Sommerfeld condition). One way to obtain reasonable representation spaces, nevertheless, is to consider generalized sections in the sense of distributions.

The next step in Geometric Quantization is Half-Density Quantization. In addition to the prequantum bundle L on M and the polarization P a half-density bundle S on M is considered as an additional geometric structure together with a partial connection induced on S by the polarization. The bundle L now is replaced with the line bundle $L \otimes S$ and the quantization is then based on the polarized sections of $L \otimes S$ to obtain the Half-Density Quantization roughly in the same way as in ch. 10. As a preparation a detailed exposition on r-densities on a manifold and their integration theory is given in chapter 12. Chapter 13 and 14 deal with the quantization, first for cotangent spaces $M = T^*Q$ (sometimes called momentum spaces) and then in general.

Chapter 15 deals with Half-Form Quantization which is similar to Half-Density Quantization but the additional line bundle now is a half-form bundle S, i.e. a line bundle S with the property $S \otimes S \cong \Lambda^n P^\vee$ ($\Lambda^n P^\vee$ is the line bundle of n-forms in P, the so-called canonical bundle of P). We discuss the topological condition on P and M under which such a half-bundle exists. Note, that, in contrast to the half-form case, the half-density bundle always exists as a trivial line bundle. The topological condition which ensures the existence of a half-form bundle is needed for the existence of a so-called metalinear structure on P (Chapter 16) and the existence of a metaplectic structure (Chapter 17).

Chapter 18 is dedicated to the metaplectic representation which appears in several mathematical areas. The metaplectic representation can be used to refine the preceding 3 chapters and it gives rise to generalize the procedure of geometric quantization in a number of directions.

A chapter on Chern-Simons theory is planned.

1 Hamiltonian Mechanics

Hamiltonian Mechanics is the study of conservative systems of Classical Mechanics. These systems are modeled by Hamiltonian systems. The purpose of this chapter is to introduce step by step the concept of a Hamiltonian system, first on an open subset of $\mathbb{R}^n$ as configuration space, then on the cotangent bundle of a general manifold as phase space and finally on a general symplectic manifold as phase space. A Hamiltonian system is a special case of a dynamical system induced by a function H on phase space.

Since we need general manifolds as phase spaces this chapter also serves to recall the basic notions related to a manifold and the notation used throughout these lecture notes.

1.1 A Simple Hamiltonian System

1.1.1 Canonical Equations

We begin with a special case of a system of Hamiltonian Mechanics where the configuration space ("Ortsraum") is an open subset $U \subset \mathbb{R}^n$ of $\mathbb{R}^n$ ($n \in \mathbb{N}, n > 0$):

- $U \subset \mathbb{R}^n$ open subset of $\mathbb{R}^n$, the Configuration Space,
- $M := U \times \mathbb{R}^n \cong T^*U$, the (Momentum) Phase Space, and the smooth functions $f : M \to \mathbb{R}$ or $f : M \to \mathbb{C}$ are the Observables,
- $H \in \mathcal{C}^\infty(M)$, the Hamiltonian Function.

The Equations of Motion are

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}, \tag{1}$$

i.e.

$$\dot{q}^j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q^j},$$

for $j = 1, \ldots n$.

Here, $q = (q^1, \ldots, q^n)$ are the Position Coordinates in U ("Ortskoordinaten") and $p = (p_1, \ldots, p_n)$ are the Momentum Coordinates of the cotangent space $T_q^*U \cong \mathbb{R}^n$ ("Impulskoordinaten").

The equations of motion are also called Canonical Equations or Hamiltonian Equations, and (M, H) will be called a Simple Hamiltonian System with n degrees of freedom.

In many cases of simple Hamiltonian systems the function H has the interpretation of the energy as we see in the case of the harmonic oscillator:

Example 1.1. The harmonic oscillator in n dimensions can be modeled as a simple Hamiltonian System in the following way (disregarding constants):

- $U = \mathbb{R}^n$, and
- $H(q,p) := \frac{1}{2}(\|p\|^2 + \|q\|^2) = \frac{1}{2}\sum_{j=1}^n \left((p_j)^2 + (q^j)^2\right)$, $(q,p) \in M = U \times \mathbb{R}^n$,

with canonical equations

$$\dot{q} = p, \quad \dot{p} = q.$$

In general, the canonical equations can be written in the form

$$(\dot{q}, \dot{p}) = \left(\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q}\right) \tag{2}$$

In this form they look like a dynamical system on the phase space M of the type

$$\dot{z} = A(z),$$

with a vector field $A : M \to TM$ and $z = (q,p)$, where the vector field A is similar to a gradient

$$A = \nabla F = \frac{\partial F}{\partial z}$$

of a $\mathcal{C}^\infty$-function F on M. This is, in fact, true up to a "twist", the symplectic twist!

1.1.2 Symplectic Involution

To explain the symplectic twist, we define the Symplectic Structure on the phase $M = T^*U \cong U \times \mathbb{R}^n$ by the linear map σ on the tangent space $T_zM \cong \mathbb{R}^n \times \mathbb{R}^n$, $z \in M$,

$$\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad (q,p) \longmapsto (p, -q), \tag{3}$$

given by the block matrix σ

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4}$$

acting as follows:

$$\sigma : \quad \begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix}.$$

Definition 1.2. We define the SYMPLECTIC GRADIENT $\nabla^\sigma F$ of a function $F \in \mathcal{C}^\infty(M)$ to be

$$\nabla^\sigma F = \sigma \circ \nabla F = \left(\frac{\partial F}{\partial p}, -\frac{\partial F}{\partial q}\right).$$

The vector field $\nabla^\sigma H$ will often be denoted by X_H, $X_H := \nabla^\sigma H$, and X_H will be called the HAMILTONIAN VECTOR FIELD associated with H.

With these notations, the canonical equations obtain the form

$$\dot{z} = \nabla^\sigma H(z) \quad \text{for} \quad z = (q, p) \in M\,,$$

or

$$\boxed{\dot{z} = X_H(z).} \tag{5}$$

Because of $\sigma^2 = \sigma \circ \sigma = -\mathrm{id}_{\mathbb{R}^{2n}}$ the map σ and ist matrix is called the SYMPLECTIC INVOLUTION.

1.1.3 Symplectic Form

The symplectic structure on $M = T^*U$ can also be given by the symplectic form or by the Poisson bracket, as will be explained in the following.

Definition 1.3. The SYMPLECTIC FORM ω on the tangent space $T_zP = T_z(T^*U) \cong \mathbb{R}^n \times \mathbb{R}^n$ at $z \in M$ is

$$\omega := \sum_{j=1}^{n} dq^j \wedge dp_j = dq^j \wedge dp_j$$

(we use Einstein summation in the following).

Hence, the bilinear and alternating map

$$\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

is given by

$$\omega(X, \bar{X}) = X^j \bar{Y}_j - \bar{X}^j Y_j\,,$$

when

$$X = (X^1, \ldots, X^n, Y_1, \ldots, Y_n)\ ,\ \bar{X} = (\bar{X}^1, \ldots, \bar{X}^n, \bar{Y}_1, \ldots, \bar{Y}_n) \in \mathbb{R}^n \times \mathbb{R}^n.$$

with respect to the standard coordinates of $\mathbb{R}^n \times \mathbb{R}^n$.

The corresponding standard vector space basis $B = \{a_1, \ldots, a_n, b^1, \ldots, b^n\}$ of $\mathbb{R}^n \times \mathbb{R}^n$ determines the coordinates $X = (X^1, \ldots, X^n, Y_1, \ldots, Y_n) = X^j a_j + Y_j b^j$ which we have just used. ω satisfies $\omega(a_i, b^j) = \delta_i^j = -\omega(b^j, a_i)$ and for all other basis vectors $v, w \in B$: $\omega(v, w) = 0$. Such a basis is called a symplectic frame. The induced matrix representing the symplectic form ω is given by the coefficients $\omega(v, w), v, w \in B$. It is the symplectic involution σ(cf. (3), (4)). Therefore, the symplectic form can also be described by matrix multiplication

$$\omega(X, \bar{X}) = X^\top \sigma \bar{X}.^1$$

1.1.4 Poisson Bracket

The Poisson Bracket $\{F, G\}$ of two functions (i.e. observables) $F, G \in \mathcal{C}^\infty(M)$ is defined by

$$\{F, G\} := \omega(X_F, X_G)\,,$$

which is also given by the well-known expression

$$\boxed{\{F, G\} = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} = \frac{\partial F}{\partial q^j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q^j}\,.}$$

Remark 1.4. Other sign convention are used in the literature, for example $dp_j \wedge dq^j$ which is $-\omega$ in our notation. See Table 18.3 for more conventions.

A straightforward and remarkable property of the Poisson bracket is the following result

Proposition 1.5 (Equations of motion in Poisson form)**.** *A curve $z : I \to M$ in M (i.e. $z \in \mathcal{E}(I, M)$) is a solution of the canonical equations $\dot{z} = X_H(z)$ if and only if*

$$\dot{F} = \{F, H\}$$

for all observables $F \in \mathcal{E}(M)$, i.e.

$$\frac{d}{dt}F(z(t)) = \{F(z(t)), H(z(t))\}\,,\ t \in I.$$

Proof. If $z(t) = (q(t), p(t))$ is a solution of $\dot{z} = X_H(z)$ one obtains for every $F \in \mathcal{E}(M)$

$$\dot{F} = \frac{d}{dt}F(z(t)) = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial p}\dot{p} = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q}.$$

Hence,

$$\dot{F} = \{F, H\}\,.$$

[1]The vectors $X, \bar{X}$ are tangent vectors written as column vectors and '$\top$' denotes transposition.

The converse follows by choosing q^j and p_j for F:

$$\dot{q}^j = \{q^j, H\} = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \dot{p}_j = \{p_j, H\} = -\frac{\partial H}{\partial q^j}.$$

□

Corollary 1.6. *$F \in \mathcal{E}(M)$ is a* FIRST INTEGRAL *or* CONSTANT OF MOTION *("Bewegungskonstante") if and only if*

$$\{F, H\} = 0\,.$$

Observation 1.7. The symplectic structure of $M = T^*U \cong U \times \mathbb{R}^n$ is given either

1. by the symplectic involution σ, or
2. by the symplectic form ω,
3. by the Poisson bracket $\{\ ,\ \}$.

What we have described so far in this chapter presents only the local models of conservative classical mechanics where a configuration space can be detected as an open subset U of $\mathbb{R}^n$. For global considerations which are, in particular, needed for the program of Geometric Quantization one has to redefine the above concepts for general manifolds.

Note, that in Classical Mechanics the reduction of degrees of freedom by first integrals, by constraints or by symmetry considerations leads to general manifolds in a natural way (cf. Subsection 17.8).

Example 1.8. The reduction of a Simple Hamiltonian System $M = T^*U$ with Hamiltonian function H with respect to a first integral H (e.g. $F = H$): As a first step the "surface" $S_c := F^{-1}(c)$ for a $c \in F(M)$ is considered. When the gradient of F does not vanish on S_c, the surface will be a $(2n-1)$-dimensional manifold. Identifying points in S_c which lie on the same solution of the canonical equations we get an equivalence relation $\sim$ on S_c. The quotient $S_c/\sim$ is the space of orbits (= motions) with $F = c$. If this quotient space is also a differentiable quotient the study of the Hamiltonian system continues by investigating the reduced space $S_c/\sim$ of dimension $2n-2$. In general, the reduced space will not be of the form $T^*\mathbb{R}^{n-1}$ or an open subset thereof. But the reduced space obtains a natural symplectic form (pushforward of ω) and so generates a general Hamiltonian systems, as we explain in the next section. Concrete examples of reduction are given in Section 1.3.

1.2 Symplectic Manifolds and Hamiltonian Systems

In these lecture notes, a manifold will always be a differentiable (i.e. $\mathcal{C}^\infty$-) real manifold with countable topology and finite dimension. In most cases the manifold is also assumed to be connected. Later we consider also complex manifolds.

Remark 1.9. In physics there appear also infinite dimensional manifolds having their local models in a fixed Hilbert, Banach, or Fréchet space (see e.g. [AM78, Put93]). In this course, however, to simplify matters, we concentrate on the finite dimensional case.

Relevant manifolds in geometry and physics are

- Open subsets $U \subset \mathbb{R}^m$
- Tangent and cotangent bundles TM, T^*M over a manifold M
- The rotation group SO(3) and other Lie groups
- Products $M_1 \times M_2$ of manifolds M_1, M_2
- Submanifolds of the above like the spheres $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ (of radius 1), or matrix groups like $\mathrm{SO}(3) \subset \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \cong \mathbb{R}^9$
- Quotients of the above like the projective spaces $\mathbb{P}^n(\mathbb{R}), \mathbb{P}^n(\mathbb{C})$

Exercise 1.10. Describe the projective spaces $\mathbb{P}^n(\mathbb{R})$, resp. $\mathbb{P}^n(\mathbb{C})$ as quotient manifolds (cf. A.10) of $\mathbb{R}^{n+1} \setminus \{0\}$, resp. $\mathbb{C}^{n+1} \setminus \{0\}$ and of $\mathbb{S}^n \subset \mathbb{R}^n$, resp. $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ by explicitly presenting suitable charts and confirming the universal property.

1.2.1 Notations for Manifolds

Let us recall some notations for manifolds and related basic concepts. (The notion of a differentiable (or smooth) manifold as well as related concepts are collected together in the Appendix, Section A on Manifolds, particularly in A.16, A.19, ff.):

Notation 1.11 (Local view of tangent vectors)**.** Let M be an n-dimensional manifold.

1. The CHARTS on M defining the differentiable structure of M will often be denoted as follows
$$q = (q^1, q^2, \dots, q^n) : U \to V\,,$$
where $U \subset M$ is an open subset in M, $V \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$ and q is differentiable with differentiable inverse. The $q^j : U \to \mathbb{R}, j = 1, \dots, n$ are the (local) COORDINATES given by the chart q.

2. Moreover, such a chart q provides for each $a \in U$ a natural vector space basis

$$\left(\frac{\partial}{\partial q^1}(a), \frac{\partial}{\partial q^2}(a) \ldots, \frac{\partial}{\partial q^n}(a)\right)$$

of the TANGENT SPACE $T_a M$ at a, where

$$\frac{\partial}{\partial q^j}(a) := \left[q^{-1}\left((q(a) + te_j)\right)\right]_a$$

is the tangent vector of the curve $q^{-1}(q(a) + te_j)$ through $a \in U$ and where $(e_1, \ldots, e_n)$ is the standard unit vector space basis of $\mathbb{R}^n$. Sometimes

$$\frac{\partial}{\partial q^j}(a) \text{ is abbreviated as } \frac{\partial}{\partial q^j}, \partial_j(a) \text{ or } \partial_j\,,$$

whenever it is clear from the context for which chart q resp. for which point $a \in M$ the expressions are employed.

3. The corresponding vector fields

$$\frac{\partial}{\partial q^j} : U \to TU\,,\; a \mapsto \frac{\partial}{\partial q^j}(a)\,,$$

can be used to represent every vector field $X : U \to TU$ over U through the uniquely determined coefficients X^j:

$$X = X^j \frac{\partial}{\partial q^j}.$$

4. For a vector field X over U the action of X on f (the DIRECTIONAL DERIVATIVE; "Richtungsableitung") is

$$L_X f(a) := \frac{d}{dt} f \circ x(t)|_{t=t_0},$$

where the curve x represents $X(a)$ at the point $a = x(t_0)$: $X(a) = [x]_a$.
With the abbrevation

$$\frac{\partial f}{\partial q^j}(a) = \left(\frac{\partial}{\partial q^j} f\right)(a) = \frac{d}{dt} f \circ q^{-1}(q(a) + te_j)|_{t=0}$$

we obtain the formula for the action of X in local coordinates

$$L_X f = X^j \frac{\partial f}{\partial q^j}\,.$$

L_X is often called LIE DERIVATIVE in the direction of X. Sometimes the notation Xf instead of $L_X f$ is used.

Notation 1.12 (Local view of cotangent vectors)**.** Let M be again an n-dimensional manifold.

1. A chart $q : U \to V$ provides for each $a \in U$ a natural vector space basis
$$\big(dq^1(a), dq^2(a) \ldots, dq^n(a)\big)$$
of the COTANGENT SPACE $T_a^*M = (T_aM)^*$ of M at a, where
$$dq^j(a)\left([x]_a\right) := \left(\frac{d}{dt}(q^j \circ x)|_{t=t_0}\right), \tag{6}$$
when $a = x(t_0)$. For convenience, $dq^j(a)$ is often abbreviated as dq^j when it is clear for which point a the expressions are employed.
The basis is dual to the above basis if T_aM:
$$dq^k(\frac{\partial}{\partial q^j}) = \delta_j^k \,.$$

2. Note that
$$dq^j : U \to T^*U\,,\ a \mapsto dq^j(a)\,,$$
is a 1-FORM on U. Moreover, every 1-form $\alpha : U \to T^*U$ over U can be described uniquely by
$$\alpha = \alpha_j dq^j \,,$$
where the coefficients α^j are smooth. α_j can be obtained by
$$\alpha_j(a) = \alpha(a)\left(\frac{\partial}{\partial q^j}(a)\right) = \alpha(\partial_j)(a)\,.$$
And $\mathrm{d}q_j(X) = X^j$ for a vector field $X \in \mathfrak{V}(U)$.

3. A smooth function $f : W \to \mathbb{R}$ on an open subset $W \subset M$ induces a 1-form df in the same way as dq^j (cf. formula (6)):
$$df(a)\left([x]_a\right) := \left(\frac{d}{dt}(f \circ x(t))|_{t=t_0}\right).$$
In local coordinates
$$df = \frac{\partial f}{\partial q^j} dq^j \,.$$
When the 1-form df is applied to the vector field X the result will be:
$$df(X) = \frac{\partial f}{\partial q^j} X^j \,.$$
In particular, $df(X) = L_X(f)$.

Notation 1.13 (Global view of fields and forms)**.**

1. For manifolds M, N

$$\mathcal{E}(M,N) = \{f : M \to N \mid f \text{ smooth}\}$$

denotes the set of smooth mappings from M to N. And

$$\mathcal{E}(M) = \mathcal{E}(M,\mathbb{K})$$

denotes the corresponding set of functions, where $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. Pointwise addition and multiplication defines on $\mathcal{E}(M)$ the structure of a commutative $\mathbb{K}$-algebra over $\mathbb{K}$ (i.e. a $\mathbb{K}$-vector space with commutative ring multiplication).

From the point of physics, the smooth functions $f \in \mathcal{E}(M,\mathbb{K})$ are the Classical Observables.

2. For a commutative algebra R, a Derivation is a $\mathbb{K}$-linear map $D : R \to R$ with $D(fg) = D(f)g + fD(g)$ for all $f, g \in R$. The set $\mathrm{Der}(R)$ of derivations of R is a natural R-module by pointwise addition and multiplication. Moreover, with respect to the commutator $[D, D'] : D \circ D' - D' \circ D$ for $D, D' \in \mathrm{Der}(R)$ this R-module is also a Lie-algebra over $\mathbb{K}$.

Applied to $R = \mathcal{E}(M)$ we see that every vector field X can be interpreted to be a derivation since the Lie derivative L_X is, in fact, a derivation. Conversely, every derivation $D \in \mathrm{Der}((M))$ is induced by a unique vector field X, i.e. $D = L_X$. As a result, the Lie algebra $\mathrm{Der}(\mathcal{E}(M))$ can be identified with the space of vector fíelds, i.e. $\mathfrak{V}(M) = \mathrm{Der}(\mathcal{E}(M)$. In this way $\mathfrak{V}(M)$ obtains the structure of a Lie algebra: $[X, Y]$ is the unique vector field satisfying

$$[X, Y] = L_X \circ L_Y - L_Y \circ L_X$$

for $X, Y \in \mathfrak{V}(M)$.

3. The s-forms (Differential Forms of degree s, $s \in \mathbb{N}$) on M are the maps $\eta : (\mathfrak{V}(M))^s \to \mathcal{E}(M)$ which are s-multilinear over the ring $\mathcal{E}(M)$ and alternating[2]. Finally, the $\mathcal{E}(M)$-module of all s-forms is denoted by

$$\mathcal{A}^s(M) = \{\eta : (\mathfrak{V}(M))^s \to \mathcal{E}(M) \mid \ s\text{-multilinear over } \ \mathcal{E}(M) \ \text{ and alternating}\}$$

Particular cases:
$\mathcal{A}^0(M) = \mathcal{E}(M)$ are the 0-forms or functions, and
$\mathcal{A}^1(M) =: \mathcal{A}(M)$ are the 1-forms.

[2]In the same way one can define the (r, s)-tensor fields.

4. The Wedge Product $\alpha \wedge \beta$ of two 1-forms $\alpha, \beta \in \mathcal{A}(M)$ is given by

$$\alpha \wedge \beta := \alpha \otimes \beta - \beta \otimes \alpha \,,$$

i.e. $\alpha \wedge \beta(X,Y) = \alpha(X)\beta(Y) - \beta(X)\alpha(Y)$ for $X, Y \in \mathfrak{V}(M)$. Similarly one obtains the general wedge product

$$\wedge : \mathcal{A}^r(M) \times \mathcal{A}^s(M) \to \mathcal{A}^{r+s}(M) \,.$$

5. The Exterior Derivative

$$d = d^s : \mathcal{A}^s(M) \to \mathcal{A}^{s+1}(M)$$

is globally given by

$$d\eta(X_0, X_1, \ldots, X_s) := \sum_{j=0}^{s} (-1)^j L_{X_j} \Big(\eta(X_0, \ldots \widehat{X_j}, \ldots, X_s) \Big) +$$

$$+ \sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_0, \ldots \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_s).$$

Here, $\widehat{X_j}$ means that X_j has to be deleted.
For a 1-form $\alpha \in \mathcal{A}(M)$ the definition leads to

$$d\alpha(X,Y) = L_X\left(\alpha(Y)\right) - L_Y\left(\alpha(X)\right) - \alpha\left([X,Y]\right) \tag{7}$$

for $X, Y \in \mathfrak{V}(M)$.

In local coordinates $q : U \to V$, $\eta \in \mathcal{A}^s(U)$ has the presentation

$$\eta = \sum_{j_1 < j_2 < \ldots < j_s} \eta_{j_1 j_2 \ldots j_s} dq^{j_1} \wedge \ldots \wedge dq^{j_s}$$

and the exterior derivative of η is then

$$d\eta = \sum_{j_1 < j_2 < \ldots < j_s} \sum_{i=1}^{n} \frac{\partial \eta_{j_1 j_2 \ldots j_s}}{\partial x^j} dx^j \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_s}. \tag{8}$$

1.2.2 Symplectic Manifolds

Definition 1.14. A Symplectic Form ("Symplektische Form") on a manifold M is a 2-form $\omega \in \mathcal{A}^2(M)$ which is non-degenerate, i.e. $\omega(a) : T_a M \times T_a M \to \mathbb{R}$ is non-degenerate for all $a \in M$, and which is closed, i.e. $d\omega = 0$.

Recall from Linear Algebra that a bilinear map $g : V \times V \to \mathbb{R}$ on a finite dimensional real vector space is non-degenerate if for each $v \in V$ the condition $g(v, w) = 0$ for all $w \in W$ implies $v = 0$. Or, equivalently if and only if the induced map

$$g^\flat : V \to V^\vee \, , \, v \mapsto (w \mapsto g(v, w)) \, , \, v, w \in V,$$

is an isomorphism of vector spaces. Here, $V^\vee$ denotes the dual of a vector space, the space of linear forms, also denoted by V^*.
Also equivalent when we describe g with respect to a basis e_i by $g(e_i, e_j) =: g_{ij}$ is the condition that the matrix (g_{ij}) has non-zero determinant.

In the case of g being alternating, this implies that the dimension of V has to be even. In fact,

$$\det(g_{ij}) = \det\left((g_{ij})^T\right) = \det(-(g_{ij})) = (-1)^d \det(g_{ij})$$

if $\dim V = d$, hence $1 = (-1)^d$ when $\det(g_{ij}) \neq 0$.

Observation 1.15. As a result, if ω is a symplectic form on M the dimension of M is even, as we have seen in the fundamental example of the special symplectic form $\omega = dq^j \wedge dp_j$ on the phase space T^*U (see 1.3).

Definition 1.16. A SYMPLECTIC MANIFOLD ("Symplektische Mannigfaltigkeit") is a manifold M together with a symplectic form ω.
The corresponding maps which preserve the symplectic structure are the CANONICAL TRANSFORMATIONS ("Kanonische Transformation") or SYMPLECTOMORPHISMS ("Symplektomorphismus"), i.e. the diffeomorphisms $\Phi : M \to M'$ between symplectic manifolds (M, ω) and (M', ω') which satisfy $\Phi^*\omega' = \omega$. Recall (cf. Definition A.33:

$$\Phi^*\omega'(X, Y)|_a = \omega'(a)(T_a\Phi(X), T_a\Phi(Y)) \, , \ \text{for} \ X, Y \in T_a(M) \, , \, a \in M \, .$$

For a symplectic manifold (M, ω) we know that the dimension of M,

$$\dim M = \dim_{\mathbb{R}} T_a M \in 2\mathbb{N} \, ,$$

is even and we write, in general, $\dim M = 2n$.

Examples are the phase spaces $M = T^*U$ for open $U \subset \mathbb{R}^n$ as considered in Section 1.1. We call such an example a SIMPLE PHASE SPACE. Slightly more general examples are the cotangent bundles $M = T^*Q$ of general manifolds Q as will be explained below in the next example. Further examples occur as quotients of $M = T^*Q$ in the process of reduction of degrees of freedom. Other examples are coadjoint orbits (see Section 1.3.5) and Kähler manifolds (see 9.28).

Construction 1.17 (Cotangent bundle)**.** Let Q be a manifold of dimension n. Recall that the COTANGENT BUNDLE T^*Q is given (as a set) by

$$T^*Q := \bigcup_{a \in Q} T_a^* Q$$

$T_a^*Q := (T_aQ)^* = \mathrm{Hom}_{\mathbb{R}}(T_aQ, \mathbb{R})$, with projection $\tau^* : T^*Q \to Q$, $\tau^*(T_a^*Q) = \{a\}$, $a \in Q$. The structure of a $2n$-dimensional manifold on the cotangent bundle $M := T^*Q$ is defined by the bundle charts (cf. A.19)

$$\tilde{q} = (q^1, \dots q^n, p_1, \dots, p_n) : T^*U \to V \times \mathbb{R}^n ,$$

induced by the charts $q : U \to V$ of the manifold Q, $U \subset Q$ open, where

$$p_j(\mu) = \mu\left(\frac{\partial}{\partial q^j}\right) , \; \mu \in T^*U .$$

The Liouville form on $M = T^*Q$ is, by definition,

$$\lambda := p_j dq^j$$

in local bundle coordinates.

It can be defined globally: For $X \in T_\mu M$ and $\mu \in T_a^*Q \subset M$ we define $\lambda_\mu : T_\mu M \to \mathbb{R}$ by

$$\lambda_\mu(X) := \mu\left(T_\mu \tau^*(X)\right) ,$$

where $T_\mu \tau^* : T_\mu M \to T_aQ$ is the tangent map (derivative) of τ^* at $\mu \in M$ (see Observation A.14). The same formula is well-defined for vector fields $X \in \mathfrak{V}(M)$ and 1-forms $\mu \in \mathcal{A}(Q)$ providing a map $\lambda : \mathfrak{V}(M) \to \mathcal{E}(M)$, $\mu \mapsto \lambda_\mu(X)$. Since λ is $\mathcal{E}(M)$-linear, it is a 1-form $\lambda \in \mathcal{A}(M)$.

With respect to bundle charts $(q, p) : T^*U \to V \times \mathbb{R}^n$ one sees

$$\lambda|_{T^*U} := p_j dq^j .$$

In fact, $\mu \in \mathcal{A}(Q)$ and $X \in \mathfrak{V}(M)$ have locally the following representations:

$$\mu = \mu_k dq^k \quad \text{and} \quad X = X^j \frac{\partial}{\partial q^j} + X_k \frac{\partial}{\partial p_k} .$$

Hence,

$$T_\mu \tau^*(X)(\tau^*(\mu)) = [\tau^* \circ X]_{\tau^*(\mu)} = X^j \frac{\partial}{\partial q^j}|_{\tau^*(\mu)} ,$$

with respect to bundle coordinates, which implies $\lambda_\mu(X) = \mu\left(T_\mu \tau^*(X)\right) = \mu_j X^j = (p_j dq^j)_\mu(X)$. Therefore, $\lambda|_U = p_j dq^j$.

Note, that λ can be characterized by the following property: $\lambda \in \mathcal{A}(T^*Q)$ is the unique one form on T^*Q such that for any one form $\alpha \in \mathcal{A}(Q)$ on Q the pullback of λ via α gives back λ:

$$\alpha^* \lambda = \lambda .$$

The corresponding natural Symplectic Form ω on the cotangent bundle is $\omega := -d\lambda$. In local bundles charts ω has the form

$$\omega|_{T^*U} = dq^j \wedge dp_j .$$

The local expression shows that ω is indeed non-degenerate and closed. Therefore, (T^*Q, ω) is a symplectic manifold, often called MOMENTUM PHASE SPACE.

Remark 1.18. In the case of the cotangent bundle T^*Q the symplectic form is exact: $d\omega = -dd\lambda = 0$. The 1-form $-\lambda$ is called the SYMPLECTIC POTENTIAL.

In general, since a symplectic form ω is closed by definition, ω has a symplectic potential locally, i.e. (by the Lemma of Poincaré) for each point $a \in M$ there exists a neighbourhood U and a 1-form $\alpha \in \mathcal{A}(U)$ with $d\alpha = \omega|_U$. A global potential always exists if $\mathrm{H}^2_{\mathrm{dR}}(M, \mathbb{R}) = 0$. But many naturally defined symplectic manifolds do not have a symplectic potential, this holds, for instance, for all compact symplectic manifolds.

Locally all symplectic manifolds look like open subspaces of the simple phase spaces $T^*U \cong U \times \mathbb{R}^n$ with $\omega = dq^j \wedge dp_j$ as the following result confirms.

Theorem 1.19 (DARBOUX'S Theorem)**.** *Every point $a \in M$ of a symplectic manifold (M, ω) has an open neighbourhood $U \subset M$ and a chart*

$$\varphi = (q, p) = (q^1, \ldots, q^n, p_1, \ldots, p_n) : U \to V \subset T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n,$$

such that in these coordinates

$$\omega|_U = dq^j \wedge dp_j \, .$$

The proof of the theorem can be found e.g. in [LM87] or [Put93].

The q^j, p_j are called (local) CANONICAL COORDINATES. Note, that the chart $\Phi = (q, p) : U \to V$ is a canonical transformation from $(U, \omega|_U)$ to $(V, \omega'|_V)$, where ω' is the standard symplectic form on $T^*\mathbb{R}^n$, $\omega' = dq^j \wedge dp_j$ or $\omega'(X, Y) = X^\top J\sigma Y$ (cf. Definition 1.3). As a consequence, $(U, \omega|_U)$ is symplectomorphically equivalent to the open subspace (V, ω') of $T^*\mathbb{R}^n$ with the standard symplectic form.

Remark 1.20. 1. This result is in sharp contrast to Riemannian geometry: In case of a semi-Riemannian manifold (M, g) in every point one can find a chart q such that the metric tensor $g(a)$ has the form $\sum_{j=0}^n \eta_j dq^j \otimes dq^j$ with $\eta_j \in \{+1, -1\}$. In general, this cannot be achieved in a full neighbourhood of a. The measure of this deviation from the "flat" case is the curvature of the Riemannian manifold at $a \in M$. In this sense a symplectic manifold has no curvature, it is locally flat.

2. Moreover, this result allows to transfer the notions of Hamiltonian vector fields X_H and that of Poisson brackets $\{\ ,\ \}$ locally to a symplectic manifold. That these notions have a global description will be shown in Subsection 1.2.3 below.

3. As we will see in the next subsection, a symplectic manifold serves as a general phase space for Hamiltonian Mechanics. The case of a cotangent bundle $M = T^*Q$ will be called MOMENTUM PHASE SPACE with respect to the configuration space Q. In $M = T^*Q$ we have a special class of canonical coordinates, the bundle charts

generated by the charts of Q: To each chart $q : U \to V$ we have the bundle chart $(q^1, \dots, q^n, p_1, \dots, p_n) : T^*U \to V \times \mathbb{R}^n$, where

$$p_j(\mu) := \mu\left(\frac{\partial}{\partial q^j}\right) \ , \ \mu \in T^*U \, .$$

The p_j are called Generalized Momenta, and this explains why T^*Q is called momentum phase space. For a general symplectic manifold we have local canonical coordinates q, p according the the theorem of Darboux. But these q and p are interchangeable and neither of the two can be regarded to describe momenta.

Since in the case of a symplectic manifold the symplectic form ω is assumed to be non-degenerate, at each point $a \in Q$ we obtain vector space isomorphisms according to Definiton 1.14:

$$\omega^\flat(a) : T_aM \to T_a^*M \, , \ X \mapsto (Y \mapsto \omega(X,Y)) \ , \ X, Y \in T_aM \, ,$$

and its inverses $\omega^\sharp := (\omega^\flat)^{-1}$. These isomorphisms induce a vector bundle isomorphisms $\omega^\flat : TM \to T^*M$ and $\omega^\sharp : T^*M \to TM$.

The following proposition is easy to show

Proposition 1.21. *Let $\omega \in \mathcal{A}^2(M)$ be a 2-form. The following conditions are equivalent:*

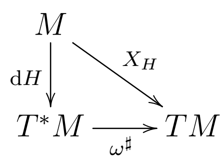
1. *ω is non-degenerate*
2. *For every $X \in \mathfrak{V}(M)$: X vanishes everywhere $\iff$ $\omega(X,Y) = 0$ for all $Y \in \mathfrak{V}(M)$.*
3. *$\omega^\flat : TM \to T^*M$ is a vector bundle isomorphism.*
4. *$\omega^n = \omega \wedge \omega \wedge \ldots \wedge \omega$ is a nowhere vanishing $2n$-form. Thus it is a volume form.*

1.2.3 Hamiltonian Systems

Definition 1.22. Let (M, ω) be a symplectic manifold. To every observable $H \in \mathcal{E}(M)$ there corresponds the Hamiltonian Vector Field

$$X_H := \omega^\sharp \circ dH.$$

The diagram



is commutative and illustrates the definition of X_H.

Observation 1.23. $X_H := \omega^\sharp \circ dH$ implies $\omega^\flat \circ X_H = dH$. Hence, X_H is also determined as being the unique vector field satisfying

$$\omega(X_H, Y) = \mathrm{d}H(Y) \quad \text{for all} \quad Y \in \mathfrak{V}(M).$$

Proposition 1.24. *In local canonical coordinates* $(q,p) : U \to V$ *the Hamiltonian vector field* X_H *can be written as*

$$\boxed{X_H|_U = \frac{\partial H}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial H}{\partial q^j}\frac{\partial}{\partial p_j}.} \tag{9}$$

Proof. In these local canonical coordinates $\omega|_U$ has the form $dq^j \wedge dp_j$ (cf. Theorem 1.19). With the use of the representation

$$Y = Y^j \frac{\partial}{\partial q^j} + \bar{Y}_j \frac{\partial}{\partial p_j}$$

of vector fields Y on U we deduce

$$\omega^\flat\left(\frac{\partial}{\partial q^j}\right)(Y) = \omega\left(\frac{\partial}{\partial q^j}, Y\right) = \bar{Y}_j,$$

which implies

$$\omega^\flat\left(\frac{\partial}{\partial q^j}\right) = dp_j.$$

Similarly, using

$$\omega^\flat\left(\frac{\partial}{\partial p_j}\right)(Y) = -Y^j$$

one deduces

$$\omega^\flat\left(\frac{\partial}{\partial p_j}\right) = -dq^j.$$

As a consequence, $\omega^\sharp$ acts with respect to the basis (dq^j, dp_k) of T_a^*U as

$$\omega^\sharp(dq^j) = -\frac{\partial}{\partial p_j}\,;\ \omega^\sharp(dp_k) = \frac{\partial}{\partial q^k}\,.$$

Thus, in local canonical coordinates $\omega^\sharp$ has the form

$$\omega^\sharp(\alpha_j dq^j + \bar{\alpha}^k dp_k) = \bar{\alpha}^k \frac{\partial}{\partial q^k} - \alpha_j \frac{\partial}{\partial p_j}. \tag{10}$$

(This resembles the symplectic involution $\sigma(\alpha, \bar{\alpha}) = (\bar{\alpha}, -\alpha)$, cf. formula (3)). Now,

$$dH = \frac{\partial H}{\partial q^j} dq^j + \frac{\partial H}{\partial p_k} dp_k$$

gives the desired result

$$X_H|_U = \omega^\sharp\left(\frac{\partial H}{\partial q^j}dq^j + \frac{\partial H}{\partial p_k}dp_k\right) = \frac{\partial H}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial H}{\partial q^j}\frac{\partial}{\partial p_j}.$$

□

Definition 1.25. (X, ω, H) is called a HAMILTONIAN SYSTEM whenever ω is a symplectic form. A MOTION of the Hamiltonian system is a curve $z \in \mathcal{E}(I, U)$ on an open interval I satisfying

$$\boxed{\dot{z} = X_H(z)\,,}$$

where $\dot{z}(t) := [z]_{z(t)}$ is the tangent vector given by the curve z in the point $z(t)$.

Using

$$\dot{z}(t) = (\dot{q}(t), \dot{p}(t)) = \frac{dq^j}{dt}\frac{\partial}{\partial q^j} + \frac{dp_k}{dt}\frac{\partial}{\partial p_k}$$

in local canonical coordinates the preceding Proposition 1.24 immediately implies:

Corollary 1.26. *In local canonical coordinates* (q, p) *the equations of motion for* $z(t) =: (q(t), p(t))$ *have the form*

$$\boxed{\dot{q} = \frac{\partial H}{\partial p}\,,\ \dot{p} = -\frac{\partial H}{\partial q}.}$$

As before in the simple case where $M = T^*U$, $U \subset \mathbb{R}^n$, the symplectic form induces Poisson brackets on a general symlpectic manifold (M, ω):

Definition 1.27. The POISSON BRACKET on $\mathcal{E}(M)$, given by the symplectic form $\omega \in \mathcal{A}^2(M)$, is defined as

$$\{F, G\} := \omega(X_F, X_G)\,,\ F, G \in \mathcal{E}(M).$$

Proposition 1.28 (Equations of motion in Poisson form)**.** *The equations of motion in case of a Hamiltonian system can again be written in the so called* POISSON FORM

$$\boxed{\dot{F} = \{F, H\}.}$$

Proof. In local canonical coordinates $(q, p) : U \to V \subset \mathbb{R}^n \times \mathbb{R}^n$ (cf. theorem 1.19) the Poisson bracket has the form

$$\boxed{\{F, G\}|_U = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q} = \frac{\partial F}{\partial q^j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q^j}} \tag{11}$$

which is well-known from the simple case. Hence, the proof reduces to the proof of Proposition 1.5. □

Corollary 1.29. *An observable $F \in \mathcal{E}(M)$ is a first integral of the Hamiltonian system (M, ω, H) if and only if $\{F, H\} = 0$.*

1.2.4 Hamiltonian Vector Fields

We study relations between the Poisson brackets and Hamiltonian vector fields. The fact that the Poisson bracket satisfies the Jacobi identity and therefore induces on $\mathcal{E}(M)$ the structure of a Lie algebra is of fundamental importance for the program of Geometric Quantization. We provide two proofs of this result.

Theorem 1.30. *The Poisson bracket $\{\ ,\ \} : \mathcal{E}(M) \times \mathcal{E}(M) \to \mathcal{E}(M)$ of a symplectic manifold (M, ω) is a Lie bracket, in other words $\mathcal{E}(M)$ with the Poisson bracket is a Lie algebra over $\mathbb{R}$*[3]*, the* Poisson Algebra*, that is*

1. *$\{\ ,\ \}$ is bilinear over $\mathbb{R}$.*
2. *$\{F, G\} = -\{G, F\}$ for $F, G \in \mathcal{E}(M)$, i.e. $\{\ ,\ \}$ is alternating.*
3. *$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$ for $F, G, H \in \mathcal{E}(M)$. (*Jacobi Identity*)*

   In addition:

4. *$\{F, GH\} = G\{F, H\} + \{F, G\}H = G\{F, H\} + H\{F, G\}$ (*Product Rule*). Equivalently, $\{F,\ \}$ is a derivation on $\mathcal{E}(M)$.*
5. *For connected M: $G \in \mathcal{E}(M)$ is constant, iff $\{F, G\} = 0$ for all $F \in \mathcal{E}(M)$ (*Completeness*).*
   In general: $\{F, G\} = 0$ for all $F \in \mathcal{E}(M)$ iff $dG = 0$.

Proof. 1. and 2. are immediately clear.

First proof of 3.: We apply formula (11) to obtain in local canonical coordinates

$$\begin{aligned}
\{F, \{G, H\}\} &= \frac{\partial F}{\partial q}\frac{\partial}{\partial p}\{G, H\} - \frac{\partial F}{\partial p}\frac{\partial}{\partial q}\{G, H\} \\
&= \frac{\partial F}{\partial q}\frac{\partial}{\partial p}\left(\frac{\partial G}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial G}{\partial p}\frac{\partial H}{\partial q}\right) - \frac{\partial F}{\partial p}\frac{\partial}{\partial q}\left(\frac{\partial G}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial G}{\partial p}\frac{\partial H}{\partial q}\right) \\
&= \frac{\partial F}{\partial q}\left(\frac{\partial^2 G}{\partial p \partial q}\frac{\partial H}{\partial p} + \frac{\partial G}{\partial q}\frac{\partial^2 H}{\partial p^2} - \frac{\partial^2 G}{\partial p^2}\frac{\partial H}{\partial q} - \frac{\partial G}{\partial p}\frac{\partial^2 H}{\partial p \partial q}\right) \\
&\quad - \frac{\partial F}{\partial p}\left(\frac{\partial^2 G}{\partial q^2}\frac{\partial H}{\partial p} + \frac{\partial G}{\partial q}\frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 G}{\partial q \partial p}\frac{\partial H}{\partial q} - \frac{\partial G}{\partial p}\frac{\partial^2 H}{\partial q^2}\right)
\end{aligned}$$

[3] the Lie algebra properties are 1.-3.

In the same way we get expressions for $\{G,\{H,F\}\}$ and $\{H,\{F,G\}\}$. Summing up all the terms one sees that the Jacobi identity is satisfied.

Second proof of 3.: A more conceptual proof which, moreover, does not use local canonical coordinates, is the following: We introduce the cyclic summation $\sum_{ijk} T_{ijk}$ of summable terms as $\sum_{ijk} T_{ijk} := T_{ijk} + T_{jki} + T_{kij}$. In particular, he Jacobi identity has the form

$$\sum_{FGH} F, G, H = 0\,.$$

The exterior derivative of ω vanishes, hence, from

$$\begin{aligned} 0 &= d\omega(X_F, X_G, X_H) \\ &= L_{X_F}(\omega(X_G, X_H)) - L_{X_G}(\omega(X_F, X_H)) + L_{X_H}(\omega(X_F, X_G)) \\ &\quad - \omega(([X_F, X_G], X_H) + \omega(([X_F, X_H], X_G) - \omega(([X_G, X_H], X_F) \end{aligned}$$

we obtain

$$\begin{aligned} 0 &= L_{X_F}(\omega(X_G, X_H)) - \omega(([X_F, X_G], X_H) \\ &\quad + L_{X_G}(\omega(X_H, X_F)) - \omega(([X_G, X_H], X_F) \\ &\quad + L_{X_H}(\omega(X_F, X_G)) - \omega(([X_H, X_F], X_G) \\ &= \sum\nolimits_{FGH} L_{X_F}(\omega(X_G, X_H)) - \omega(([X_F, X_G], X_H)\,. \end{aligned}$$

Now,

$$L_{X_F}(\omega(X_G, X_H)) = \{\omega(X_G, X_H), F\} = \{\{G, H\}, F\} = -\{F, \{G, H\}\}$$

since, in general,

$$L_{X_F} I = dI(X_F) = \omega(X_I, X_F) = \{I, F\}\,,$$

for a function $I \in \mathcal{E}(M)$. Applying this again we obtain

$$\begin{aligned} -\omega([X_F, X_G], X_H) &= L_{[X_F, X_G]} H = L_{X_F} L_{X_G} H - L_{X_G} L_{X_F} H \\ &= \{\{H, G\}, F\} - \{\{H, F\}, G\} \\ &= \{F, \{G, H\}\} + \{G, \{H, F\}\} \end{aligned} \tag{12}$$

Implementing these identities in the above cyclic sum gives

$$\begin{aligned} 0 &= \sum\nolimits_{FGH} L_{X_F}(\omega(X_G, X_H)) - \omega(([X_F, X_G], X_H) \\ &= \sum\nolimits_{FGH} -\{F, \{G, H\} + \{F, \{G, H\}\} + \{G, \{H, F\}\} \\ &= \sum\nolimits_{FGH} \{G, \{H, F\} = \sum\nolimits_{FGH} \{F, \{G, H\}. \end{aligned}$$

And this is the Jacobi identity!

The statement in 4. follows immediately from the chain rule $d(GH) = HdG + GdH$.

To show 5., observe that for $G \in \mathcal{E}(M)$ the condition $\{F, G\} = \omega(X_F, X_G) = 0$ for all $F \in \mathcal{E}(M)$ is equivalent to $X_G = 0$ by the non-degeneracy of ω and this is in turn equivalent to $dG = 0$. □

The statement of 3. in the preceding theorem is essentially equivalent to the next proposition.

Corollary 1.31. *The mapping*

$$\Phi : \mathcal{E}(M) \to \mathfrak{V}(M)\,,\ F \mapsto -X_F\,,$$

is a Lie algebra homomorphism, i.e. Φ *is* $\mathbb{R}$*-linear and satisfies*

$$\Phi(\{F,G\}) = -X_{\{F,G\}} = [X_F, X_G] = [\Phi(F), \Phi(G)]\,.$$

Proof. For $F, G, H \in \mathcal{E}(M)$ we just have shown in formula (12):

$$L_{[X_F,Y_G]}\,H = \{F, \{G,H\}\} + \{G, \{H,F\}\}.$$

By the Jacobi identity this is $-\{H, \{F,G\}\}$ and we conclude

$$[X_F, X_G]\,H = -X_{\{F,G\}}H.$$

□

Corollary 1.32. *The Lie bracket of two first integrals is again a first integral.*

Proof. Let F, G be first integrals of (M, ω, H). Then $\{G,H\} = \{F,H\} = 0$ (cf. Corollary 1.29). By the Jacobi identity,

$$\{\{F,G\},H\} = -\{H,\{F,G\}\} = \{F,\{G,H\}\} + \{G,\{H,F\}\} = 0.$$

As a consequence, $\{F,G\}$ is a first integral by Corollary 1.29. □

Observation 1.33. The second proof of the Jacobi identity (cf. proof of 3. in Theorem 1.30) yields more than merely the identity. Note, that for a non-degenerate (and not necessarily closed) two-form $\omega \in \mathcal{A}^2(M)$ on a manifold M the generation of Hamiltonian vector fields X_H and the introduction of the Poisson bracket is possible in the same way as it is done in the preceding subsections. The above mentioned second proof of 3. in Theorem 1.30 now shows that for $F, G, H \in \mathcal{E}(M)$ the statement $d\omega(X_F, X_G, X_H) = 0$ is equivalent to F, G, H satisfying the Jacobi identity, i.e. $\sum_{FGH}\{F,\{G,H\}\} = 0$. Since the Hamiltonian vector fields generate the tangent spaces T_aM we have proven the following remarkable result:

Proposition 1.34. *A manifold* M *with a non-degenerate* $\omega \in \mathcal{A}^2(M)$ *is symplectic if and only if the Poisson bracket induced by* ω *satisfies the Jacobi identity.*

Observation 1.35. As a result, the Hamiltonian vector fields

$$\mathfrak{Ham}(M) := \{X_F \mid F \in \mathcal{E}(M)\}$$

form a Lie subalgebra of the Lie algebra $\mathfrak{V}(M)$ of vector fields. The kernel $\operatorname{Ker}\Phi \subset \mathcal{E}(M)$ consists of the locally constant functions. Hence, for connected manifolds M one has $\mathbb{R} = \operatorname{Ker}\Phi$ and one obtains the following exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{E}(M) \xrightarrow{\Phi} \mathfrak{Ham}(M) \longrightarrow 0,.$$

This exact sequence exhibits the Lie algebra $\mathcal{E}(M)$ as a CENTRAL EXTENSION of the Lie algebra $\mathfrak{Ham}(M)$ of Hamiltonian vector fields.

Locally Hamiltonian vector fields form a Lie algebra (subalgebra of $\mathfrak{V}(M)$) as well and have $\mathfrak{Ham}(M)$ as an ideal.

1.3 Examples of Hamiltonian Systems

1.3.1 Harmonic Oscillator

In this case the phase space reads:

$$M = T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}.$$

The symplectic form is given by:

$$\omega = dq^j \wedge dp_j,$$

The hamiltonian (total energy) takes the form:

$$H(q,p) = \frac{1}{2}\left(||q||^2 + ||p||^2\right), \quad X_H = p_j \frac{\partial}{\partial q^j} - q^j \frac{\partial}{\partial p_j}.$$

The equations of motion are well known:

$$\dot{q} = p, \dot{p} = -q.$$

H is a first integral, that is, every motion $(q,p) : I \to M$ with $H(q(t_0), p(t_0)) = E \geq 0$ satisfies

$$H(q(t), p(t)) = E \quad \forall t \in I.$$

Hence, the points $(q(t), p(t))$ of the solution (q,p) remain in the hypersurface

$$\Sigma_E = H^{-1}(E)$$

for all $t \in I$. Σ_E is in fact a submanifold of dimension $2n-1$ since $\nabla H = (q,p) \neq 0$ for $E > 0$. We see

$$\Sigma_E = \left\{(q,p) \mid ||q||^2 + ||p||^2 = 2E\right\} = \mathbb{S}^{2n-1}\left(\sqrt{2E}\right),$$

where

$$\mathbb{S}^{k-1}(r) := \left\{x \in \mathbb{R}^k \mid \ ||x||^2 = r^2\right\}$$

denotes the $(k-1)$-sphere of radius r in $\mathbb{R}^k$. This example is a reduction in the sense of the following subsection.

1.3.2 Reduction with respect to first integrals

Let F be a first integral of a Hamiltonian system (M, ω, H), i.e. $F \in \mathcal{E}(M)$ and $\{F, H\} = 0$ (c.f. Corollary 1.29). Let $c \in \mathbb{R}$ be a value with

$$\Sigma_c := F^{-1}(c) \neq \varnothing ,$$

i.e. $c \in F(M)$. Assume that the level set Σ_c is a smooth hypersurface, this holds e.g. if $\nabla F \neq 0$ on Σ_c. Then the space of orbits with $F = c$ is the quotient

$$O_c := \Sigma_c / \sim,$$

with respect to the equivalence relation

$$a \sim b \iff \exists \text{ motion } x : I \to \Sigma_c \text{ with } x(t_1) = a \text{ and } x(t_2) = b \text{ for } t_1, t_2 \in I.$$

Assume, moreover, that the orbit space O_c has a differentiable structure, that is, the differentiable quotient exists as a manifold (cf. A.10). Then it is a $(2n-2)$-dimensional manifold. Furthermore, assume that $\omega|_{\Sigma_c}$ induces on O_c a natural symplectic form $\omega_c \in \mathcal{A}^2(O_c)$ (such that $\omega|_{O_c} = \pi^*(\omega_c)$ for the projection $\pi : \Sigma_c \to O_c$).
Since the hamiltonian H is constant on the orbits it descends to O_c as $H_c \in \mathcal{E}(O_c)$ with $H = H_c \circ \pi$ on Σ_c.
As a result, the original system (M, ω, H) has been reduced (by one degree of freedom) to (O_c, ω_c, H_c). In general, this procedure can be repeated. In good cases ("completely integrable systems") one can go down to n-dimensional reductions, which then gives the solution.

In case of the harmonic oscillator of dimension n (see above), the orbit space O_E[4]

$$\Sigma_E / \sim = \mathbb{S}^{2n-1}\left(\sqrt{2E}\right) / \sim$$

is isomorphic to the complex projective space $\mathbb{P}^{n-1}(\mathbb{C})$ of all complex lines going through the origin $0 \in \mathbb{C}^n$ in $\mathbb{C}^n$. And the symplectic form ω_E is the usual Kähler form on $\mathbb{P}^{n-1}(\mathbb{C})$.

Here, we introduce complex coordinates (resp. the structure of a complex vector space on $\mathbb{R}^n \times \mathbb{R}^n$) by defining $z := p + iq, z^j := p_j + iq^j$. Observe that the canonical equations $(\dot{q}, \dot{p}) = (p, -q)$ are now

$$\dot{z} = iz.$$

Note, that multiplication by i is the symplectic involution of Subsection 1.1: $i = \sigma$.

The motions are $z(t) = e^{it} z_0 \; t \in \mathbb{R}$, where $z_0 = z(0) \in \mathbb{C}^n$. Moreover, the observables

$$H_j := \frac{1}{2}\left(p_j^2 + (q^j)^2\right) = \frac{1}{2}\left\|z^j\right\|^2 = \frac{1}{2} z^j \bar{z}^j, j = 1, \ldots, n, \quad \text{(no summation!)}$$

[4] The equivalence relation given by the orbits is in this case also given by the action of the group $\mathrm{U}(1) \cong \mathbb{S}^1$, so that $O_E = \mathbb{S}^{2n+1}/\mathrm{U}(1)$.

on $\mathbb{R}^{2n}$ are first integrals:

$$\begin{aligned}\frac{d}{dt}H_j(z(t)) &= \frac{1}{2}(\dot{z}^j\bar{z}^j + z^j\dot{\bar{z}}^j)\\ &= \frac{1}{2}(iz^j\bar{z}^j + z^j(-i\bar{z}^j)) = 0.\end{aligned}$$

With the values $E = \sum E_j$, $E_j = H_j(z(t_0))$, for some $t_0 \in I$, and $\vec{E} := (E_1, \dots, E_n)$, we obtain the level set for $(H_1, \dots, H_n)$, an n-dimensional manifold,

$$M_{\vec{E}} = \bigcap_{j=1}^{n} H_j^{-1}(E_j) = \prod_{j=1}^{n} \mathbb{S}^1\left(\sqrt{2E_j}\right).$$

which is an n dimensional torus. Again, the level $M_{\vec{E}}$ set is invariant in the sense that the motion $z(t)$ remains in $M_{\vec{E}}$, if $z(t_0) \in M_{\vec{E}}$.

This "reduction" gives a complete solution: Every motion $z = z(t)$ satisfies $z^j(t) \in \mathbb{S}\left(\sqrt{2E_j}\right)$ for $j = 1, \dots, n$ and all $t \in I$. It is determined by $z(t_0)$, and if $t_0 = 0$ it will be of the form

$$z^j(t) = e^{it}z^j(0) = \left(\cos t\, p_j(0) - \sin t\, q^j(0) + i(\sin t\, p_j(0) + \cos(t)\, q^j)\right)$$

or

$$\left(q^j(t), p_j(t)\right) = \left(\cos t\, q^j(0) + \sin t\, p_j(0), \cos t\, p_j(0) - \sin t\, q^j(0)\right).$$

This is a rather simple example of reduction of a completely integrable system. A COMPLETELY INTEGRABLE SYSTEM is a hamiltonian system of dimension $2n$ with n first integrals in involution, i.e. $\{F_j, F_k\} = 0$ for $1 \le j, k \le n$ which are independent, i.e. the map $F = (F_1, \dots, F_n) : M \to \mathbb{R}^n$ is of rank n. The theorem of Arnold-Liouville says that in case the level sets are compact they are finite unions of tori.

1.3.3 Kepler Problem (Hydrogen Atom)

In this case the configuration and phase space is given by:

$$Q = \mathbb{R}^3 \setminus \{0\}, M = T^*Q = Q \times \mathbb{R}^3.$$

The symplectic form is the usual one:

$$\omega = dq^j \wedge dp_j.$$

And the hamiltonian reads:

$$H(q,p) = \frac{1}{2m}||p||^2 - \frac{k}{||q||}, \quad m, k > 0.$$

We have

$$\nabla H = \left(\frac{kq}{||q||^3}, -\frac{p}{m} \right) \neq 0$$

in all of M. Hence, the energy hypersurface

$$\Sigma_E = H^{-1}(E)$$

is a smooth submanifold of dimension 5 for all $E \in \mathbb{R}$.

Let $E \in \,]-\infty, 0\,[$. The orbits in Σ_E are ellipses and one can show (cf. Example 7.7), that the orbit space $O_E = \Sigma_E/\sim$ is isomorphic (as a differentiable manifold) to $\mathbb{S}^2(mk) \times \mathbb{S}^2(mk)$. The symplectic form ω descends to a form ω_E on $\Sigma_E/\sim$. And on $S_E := \mathbb{S}^2(mk) \times \mathbb{S}^2(mk)$ it has the form

$$\omega_E = \frac{1}{2\rho} \left(\frac{dx_1 \wedge dx_2}{x_3} + \frac{dy_1 + dy_2}{y_3} \right),$$

with respect to the chart with $x_3 \neq 0 \neq y_3$ on S_E. Here $\rho = \sqrt{-2mE}$.

It is interesting to ask which energy values occur if we quantize the system (S_E, ω_E) according to the program of Geometric Quantization.
In Example 7.7 we show: After adjusting the constants the energy levels – predicted by Geometric Quantization – are $E_N = -2\pi^2 m\hbar^2 N^{-2}, N \in \mathbb{N}, N \geq 1$, the values known for the hydrogen atom from experiments!

1.3.4 Particle in a Field

Twist of the Cotangent Bundle

In this general example the phase space is the cotangent bundle

$$M = T^*Q.$$

of an n-dimensional manifold Q. However, the symplectic structure is not given by the previously considered standard form

$$\omega_0 = dq^j \wedge dp_j\,,$$

but by a Twist

$$\omega_F = \omega := \omega_0 + \tau^* F\,,$$

where $F \in \mathcal{A}^2(Q)$ is a closed two-form and $\tau^* F$ is the pull-back with respect to the natural projection $\tau : T^*Q \cong Q \times \mathbb{R}^n \to Q$. This change of the symplectic structure is sometimes called Deformation of the Symplectic Structure.

As a special case we describe a Relativistic Charged Particle in the following example

Example 1.36. The configuration space is a spacetime Q with Lorentzian metric g, for instance, an open subset Q of the Minkowski space $\mathbb{R}^4$. The symplectic manifold (T^*Q, ω_0) is the phase space for a relativistic particle. The function

$$H := \frac{1}{2}\, g(p,p)\,,\, p \in T^*Q\,,$$

determines the dynamics of a relativistic particle by its Hamiltonian equations.

Let us assume that in addition to the above structure an electromagnetic field in form of a closed 2-form $F \in \mathcal{A}^2(Q)$ on Q is present. Then the Hamiltonian dynamics is given by the Hamiltonian vector field X_H with the same Hamiltonian function H but with respect to the modified symplectic form

$$\omega := \omega_0 + e\tau^*(F)\,,$$

where e is the charge of the particle.

The configuration space Q can also be a Riemannian manifold with metric tensor g, and for F one can take a geometrically induced closed 2-form.

Since F is closed, $\tau^*(F)$ is closed as well, and consequently ω_F is closed. This shows one part of the following assertion.

Proposition 1.37. *For a two-form $F \in \mathcal{A}^2(Q)$ the twisted two-form $\omega_F := \omega_0 + \tau^* F$ is a symplectic form on $T^*Q = M$ if and only if F is closed.*

Proof. We just have seen, that ω_F is closed, whenever F is closed. And, of course, when ω_F is closed, $\tau^*(F)$ has to be closed and, in turn, F is closed.

To investigate the non-degeneracy we use local coordinates q in an open subset U of Q and see

$$F|_U = F_{jk} dq^j \wedge dq^k\,,$$

where $F_{jk} \in \mathcal{E}(U)$ are suitable functions. On $V := \tau^{-1}(U) = U \times \mathbb{R}^n$ this implies with respect to the canonical coordinates (q,p) (of the bundle chart tu q)

$$\begin{aligned}\omega_F|_V &= \omega_0|_V + \tau^*(F)|_V = dq^j \wedge dp_j + \tau^*(F_{jk} dq^j \wedge dq^k) \\ &= dq^j \wedge dp_j + \tau^* F_{jk} dq^j \wedge dq^k = dq^j \wedge dp_j + (F_{jk} \circ \tau) dq^j \wedge dq^k\end{aligned}$$

The action of ω_0 on the tangent space $T_a M \cong \mathbb{R}^n \times \mathbb{R}^n$ at $a \in V$ is given by the block matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and, by the above local expression of ω_F, the corresponding action of ω_F at a is given by the block matrix

$$\begin{pmatrix} F(a) & 1 \\ -1 & 0 \end{pmatrix},$$

where $F(a)$ is the matrix $(F_{jk}(\tau(a)))$. This expression shows that ω_F is non-degenerate and the proposition is proven. □

Twisted Canonical Coordinates

By Darboux's theorem every symplectic form can locally be written as $d\tilde{q}^k \wedge d\tilde{p}_k$ with respect to local canonical coordinates $\tilde{q}^j, \tilde{p}_j$. Let us find local canonical coordinates in the case of the twisted symplectic form ω_F on $M = T^*Q$. Locally, on suitable open subsets $U \subset Q$ the closed form F can be expressed as $F|_U = dA$, where $A \in \mathcal{A}^1(U)$ is a one-form with $A = A_j dq^j$. Then, in the coordinates (q, p) of the bundle chart, which are canonical coordinates of ω_0, we have:

$$\begin{aligned}
\omega_F|_U &= dq^k \wedge dp_k + \tau^* dA = dq^k \wedge dp_k + \tau^* d(A_k dq^k) \\
&= dq^k \wedge dp_k + \tau^*(dA_k \wedge dq^k + 0) = dq^k \wedge dp_k + \tau^*(dA_k) \wedge \tau^* dq^k \\
&= dq^k \wedge dp_k + d(\tau^* A_k) \wedge d(\tau^* q^k) = dq^k \wedge dp_k + d(\tau^* A_k) \wedge dq^k \\
&= dq^k \wedge dp_k - dq^k \wedge d(\tau^* A_k) = dq^k \wedge d(p_k - \tau^* A_k).
\end{aligned}$$

This result implies, that the following definition yields canonical coordinates for the twisted symplectic structure.

$$\tilde{q}^k = q^k, \quad \tilde{p}_k = p_k - \tau^* A_k. \tag{13}$$

The formulas can be interpreted as describing a particle in a generalized magnetic field, where A corresponds to the vector potential of electrodynamics.

Twisted Hamiltonian Vector Field

We want to determine the Hamiltonian vector fields corresponding to $\omega_F = \omega$ and to give a physical interpretation of the twist. We use local canonical coordinates (q, p) with respect to the standard 2-form $\omega_0 = dq^j \wedge dp_j$. Let $H \in \mathcal{E}(M)$. The Hamiltonian vector field related to ω_0 will be denoted by X_H^0 and the one related to ω will be denoted by X_H. According to Proposition 1.24 X_H^0 takes the form

$$X_H^0\big|_U = \frac{\partial H}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial H}{\partial q^j}\frac{\partial}{\partial p_j}. \tag{14}$$

ω can be expanded locally as:

$$\omega|_U = dq^j \wedge dp_j + (F_{jk} \circ \tau) dq^j \wedge dq^k.$$

In the following, we omit "$\circ\,\tau$" in order to have simpler formulas and understand F_{jk} as functions on open subsets $U \subset M$. Similarly, we write A_k instead of $\tau^*(A_k) = A_k \circ \tau$.

The (local) transformation

$$(\tilde{q}, \tilde{p}) \mapsto (q, p) = G(\tilde{q}, \tilde{p}) = (\tilde{q}, \tilde{p} + A)\,,$$

which is the inverse of (13), leads to the following identity for functions $f \in \mathcal{E}(U)$:

$$\frac{\partial f}{\partial \tilde{q}^k} = \frac{\partial f}{\partial q^j}\frac{\partial G^j}{\partial \tilde{q}^k} + \frac{\partial f}{\partial p_j}\frac{\partial G_j}{\partial \tilde{q}^k} = \frac{\partial f}{\partial q^j}\frac{\partial \tilde{q}^j}{\partial \tilde{q}^k} + \frac{\partial f}{\partial p_j}\frac{\partial(\tilde{p}_j + A_j)}{\partial \tilde{q}^k} = \frac{\partial f}{\partial q^k} + \frac{\partial f}{\partial p_j}\frac{\partial A_j}{\partial q^k}$$

since

$$\frac{\partial A_j}{\partial \tilde{q}^k} = \frac{\partial A_j}{\partial q^k} .$$

As a consequence,

$$\frac{\partial}{\partial \tilde{q}^k} = \frac{\partial}{\partial q^k} + \frac{\partial A_j}{\partial q^k}\frac{\partial}{\partial p_j} .$$

In the same way, we obtain

$$\frac{\partial}{\partial \tilde{p}_k} = \frac{\partial}{\partial p_k} .$$

Applying these identities, the Hamiltonian X_H can be expressed in the canonical coordinates related to ω_0:

$$\begin{aligned} X_H &= \frac{\partial H}{\partial \tilde{p}_k}\frac{\partial}{\partial \tilde{q}^k} - \frac{\partial H}{\partial \tilde{q}^j}\frac{\partial}{\partial \tilde{p}_j} \\ &= \frac{\partial H}{\partial p_k}\left(\frac{\partial}{\partial q^k} + \frac{\partial A_j}{\partial q^k}\frac{\partial}{\partial p_j}\right) - \left(\frac{\partial H}{\partial q^j} + \frac{\partial A_k}{\partial q^j}\frac{\partial H}{\partial p_k}\right)\frac{\partial}{\partial p_j} \\ &= \left(\frac{\partial H}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial H}{\partial q^j}\frac{\partial}{\partial p_j}\right) + \left(\frac{\partial H}{\partial p_k}\frac{\partial A_j}{\partial q^k} - \frac{\partial A_k}{\partial q^j}\frac{\partial H}{\partial p_k}\right)\frac{\partial}{\partial p_j} \\ &= X_H^0 + \frac{\partial H}{\partial p_k}\left(F_{kj} - F_{jk}\right)\frac{\partial}{\partial p_j}. \end{aligned}$$

We have shown:

Lemma 1.38.

$$X_H - X_H^0 = \frac{\partial H}{\partial p_k}\left(F_{kj} - F_{jk}\right)\frac{\partial}{\partial p_j} .$$

And the equations of motion in (M, ω_F, H) with respect to the coordinates of the bundle chart have the following form:

Proposition 1.39.

$$\dot{q}^k = \frac{\partial H}{\partial p_k} \ , \ \ \dot{p}_k = -\frac{\partial H}{\partial q^k} + \left(F_{jk} - F_{kj}\right)\frac{\partial H}{\partial p_j} .$$

Proof. It is easy to see that

$$\dot{q}^k = \dot{\tilde{q}}^k = \frac{\partial H}{\partial \tilde{p}_k} = \frac{\partial H}{\partial p_k} .$$

The second set of equations reads

$$\dot{\tilde{p}}_k = -\frac{\partial H}{\partial \tilde{q}^k} = -\frac{\partial H}{\partial q^k} - \frac{\partial A_j}{\partial q^k}\frac{\partial H}{\partial p_j} .$$

Moreover,

$$\dot{\tilde{p}}_k = \dot{p}_k - \frac{d}{dt}A_k = \dot{p}_k - \frac{\partial A_k}{\partial q^j}\dot{q}^j = \dot{p}_k - \frac{\partial A_k}{\partial q^j}\frac{\partial H}{\partial p_j}\,.$$

As a consequence,

$$\dot{p}_k = \frac{\partial A_k}{\partial q^j}\frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial q^k} - \frac{\partial A_j}{\partial q^k}\frac{\partial H}{\partial p_j} = -\frac{\partial H}{\partial q^k} + (F_{jk} - F_{kj})\frac{\partial H}{\partial p_j}\,.$$

□

Observation. As we have seen in the previous section, the Hamiltonian vector fields determine the classical equations of motion. Therefore, the difference $\Phi := X_H - X_H^0$ can be interpreted as a force. A possible physical interpretation: Φ looks like the magnetic part of a generalized Lorentz force. In dimension 3 it can rewritten as $v \times B$, as we see in the special example below.

Charged Particle in $\mathbb{R}^3$

Let $B = B^j dq^j \in \mathfrak{V}(\mathbb{R}^3)$ the divergence-free vector field representing the magnetic field, where q^j , $j = 1, 2, 3$, are the cartesian coordinates. The classical equations of motion for a particle with charge e and mass m are given by the Lorentz Force Law (we set $c = 1$):

$$m\frac{dv}{dt} = e(v \times B). \tag{15}$$

We recover this law as the equations of motion of a suitable Hamiltonian system which is given by the following following twisted symplectic form on $M = T^*\mathbb{R}^3 \cong \mathbb{R}^3 \times \mathbb{R}^3$:

$$\omega := \omega_0 + eF\,, \tag{16}$$

where

$$F := \frac{1}{2} i_{\vec{B}}(dq^1 \wedge dq^2 \wedge dq^3)\,,$$

such that

$$F = B^1 dq^2 \wedge dq^3 + B^2 dq^3 \wedge dq^1 + B^3 dq^1 \wedge dq^2.$$

Note, that F is closed, since B is assumed to be divergence-free.

The Hamiltonian system is (M, ω, H) with H as the kinetic energy. The kinetic energy is:

$$H = \frac{1}{2m}\left(p_1^2 + p_2^2 + p_3^2\right)\,,$$

where $p = mv$.

According to Proposition 1.39 the equations of motion include in particular

$$\dot{p}_k = -\frac{\partial H}{\partial q^k} + e\,(F_{jk} - F_{kj})\frac{\partial H}{\partial p_j}\,.$$

which is, in our case,

$$\dot{p}_k = e\left(F_{jk} - F_{kj}\right)\frac{1}{m}p_j\,.$$

These three equations are equivalent to (15). We check the case $k = 1$:

$$m\dot{v}_1 = \dot{p}_1 = e(F_{j1} - F_{1j})v_j = ev_j 2F_{j1} = e(v_2B^3 - v_3B^2) = e(v \times B)_1\,.$$

Remark 1.40. F has to be closed, which in this special case means that F is exact. On more general spaces, not every two-form, which is closed, need be exact. $F = dA$ holds true only locally and, in general, not globally. However, we can understand A as being a connection one-form on a U(1)-bundle or on a line bundle, and F as being the curvature 2-form of the connection. This already is the topic of gauge theory treated in the Chapters 3, 4 and 5.

1.3.5 Coadjoint Orbits

This class of examples of symplectic manifolds yields a close, but not obvious connection between the representation theory of Lie groups and Lie algebras and Geometric Quantization.

In the following (see Appendix, Chapter C):

- G is a connected Lie group of finite dimension (for instance a closed matrix group $G \subset \mathrm{GL}(k, \mathbb{R})$).
- $\mathfrak{g} = \mathrm{Lie}\, G$ is the associated Lie algebra.
- the *conjugation* with respect to $g \in G$ yields the smooth map

$$\tau_g : G \to G, x \mapsto gxg^{-1}, x \in G,$$

- and the *adjoint representation* $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ of the group G is defined as the derivative $T_e\tau_g$ of τ_g at the unit e of G;

$$\mathrm{Ad}_g := T_e\tau_g : T_eG = \mathfrak{g} \to \mathfrak{g} = T_eG.$$

  Ad_g is the map $X \mapsto gXg^{-1}$ in case of a matrix group G.

From $\mathrm{Ad}_{gh} = \mathrm{Ad}_g \circ \mathrm{Ad}_h$ for $g, h \in G$, one deduces that

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$

is a Lie group homomorphism.

Definition 1.41. The Coadjoint Representation is the "dual" or "adjoint" of the adjoint representation:

$$\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*),$$

which is given by:

$$\mathrm{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^*, \mu \to \mathrm{Ad}^*_g(\mu) \in \mathfrak{g}^*,$$

with

$$\mathrm{Ad}^*_g(\mu)(X) := \mu\left(\mathrm{Ad}_{g^{-1}}(X)\right)$$

for $\mu \in \mathfrak{g}^* = \{\nu : \mathfrak{g} \to \mathbb{R} \mid \mathbb{R}-\text{linear}\}$ and $X \in \mathfrak{g}$.

It is easy to check that $\mathrm{A}^*_{gh} = \mathrm{Ad}^*_g \circ \mathrm{Ad}^*_h, g, h \in G$, i.e. A^* is again Lie group homomorphism.

As a result, we have an action of the Lie group G on the dual of its own Lie algebra $\mathfrak{g}^*$

$$G \times \mathfrak{g}^* \to \mathfrak{g}^*, \;\; (g, \mu) \mapsto \mathrm{Ad}^*_g(\mu).$$

This is called the Coadjoint Action and has the orbits:

$$M_\mu := \left\{\mathrm{Ad}^*_g \mu \mid g \in G\right\}, \; \mu \in \mathfrak{g}^*.$$

One can show:

1. M_μ is a smooth submanifold of $\mathfrak{g}^* \cong \mathbb{R}^m$ with a natural symplectic form ω_μ and with symmetry group G.
2. Every symplectic manifold M on which G acts transitively by symplectomorphisms looks locally like an open part of smooth orbit M_μ. More precisely,M can be realized as a covering $M \to M_\mu$ of M_μ.

Here $\phi : (M, \omega) \to (M', \omega')$ is a Symplectomorphism (or canonical transformation), if ϕ is a diffeomorphism preserving the symplectic structures, i.e. with $\phi^*\omega' = \omega$.

1.4 Lagrangian Mechanics

In many situations a system of Classical Mechanics is given as a Lagrangian system. In the following we discuss when a Lagrangian system induces a Hamiltonian system in a natural way and vice versa.

The ingredients of a Lagrangian System (M, L) are:

An n-dimensional manifold Q as the Configuration Space.

The tangent bundle $M = TQ$, called the Velocity Phase Space.

And a function $L \in \mathcal{E}(M)$, called the Lagrangian.

Example 1.42. A NATURAL SYSTEM[5] is a Lagrangian system with a Lagrangian of the form $L(v) = \frac{1}{2}g(v,v) - U(\tau(v))$, $v \in TQ = M$, where g is a Riemannian metric on Q and $U \in \mathcal{E}(Q)$ is a so-called potential.

In local coordinates

$$v = v^j \frac{\partial}{\partial q^j} \quad \text{and} \quad g(v,w) = g_{jk}v^j w^k \,,\; g_{jk} = g_{jk}(\tau(v)) = g_{jk}(q) \,,$$

$v, w \in T_{\tau(v)}Q$. Hence,

$$\frac{1}{2}g(v,v) = \frac{1}{2}g_{jk}v^j v^k \,.$$

Definition 1.43. A curve $q : I \to Q$ on an interval $I \subset \mathbb{R}$ is called a MOTION of the Lagrangian system (M, L), if q satisfies the EULER-LAGRANGE EQUATIONS, that is:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}(\dot{q})\right) = \frac{\partial L}{\partial q}(\dot{q}) \text{ (in bundle coordinates) .}$$

Here,

$$\dot{q} = \frac{dq}{dt} \in TQ \,.$$

In the case of a natural system the equations have the form

$$\frac{d}{dt}\left(g_{jk}(q)\dot{q}^k\right) = \frac{\partial U}{\partial q^j}(q)$$

or

$$\frac{\partial g_{jk}}{\partial q^i}\dot{q}^i q^k + g_{jk}\ddot{q}^k = \frac{\partial U}{\partial q^j}(q) \,.$$

Fact 1.44. *In case of a natural system without potential, i.e.* $L(v) = \frac{1}{2}g(v,v)$ *the geodesics of the Riemannian manifold* (Q, g) *are essentially the motions in the level set* $L^{-1}(\frac{1}{2})$.

Construction 1.45. Given a chart

$$\phi = \left(q^1, \ldots, q^n\right) : U \to V \subset \mathbb{R}^n$$

with associated bundle chart

$$\tilde{\phi} = \left(q^1, \ldots, q^n, v^1, \ldots, v^n\right) : TU \to V \times \mathbb{R}^n$$

the local 1-form, called LIOUVILLE FORM induced by L is:

$$\lambda_L := \frac{\partial L}{\partial v^k}dq^k, \quad \left(\frac{\partial L}{\partial v^k} \text{ "generalized momenta"}\right)$$

[5] called "mechanical" system in [Arn89]

defining a 2-form

$$\begin{aligned}\omega_L &:= -d\lambda_L \\ &= -\frac{\partial^2 L}{\partial q^j \partial v^k} dq^j \wedge dq^k - \frac{\partial^2 L}{\partial v^j \partial v^k} dv^j \wedge dq^k.\end{aligned}$$

ω_L is well-defined on all of M and it is closed. Therefore it is a symplectic form if it is non-degenerate.

Let us call L (or the Lagrangian system (TQ, L)) *regular* when ω_L is non-degenerate.

For a regular Lagrangian L, (M, ω_L) is a symplectic manifold and with

$$H_L(v) := v^k \frac{\partial L}{\partial v^k}(v) - L(v) ,$$

the Hamiltonian system

$$(M, \omega_L, H_L)$$

has the same motions as (M, L).

Example 1.46. For a natural system the induced Hamiltonian H_L is of the form

$$H_L(v) = \frac{1}{2} g(v, v) + U(\tau(v)) ,$$

$v \in M = TQ$.

To understand regular Lagrangian systems from the Hamiltonian viewpoint the so-called fibre derivative of L is helpful: For $L \in \mathcal{E}(TQ, \mathbb{R})$ let $L_q := L|_{T_qQ} : T_qQ \to \mathbb{R}$.

Definition 1.47. The map

$$FL : TQ \to T^*Q ,$$

given by $FL(v)(w) := TL_q(v)(w)$ for $q \in Q , v, w \in T_qQ$, is called the *fibre derivative* of L

By direct calculations we can show:

Proposition 1.48. *The fibre derivative is a smooth map $FL : TQ \to T^*Q$ preserving the fibres.*[6]

$\omega_L = FL^*(\omega)$ where $\omega \in \mathcal{A}^2(T^*Q)$ is the standard 2-form on the momentum phase space T^*Q, see 1.17.

The following statements are equivalent:

1. *ω_L is non-degenerate, i.e. L is regular.*

[6] FL is not a homomorphism of vector bundles, in general.

2. *The fibre derivative FL is non-degenerate, i.e. its differential $T(FL) : T(TQ) \to T(T^*Q)$ is an isomorphism in all points $v \in TQ$. In particular, FL is a local diffeomorphism.*
3. $\det\left(\frac{\partial^2 L}{\partial v^j \partial v^k}\right) \neq 0$ *in local bundle coordinates.*

As a consequence, for a regular L the fibre derivative

$$FL : (TQ, \omega_L) \to (T^*Q, \omega)$$

is a symplectomorphism, i.e. FL respects the symplectic structure.

Proposition 1.49. *Moreover, if FL is a diffeomorphism the Hamiltonian systems (TQ, ω_L, H_L) and (T^*Q, ω, H) are equivalent, where $H := H_L \circ (FL)^{-1}$. Note, that in local bundle charts H has locally the familiar form*

$$H(q,p) = v^k p_k - L(q,v)\,, \quad \text{where} \;\; p_k = \frac{\partial L(v)}{\partial q^k}\,.$$

The last identity can be solved for v: $v = v(p)$, so that $H(q,p) = v^k(p)p_k - L(q, v(p))$.

FL can be called *Legendre transformation* in this situation. Classically the name is reserved for the transformation which takes L into H:

$$L(q,v) \mapsto H(q,p) = vp - L(q,v)\,.$$

Observe, that the Legendre transformation is not a mere coordinate transformation since L and H live on different spaces.

It is possible to describe the Hamiltonian systems which arise in this way:

Proposition 1.50. *Let $H \in \mathcal{E}(T^*Q, \mathbb{R})$ such that the fibre derivative $FH : T^*Q \to TQ$ is a diffeomorphism. Then $L := (\lambda(X_H) - H) \circ (FH)^{-1} : TQ \to \mathbb{R}$ is a Lagrangian such that $FH = (FL)^{-1}$.*

Summary:

This chapter introduces the concept of a Hamiltonian system as the mathematical model of a conservative system of Classical Mechanics. In order to formulate the general manifold case the basic notations for manifolds have been recalled. Moreover, several examples of Hamiltonian systems are presented, and it is shown in which way a Lagrangian system generates a corresponding Hamiltonian system.

For the program of Geometric Quantization the notion of a symplectic manifold with its Hamiltonian vector fields and its Poisson bracket is crucial. In particular, one needs the Lie algebra homomorphism

$$\mathcal{E}(M) \to \mathfrak{V}(M)\,,\; F \mapsto -X_F\,,$$

which satisfies

$$\boxed{[X_F, X_G] = -X_{\{F,G\}}\,.}$$

2 Ansatz Prequantization

We begin this chapter with some comments about quantization in general and proceed by presenting canonical quantization in some details in the first section.

Quantization can be viewed to be nothing more than a large set of methods, principles and procedures to "construct" quantum systems by using classical systems. Common feature: Little rigor, great freedom. Main objective: Arrive at a useful quantum system.

In particular: Quantization is not physics. There are no physical ideas or principles, which support quantization on a rigorous level. The process

$$\text{classical system} \mapsto \text{quantum system}$$

is speculation, even if it generates remarkable examples. Only the reverse process

$$\text{quantum system} \mapsto \text{classical system}$$

by taking classical or semi-classical limits can be justified by physical considerations. Nevertheless, many important quantum systems have been obtained by quantization.

Let us concentrate on quantum mechanics. The main quantum mechanical systems have been obtained by "canonical quantization" of models of Classical Mechanics, in particular, of Hamiltonian systems.

2.1 Canonical Quantization

To quantize a classical system which is given by a Hamiltonian system (X, ω, H) requires according to Dirac to fix a collection $\mathfrak{o} \subset \mathcal{E}(M, \mathbb{R})$ of observables, and find a complex Hilbert space $\mathbb{H}$ together with an $\mathbb{R}$-linear map (the QUANTIZATION MAP)

$$q : \mathfrak{o} \to \text{"End"}(\mathbb{H})^7$$

such that the following so called DIRAC CONDITIONS are satisfied:

(D1) $q(1) = \lambda \mathrm{id}_{\mathbb{H}}$,

(D2) $[q(F), q(G)] = cq(\{F, G\})$, for all $F, G \in \mathfrak{o}$.

where $\lambda \neq 0 \neq c$ are suitable constants.

In addition, all $q(F)$ should be self-adjoint (possibly unbounded) linear operators on $\mathbb{H}$.

[7] "End" means, that for $F \in \mathfrak{o}$, $q(F)$ is an operator $q(F) : D \to \mathbb{H}$ on a dense subspace $D \subset \mathbb{H}$. See Chapter F in the Appendix for operators in Hilbert spaces.

The fact that q should be $\mathbb{R}$-linear says that $\mathfrak{o}$ can be assumed to be a real vector space. So we require $\mathfrak{o}$ to be a real vector subspace of $\mathcal{E}(M,\mathbb{R})$.

Similarly, the first condition (D1) supposes that $1 \in \mathfrak{o}$. And the second condition (D2) means that $\mathfrak{o} \subset \mathcal{E}(M)$ can be assumed to be a real Lie subalgebra of the Poisson algebra $\mathcal{E}(M,\mathbb{R})$. Furthermore, the Hamiltonian H should be in the Lie algebra $\mathfrak{o}$.

The constant λ is taken as 1 in most cases. However, 2π or similar can be found in the first papers on geometric quantization. With this choice several formulas in geometry become simpler insofar that they need not the factor $\frac{1}{2\pi}$. But in these lecture notes, from now on, $\lambda = 1$.

The constant c is mostly $\frac{i}{\hbar}$ or $-i$ or similar, depending on the conventions. From the mathematical point of view the value of c is irrelevant except that the self-adjointness of the operators should not be overlooked: The requirement of self-adjointness implies that c has to be purely imaginary (c.f. Proposition 2.4 below). In our approaches later on in these lecture notes, where the assignment $F \mapsto X_F$ is used in a crucial manner and certain conventions concerning connections and their curvatures on line bundles, the constant will be $c = \frac{i}{2\pi} = -\frac{1}{2\pi i}$.

An important additional property of q is that all $q(F)$, $F \in \mathfrak{o}$, can be recovered from a common dense domain $D \subset \mathbb{H}$. This means, that for the domains of definition $D(q(F)) \subset \mathbb{H}$ of the operators $q(F)$ the condition

$$D \subset \bigcap \{D(q(F)) \mid F \in \mathfrak{o}\}$$

is satisfied. By this condition, it is possible to form the addition $q(F) + q(G)$ for $F, G \in \mathfrak{o}$ and $rq(F)$ for $r \in \mathbb{R}$.

Moreover, the image $q(F)(D)$ should be contained in D so that the composition $q(F) \circ q(G)$ can be formed. Only in this way it is possible to define the commutator

$$[q(F)|_D, q(G)|_D] \in \operatorname{End} D$$

and it makes sense that it coincides with

$$cq(\{F,G\})|_D.$$

This is the meaning of "End"($\mathbb{H}$).

In many contexts a quantization satisfying the above axioms (D1),(D2) is called a canonical quantization. But there is no prescription on how to obtain the Hilbert space $\mathbb{H}$ or the map q. Also additional axioms are common, e.g. requirements of the quantization for special observables like q^j or p_k for a simple phase space T^*U, $U \subset \mathbb{R}^n$. In particular, in several cases irreducibility is required. However, such requirements can be in conflict with the purpose to formulate quantization in an invariant way.

Geometric quantization is a canonical quantization in the sense above and gives a well-defined procedure how to find $\mathbb{H}$ and q in many cases. The first step in this procedure is PREQUANTIZATION and the purpose of this section is to motivate this concept. In Chapter 7 we pursue this issue further.

Discussion: Essentially different (even controversial) usage of the term "canonical":

1. in physics: a special choice, in a standard or common way,
2. in mathematics: a natural way, not dependent on any choice, functorial.

We now come to the subject announced in the title of this section.

2.2 Ansatz: Prequantization

First of all, we have to complexify the whole machinery replacing every $\mathbb{R}$-vector space W, which occurred so far by the complexification $W^{\mathbb{C}}$ i.e. $W^{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C} \cong W \oplus iW$. In particular, the space of observables is now $\mathcal{E}(M, \mathbb{C}) \cong \mathcal{E}(M) \otimes_{\mathbb{R}} \mathbb{C}$. More examples are $T_aM \otimes \mathbb{C}$ instead of T_aM, $\mathcal{A}^s(M) \otimes \mathbb{C}$ ($\cong \{\eta : \mathcal{E}(M, \mathbb{C})^s \to \mathcal{E}(M, \mathbb{C}) \mid \eta$ s-multilinear over $\mathcal{E}(M, \mathbb{C})$ and alternating$\}$) instead of $\mathcal{A}^s(M)$, $\mathfrak{g} \otimes \mathbb{C}$ instead of $\mathfrak{g}$, etc.
Afterwards, we omit $\mathbb{C}$ and in the following T_aM, $\mathcal{A}^s(M), \mathcal{E}(M), \mathfrak{g}, \ldots$ etc. shall denote the complexified versions.

Let (M, ω) be a symplectic manifold. We have a natural representation Φ of the Poisson algebra $(\mathcal{E}(M), \{,\})$ by forming the Hamiltonian vector field of a function $F \in \mathcal{E}(M)$ (cf. Proposition 1.31):

$$\Phi : \mathcal{E}(M) \to \mathfrak{V}(M) \ (\cong \mathrm{Der}(\mathcal{E}(M)) \subset \mathrm{End}(\mathcal{E}(M))$$

$$F \mapsto \Phi(F) := -X_F.$$

Recall that Φ is linear (meaning now $\mathbb{C}$-linear) and that for $F, G \in \mathcal{E}(M)$

$$\Phi\left(\{F, G\}\right) = -X_{\{F,G\}} = [X_F, X_G] = [\Phi(F), \Phi(G)]\,, \tag{17}$$

i.e. the respective Lie brackets are respected.

Let us now come to the "ansatz":

Attempt 1: As a first try to find an operator for a given $F \in \mathcal{E}(M)$, let us consider

$$\tilde{q}(F) := -cX_F = cL_{X_F} : \mathcal{E}(M) \to \mathcal{E}(M)\,,$$

although $\mathcal{E}(M)$ is not a Hilbert space. In the ansatz we weaken the conditions insofar as we try to obtain a construction for q with $\mathbb{H}$ replaced by the complex vector space $\mathcal{E}(M)$.

Proposition 2.1. *$\tilde{q}(F) : \mathcal{E}(M) \to \mathcal{E}(M)$ is $\mathbb{C}$-linear and satisfies*

$$[\tilde{q}(F), \tilde{q}(G)] = c\tilde{q}(\{F, G\}), \quad F, G \in \mathcal{E}(M)$$

Proof.
$$[\tilde{q}(F), \tilde{q}(G)] = (-c)^2[X_F, X_G] = c^2\left(-X_{\{F,G\}}\right) = c\tilde{q}(\{F,G\})$$
□

Evaluation of attempt 1: (D2) is satisfied (at least for $\mathcal{E}(M)$) and with $\mathfrak{o} = \mathcal{E}(M)$. But for $1 \in \mathcal{E}(M)$, we have $\tilde{q}(1) = 0$, hence the first Dirac condition (D1) is not satisfied.

Attempt 2:
In order to satisfy (D1), we can replace $\tilde{q}$ by $\hat{q}$ with $\hat{q}(F) = F + \tilde{q}(F) : g \mapsto Fg - cL_{X_F}g$. Evaluation: Now, the first axiom (D1) is satisfied $\hat{q}(1) = \mathrm{id}_{\mathcal{E}(M)}$, but the second (D2) is not. In the case $M = T^*\mathbb{R}$ for example, with the coordinate functions $q, p : M \cong \mathbb{R}$ we have $\{q,p\} = 1$:$X_q = \frac{\partial}{\partial p}$ and $X_p = \frac{\partial}{\partial q}$, and therefore: $\omega(X_q, X_p) = 1$. Consequently, $\tilde{q}(\{q,p\}) = 1$. But $\tilde{q}(q) = c\frac{\partial}{\partial p} + q$, $\tilde{q}(p) = -c\frac{\partial}{\partial q} + p$ with
$$[\tilde{q}(q), \tilde{q}(p)] \neq 1.$$

We make a further adaption and arrive at

Attempt 3:
$$q(F) := F - cL_{X_F} + \alpha(X_F), \quad F \in \mathcal{E}(M),$$
with a suitable 1-form $\alpha \in \mathcal{A}^1(M)$.

Proposition 2.2. *$q : \mathcal{E}(M) \to \mathcal{E}(M)$ is $\mathbb{C}$-linear and satisfies the first axiom (D1) of Dirac quantization (disregarding the fact that we do not have the Hilbert space yet). Moreover, it fulfills the second axiom (D2) for $\mathfrak{o} = \mathcal{E}(M)$ if and only if $d\alpha = \omega$.*

Proof. Evidently, q is $\mathbb{C}$-linear, and $q(1) = 1$ since $X_1 = 0$. So, (D1) is fulfilled.

We now check (D2). For $F \in \mathcal{E}(M)$ define $\mu(F) := F + \alpha(X_F) \in \mathcal{E}(M)$ to be the multiplication operator $\mu(F) : \mathcal{E}(M) \to \mathcal{E}(M)\,,\ H \mapsto \mu(F)H$. Then $q(F) = \tilde{q}(F) + \mu(F)$.

$$\begin{aligned}[q(F), q(G)] &= [\tilde{q}(F) + \mu(F), \tilde{q}(G) + \mu(G)] \\ &= [\tilde{q}(F), \tilde{q}(G)] + [\tilde{q}(F), \mu(G)] + [\mu(F), \tilde{q}(G)] + [\mu(F), \mu(G)]\,.\end{aligned}$$

Hence,
$$[q(F), q(G)] = c\tilde{q}(\{F,G\}) + [\tilde{q}(F), \mu(G)] + [\mu(F), \tilde{q}(G)]$$
according to first attempt (c.f. Proposition 2.1) and because of $[\mu(F), \mu(G)] = 0$.
For every $H \in \mathcal{E}(M)$
$$[\tilde{q}(F), \mu(G)]\,(H) = -cL_{X_F}(\mu(G)H) + c\mu(G)L_{X_F}H = -cL_{X_F}(\mu(G))H$$

and we obtain

$$[\tilde{q}(F), \mu(G)] = -cL_{X_F}\mu(G) = c\{F, \mu(G)\}.$$

In the same way

$$[\mu(F), \tilde{q}(G)] = cL_{X_G}\mu(F) = c\{\mu(F), G\}.$$

Altogether,

$$\begin{aligned} [q(F), q(G)] &= c\tilde{q}(\{F,G\}) + c\{F,\mu(G)\} + c\{\mu(F),G\} \\ &= c\left(\tilde{q}(\{F,G\}) + 2\{F,G\} + (\{F,\alpha(X_G)\} + \{\alpha(X_F),G\})\right) \\ &= c\left(\tilde{q}(\{F,G\}) + \{F,G\} + \alpha(X_{\{F,G\}})\right) + \\ &\quad + c\left(\{F,G\} + \{F,\alpha(X_G)\} + \{\alpha(X_F),G\} - \alpha(X_{\{F,G\}})\right) \end{aligned}$$

and we obtain the following: Condition (D2), i.e.

$$[q(F), q(G)] = cq(\{F,G\}),$$

holds if and only if the term in the second bracket vanishes, i.e. iff

$$\{F,G\} = \{\alpha(X_G), F\} - \{\alpha(X_F), G\} + \alpha(X_{\{F,G\}}). \tag{18}$$

Now, from the formula for $d\alpha$ (cf. 10), we know

$$\begin{aligned} d\alpha(X_F, X_G) &= L_{X_F}\alpha(X_G) - L_{X_G}\alpha(X_F) - \alpha([X_F, X_G]) \\ &= \{\alpha(X_G), F\} - \{\alpha(X_F), G\} + \alpha(X_{\{F,G\}}) \end{aligned}$$

and we can finally deduce:

$$(18) \text{ holds} \quad \Longleftrightarrow \quad d\alpha(X_F, X_G) = \{F,G\} = \omega(X_F, X_G)\ \forall F,G \in \mathcal{E}(M)$$

and

$$d\alpha(X_F, X_G) = \omega(X_F, X_G)\ \forall F,G \in \mathcal{E}(M) \quad \Longleftrightarrow \quad d\alpha = \omega,$$

since locally the Hamiltonian vector fields generate the $\mathcal{E}(U)$-module of vector fields. □

Because of this result, let us assume for the moment, that ω has a potential α, i.e. $d\alpha = \omega$. To proceed further, we observe that the $2n$- form

$$\omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega$$

is a volume form. Let $\mathbb{H} := \mathrm{L}^2(M, \omega^n)$ be the complex Hilbert space, which we obtain by completing the space of square integrable functions on M, i.e. the space:

$$D := \{\phi \in \mathcal{E}(M, \mathbb{C}) \mid \int_M |\phi|^2 \omega^n < \infty\}.$$

with respect to the norm $\|\phi\| = \sqrt{\int_M |\phi|^2 d\omega^n}$. $\mathbb{H}$ is a complex Hilbert space with the inner product defined by

$$\langle \phi, \psi \rangle := \int_M \overline{\phi}\psi\omega^n$$

for $\phi, \psi \in D$ and for general $\phi, \psi \in \mathbb{H}$ by continuation.

Then the $q(F)$, $F \in \mathcal{E}(M)$, which are defined on $\mathbb{H}$ (or on suitable subspaces) satisfy both the first and second Dirac condition. But there are generic defects of this approach:

1. There are important cases without symplectic potential. For example for $\mathbb{R}^N \times \mathbb{S}^2$ ("spin") or $\mathbb{S}^2 \times \mathbb{S}^2$ (hydrogen atom). Note, that for compact symplectic manifolds (M, ω) there never exists a potential $\alpha \in \mathcal{A}^1(M)$, i.e. $d\alpha = \omega$. To manage this problem, one has to generalize the ansatz, by replacing $\mathcal{E}(M)$ by sections of a complex line bundle over M, which will be explained in the next chapter.

2. Even with a symplectic potential α for ω the definition of q depends on the choice of the potential in an essential manner. To remedy this one can try to make use of the way two potentials α, α' differ: since $d(\alpha - \alpha') = 0$ there exists locally functions g such that $\alpha' = \alpha + dg$. With these local "gauges" one can build a quantization map which is no longer dependent on the choice of the potentials. But this quantization is no longer defined on proper functions $\phi \in \mathcal{E}(M)$, it is defined on generalized functions which can be given as families of local functions $(\phi_i)_{i \in I}$, $\phi_i \in \mathcal{E}(U_i)$, for an open cover $(U_i)_{i \in I}$ of M transforming suitably. This concept of generalized functions is explained in a reasonable manner by the concept of sections in a complex line bundle where the derivatives are now replaced by connections on this complex line bundle, see the next chapters.

3. The Hilbert space of wavw functions $\mathbb{H}$ is generated by functions in $2n$ variables. This should be reduced to n variables. This can be done by introducing polarizations. We will discuss this in a later chapter.

Apart from these issues, one has to find the appropriate Hilbert space $\mathbb{H}$ on the basis of these sections and then one has to check for which observables F the $q(F)$ are self-adjoint, among other considerations.

We conclude this Chapter by restricting to the special case of a simple phase space $M = T^*\mathbb{R}^n$ to figure out what properties follow when we use the PREQUANTUM OPERATOR

$$q(F) := -cL_{X_F} + F + \alpha(X_F)\,,$$

with $d\alpha = \omega$ as a first step in the program of geometric quantization..

Proposition 2.3. *Let $\mathbb{H} := L^2(M, \omega^n)$ be the Hilbert space introduced above in case of $M = T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$. For $F \in \mathcal{E}(\mathbb{R}^{2n})$ define $D := \{\phi \in \mathbb{H} \cap \mathcal{E}(\mathbb{R}^{2n}, \mathbb{C}) \mid L_{X_F}\phi \in \mathbb{H}\}$. Then $L_{X_F} : D \to \mathbb{H}$ is skew symmetric if (and only if) F is real-valued i.e. $F \in \mathcal{E}(M, \mathbb{R})$.*

Proof. We use the fact that X_F is divergence free: In standard coordinates $y^j = q^j$, $y^{n+j} = p_j$ for $j = 1, 2, \dots, n$ of $\mathbb{R}^{2n}$ let $X_F = Y^k \frac{\partial}{\partial y^k}$. (Summing over $k = 1, \dots, 2n$.) Then

$$\mathrm{div} X_F = \sum \frac{\partial Y^k}{\partial y^k} = \sum \frac{\partial}{\partial q^j}\frac{\partial F}{\partial p_m} - \frac{\partial}{\partial p_j}\frac{\partial F}{\partial q^m} = 0,$$

now summing over $j, m = 1, \dots, n$ and using the expression (9) for X_F. For $\phi, \psi \in D$:

$$\begin{aligned}\langle \phi, L_{X_F}\psi\rangle &= \int_{\mathbb{R}^{2n}} \bar{\phi} X_F(\psi)\,\omega^n \\ &= \sum_k \int_{\mathbb{R}^{2n}} \bar{\phi} Y^k \frac{\partial \psi}{\partial y^k}\,\omega^n \\ &= -\sum_k \int_{\mathbb{R}^{2n}} \frac{\partial}{\partial y^k}\left(\bar{\phi} Y^k\right)\psi\,\omega^n\end{aligned}$$

by partial integration. Hence,

$$\begin{aligned}\langle \phi, L_{X_F}\psi\rangle &= -\sum_k \int_{\mathbb{R}^{2n}} \overline{Y^k}\frac{\partial \bar{\phi}}{\partial y^k}\psi\,\omega^n - \int_{\mathbb{R}^{2n}} \bar{\phi}\left(\sum_k \frac{\partial Y^k}{\partial y^k}\right)\psi\,\omega^n \\ &= -\int_{\mathbb{R}^{2n}} \overline{L_{X_F}\phi}\,\psi\,\omega^n = -\langle L_{X_F}\phi, \psi\rangle,\end{aligned}$$

where $\sum \frac{\partial Y^k}{\partial y^k} = 0$ and $\overline{Y^k} = Y_k$ where used. □

Note, that integration with respect to ω^n is the same as Lebesgue integration over $\mathbb{R}^{2n}$ – up to a constant –, since ω^n is a constant multiple of the volume form induced by Lebesgue integration.

Proposition 2.4. *Let $\alpha \in \mathcal{A}(\mathbb{R}^{2n})$ be a real potential, i.e. $d\alpha = \omega$ and all coefficients α_k of α real-valued., and let $F \in \mathcal{E}(M)$ be real-valued. Defined $D := \{\phi \in \mathbb{H} \mid q(F)\phi \in \mathbb{H}, L_{X_F}\phi \in \mathbb{H}\}$. Then the prequantum operator*

$$q(F) = -cL_{X_F} + F + \alpha(X_F)$$

is symmetric on D if (and only if) c is purely imaginary.

Proof. We use $\langle \phi, cL_{X_F}\psi\rangle = c\langle \phi, L_{X_F}\psi\rangle = -c\langle L_{X_F}\phi, \psi\rangle$ by Proposition 2.3, hence, for purely imaginary c, $\langle \phi, cL_{X_F}\psi\rangle = \langle cL_{X_F}\phi, \psi\rangle$, i.q. $-cL_{X_F}$ is symmetric. Now, $q(F)$ is symmetric, since $T(F)\phi := (F + X_F)\phi$ can be proven to be symmetric: We have $\overline{T(F)} = T(F)$, since F and α are real. Therefore, for $\phi, \psi \in D$:

$$\langle \phi, T(F)\psi\rangle = \int_{\mathbb{R}^{2n}} \overline{\phi} T(F)\psi\,\omega^n = \int_{\mathbb{R}^{2n}} \overline{T(F)\phi}\psi\,\omega^n = \langle T(F)\phi, \psi\rangle\,.$$

□

Corollary 2.5. *If the vector field X_F is complete, $q(F)$ will be self-adjoint.*

This will be proven in a slightly more general situation in Chapter 7.

Examples 2.6. We determine some $q(F)$ in the simple phase space $M = T^*Q$, $Q \subset \mathbb{R}^n$ open, with the symplectic form $\omega = dq^j \wedge dp_j$. The canonical coordinates are q^j, p_j.

1. We choose the negative of the Liouville potential $\alpha = -p_j dq^j = -\lambda$ and set

$$c = -\frac{i}{2\pi},$$

so that

$$q(F)\phi = F\phi - \frac{i}{2\pi} L_{X_F} - p_j dq^j(X_F)\,.$$

We know

$$X_{q^j} = -\frac{\partial}{\partial p_j}\ ,\ X_{p_k} = \frac{\partial}{\partial q^k}$$

$$\alpha(X_{q^k}) = 0\,,\ \alpha(X_{p_k}) = -p_k\,.$$

So, we have:

$$\begin{aligned} q(q^j) &= q^j - \frac{i}{2\pi} L_{X_{q^j}} + 0 = q^j + \frac{i}{2\pi}\frac{\partial}{\partial p_j} =: Q^j \\ q(p_j) &= p_j - \frac{i}{2\pi} L_{X_{p_j}} - p_j = -\frac{i}{2\pi}\frac{\partial}{\partial q^j} =: P_j, \text{ and see} \\ \left[Q^j, P_k\right] &= cq\left(\{q^j, p_k\}\right) = +\frac{i}{2\pi}\delta^j_k. \end{aligned}$$

Note, that with the symplectic potential $\alpha = q^j dp_j$ the resulting operators Q^j, P_j are quite different, see below.

2. By replacing the Hilbert space $\mathbb{H} = L^2(T^*Q, \omega^n)$ introduced above with its subspace $\mathbb{H}_P$ of all square integrable functions ϕ of the form $\phi = g \circ \tau, g : Q \to \mathbb{C}$, for suitable g we arrive at the function space with the correct dependencies, namely $\mathbb{H}_P = L^2(Q)$. From 1. we conclude:

$$Q^j = q^j,\ \ P_j = -\frac{i}{2\pi}\frac{\partial}{\partial q^j}.$$

These operators turn out to be self-adjoint, which says that for the simple phase space and the algebra $\mathfrak{o}$ of observables generated by $1, p_j$ and q^j we have recovered the canonical commutation relations, CCR.

Moreover, if we include the observable $H = \frac{1}{2}\sum p_j^2$ into the discussion, we see that

$$X_H = -p_k \frac{\partial}{\partial q^k}$$

and

$$q(H) = \frac{i}{2\pi} p_k \frac{\partial}{\partial q^k} - H.$$

This is not the result we expect. The expected quantum operator is a multiple of the Laplacian Δ. It can be obtained by simply defining it as $1/2 \sum P_k P_k$. But this adhoc definition is not in the spirit of geometric quantization.

3. By replacing the potential $-\lambda$ of ω in 1. by the potential $\alpha = q^j dp_j$ (note, that $\alpha - \lambda = d(q^j p_j)$) we obtain

$$\alpha(X_{q^k}) = -q^k\,,\ \alpha(X_{p_k}) = 0\,.$$

And

$$\begin{aligned} q(q^j) &= q^j - \frac{i}{2\pi} L_{X_{q^j}} - q_j = \frac{i}{2\pi}\frac{\partial}{\partial p_j} =: Q^j \\ q(p_j) &= p_j - \frac{i}{2\pi} L_{X_{p_j}} + 0 = p_j - \frac{i}{2\pi}\frac{\partial}{\partial q^j} =: P_j\,. \end{aligned}$$

Restricting the operations to the Hilbert space $\mathbb{H}_Q$ of functions $\phi = \phi(p)$ depending only on the verical $p \in T^*\mathbb{R}^n$ this approach leads to the self-adjoint operators

$$Q^j = \frac{i}{2\pi}\frac{\partial}{\partial p^j},\quad P_j = p_j.$$

with the same commutation relations.

4. The two quantizations in 2. and 3. are closely related from a the viewpoint of representation theory. Both yield representations of the Lie algebra generated by the p_j, q^k, the so-called Canonical Commutation Relations (CCR) , and these representations are equivalent. The equivalence is given by the Fourier transform $\mathcal{F} : \mathbb{H}_Q \to \mathbb{H}_P$. In fact, let

$$\mathcal{F}(\phi)(q) := \int \phi(p) \exp(2\pi i p q) dp\,^8\,,$$

for differentiable functions $\phi \in \mathcal{E}(\mathbb{R}^n)$ with compact support. $\mathcal{F}$ can be extended uniquely to a unitary map $\mathbb{H}_Q \to \mathbb{H}_P$ which will be denoted again by $\mathcal{F}$.

Let us denote Q, P the quantizations of q, p in 2. and Q', P' the quantizations of q, p in 3. Then for $\phi = \phi(p)$ in 3. with

$$P \circ \mathcal{F}(\phi) = -\frac{i}{2\pi}\frac{\partial}{\partial q}\mathcal{F}(\phi)$$

leads to

$$P \circ \mathcal{F}(\phi) = -\frac{i}{2\pi}\int \phi(p)\frac{\partial}{\partial q} e^{2\pi i p q} dp = \int p\phi(p) e^{2\pi i p q} dp = \mathcal{F}(p\phi) = \mathcal{F} \circ P'(\phi)\,,$$

[8] written for $n = 1$ or in a schematic notation $q = (q^1, \ldots, q^n)$, $dq = dq^1 \wedge \ldots \wedge dq^n$, etc.

and

$$\mathcal{F} \circ Q'(\phi) = \int \left(\frac{i}{2\pi} \frac{\partial}{\partial p} \phi(p) \right) e^{2\pi i p q} dp = \int q \phi(p) e^{2\pi i p q} dp = q\mathcal{F}(\phi) = Q \circ \mathcal{F}(\phi) \,,$$

where we have used partial integration

$$0 = \int \frac{\partial}{\partial p} \left(\phi(p) e^{2\pi i p q} \right) dp = \int \frac{\partial \phi(p)}{\partial p} e^{2\pi i p q} dp + \int \phi(p) 2\pi i q e^{2\pi i p q} dp \,.$$

We illustrate these relations in a commutative diagram:

$$\begin{array}{ccc} \mathbb{H}_Q & \xrightarrow[Q]{P} & \mathbb{H}_Q \\ {\scriptstyle \mathcal{F}}\downarrow & & \downarrow{\scriptstyle \mathcal{F}} \\ \mathbb{H}_P & \xrightarrow[Q]{P} & \mathbb{H}_P \end{array}$$

As a result, the two representations are the same up to unitary equivalence. This result is a special case of the pairing described in Chapter 14.5, in particular in Proposition 14.20.

Summary: In this Chapter attempts are made for obtaining a canonical quantization by using the operation $F \mapsto -X_F$ for observables F on a symplectic manifold (M, ω) and the fundamental fact that it respects the Poisson bracket. As a result, to pursue these attempts in full generality one has to use complex line bundles on M which will be studied in the next chapter. Insofar, Chapter 2 serves as a motivation to study the geometry of complex line bundles.

Later we will see that starting with a symplectic manifold (M, ω) and a suitable complex line bundle L the ansatz developed in this chapter (attempt 3) leads to a quantum model. This process is called prequantization, see Chapter 7. However, the manifold M has to satisfy an integrality condition (discussed in Chapter 8), which sounds familiar regarding the principles of quantum theory.

3 Line Bundles

A line bundle over a manifold is a complex vector bundle of rank 1, i.e. with typical fibre isomorphic to $\mathbb{C}$. This chapter provides an elementary and detailed exposition of the fundamental properties of line bundles including the description of line bundles by cocycles. Moreover, the concrete examples of tautological line bundles on the complex projective spaces are studied.

3.1 Basic Definitions

Definition 3.1. A Line bundle (”Geradenbündel”) over a given manifold M is a manifold L (the Total space, ”Totalraum”) together with a map[9]

$$\pi : L \to M$$

with the following properties:

1. The fibres of π are Lines: Every fibre $L_a := \pi^{-1}(a)\,,\ a \in M$, has the structure of a one dimensional vector space over $\mathbb{C}$.

2. π is Locally trivial (”lokaltrivial”), i.e. the total space locally looks like a product $U \times \mathbb{C}$ up to isomorphisms: To each point $a \in M$ there corresponds an open neighbourhood $U \subset M$ of a and a diffeomorphism

$$\psi : L_U := L|_{\pi^{-1}(U)} \to U \times \mathbb{C}$$

such that

(a) the diagram

$$\begin{array}{ccc} L_U & \xrightarrow{\psi|_{L_U}} & U \times \mathbb{C} \\ {\scriptstyle \pi|_{\pi^{-1}(U)}} \big\downarrow & \swarrow {\scriptstyle pr_1} & \\ U & & \end{array}$$

is commutative: $pr_1 \circ \psi|_{L_U} = \pi|_{\pi^{-1}(U)}$[10],

(b) for all $b \in U$, the following induced map

$$\psi_b : L_b \xrightarrow{\psi|_{L_b}} \{b\} \times \mathbb{C} \xrightarrow{pr_2} \mathbb{C}\,,\ \psi_b := pr_2 \circ \psi|_{L_b}\,,$$

is a homomorphism (in fact an isomorphism) of vector spaces over $\mathbb{C}$.

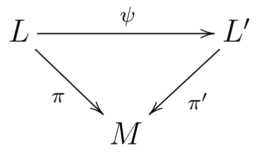
[9] assumed to be smooth, as usual

[10] pr_1, pr_2 denote the natural projections $pr_1 : W \times V \to W\,,\ (x,y) \mapsto x$ resp. $pr_2 : W \times V \to V\,,\ (x,y) \mapsto y$ for a product $W \times V$.

A line bundle is Trivial ("trivial") if $L = M \times \mathbb{C}$ with $\pi = pr_1$, and $L_b = \{b\} \times \mathbb{C}$ obtains its vector space structure through the bijection $pr_2 : L_b \to \mathbb{C}$.

However, by abuse of language, the line bundles which are isomorphic to the trivial line bundle are also called trivial, although they should better be called Trivializable. To understand "isomorphic" we have to introduce the notion of a homomorphism of line bundles.

Definition 3.2. A Line Bundle Homomorphism (homomorphism of line bundles) from $L \to M$ to $L' \to M$ is a map[11] $\psi : L \to L'$ such $\pi = \pi' \circ \psi$ and such that for each $a \in M$ the restriction $\psi_a := \psi|_{L_a} : L_a \to L'_a$ is a (vector space) homomorphism. In particular, the diagram



is commutative.

An isomorphism of line bundles is a homomorphism ψ of line bundles which is bijective such that ψ^{-1} is again a homomorphism of line bundles. In particular, an isomorphism is a homeomorphism.

Definition 3.3. A Section of a line bundle $\pi : L \to M$ over an open subset $U \subset M$ is a map[12]

$$s : U \to L$$

such that $\pi \circ s = \mathrm{id}|_U$.

The set of sections over U is denoted by $\Gamma(U, L)$. By pointwise addition and multiplication $\Gamma(U, L)$ becomes a vector space over $\mathbb{C}$ and an $\mathcal{E}(U)$-module: For $s, t \in \Gamma(U, L)$ and $f \in \mathcal{E}(U)$ we set

$$(fs + t)(a) := f(a)s(a) + t(a), a \in U,$$

and we see $fs + t \in \Gamma(U, L)$.

In case of the trivial bundle $L = M \times \mathbb{C}$ the space of sections $\Gamma(U, L)$ over an open $U \subset M$ is naturally isomorphic to $\mathcal{E}(U)$: Let $s_1(a) := (a, 1)$, $a \in U$, be the 1-section, $s_1 \in \Gamma(U, L)$. For every $f \in \mathcal{E}(U)$ one has

$$fs_1(a) = f(a)s_1(a) = f(a)(a, 1) = (a, f(a)), \ a \in U.$$

Since every section $s \in \Gamma(U, L)$ is of the form $s(a) = (a, f(a))$, $a \in U$, for some $f \in \mathcal{E}(U)$, the map

$$\mathcal{E}(U) \to \Gamma(U, L), \ f \mapsto fs_1,$$

[11] which is again smooth
[12] which is again smooth

is an $\mathcal{E}(U)$-module isomorphism.

As a consequence, in a general line bundle $\pi : L \to M$ with a local trivialisation $\psi : \pi^{-1}(U) \to U \times \mathbb{C}$ for every open subset $W \subset U$ the space of sections $\Gamma(W, L|_W)$ is isomorphic to $\mathcal{E}(W)$ as an $\mathcal{E}(W)$-module.

Proposition 3.4. *A line bundle $\pi : L \to M$ is trivial (-izable) if and only if there exists a global nowhere vanishing section of L, i.e. a section $s \in \Gamma(M, L)$ with $s(a) \neq 0$ for all $a \in M$.*

Proof. Let s be such a section. It is enough to show that

$$\psi : M \times \mathbb{C} \to L\,, \quad (a, \lambda) \mapsto \lambda s(a)\,,$$

for $(a, \lambda) \in M \times \mathbb{C}$ is a diffeomorphism and linear in the fibres. Of course, ψ is smooth and satisfies $\pi \circ \psi = \mathrm{pr}_1$: $\pi \circ \psi(a, \lambda) = \pi(\lambda s(a)) = a = \mathrm{pr}_1(a, \lambda)$. And for each $a \in M$

$$\psi_a = \psi|_{\{a\} \times \mathbb{C}} : \{a\} \times \mathbb{C} \to L_a\,, \quad (a, \lambda) \mapsto \lambda s(a)\,,$$

is an isomorphism of vector spaces, since $s(a) \neq 0$. □

The fact that for trivial line bundles L over M the sections of L are essentially the same as functions, $\Gamma(M, L) \cong \mathcal{E}(M)$, strengthens the viewpoint that for general line bundles L the sections $\Gamma(M, L)$ of L are generalized functions in M. Another such interpretation will be given after the next step in describing line bundles using cocycles (cf. Observation 3.8).

3.2 Cocycles Generating Line Bundles

The second condition in our Definition 3.1 yields an open cover $(U_j)_{j \in I}$ of M with local trivializations

$$\psi_j : L|_{U_j} \to U_j \times \mathbb{C}.$$

In particular, ψ_j is a diffeomorphism with $\pi = \mathrm{pr}_1 \circ \psi_j$, and $\psi_a : L_a \to \{a\} \times \mathbb{C}$ is an isomorphism. In addition, for each $j \in I$ one obtains a distinguished section $s_j \in \Gamma(U_j, L)$ by

$$s_j(a) := \psi_j^{-1}(a, 1)\,,\; a \in U_j,$$

with the property

$$\psi_j(z s_j(a)) = (a, z)\,, \quad \text{or} \quad \psi_j^{-1}(a, z) = z s_j(a),$$

when $(a, z) \in U_j \times \mathbb{C}$.

The s_j und s_k satisfy the identity:

$$\boxed{s_j = g_{kj} s_k,}$$

on the intersection $U_j \cap U_k$[13] where the "transition functions" ("Übergangsfunktionen") $g_{kj} : U_j \cap U_k \to \mathbb{C}$ are defined as

$$g_{kj} := \frac{s_j}{s_k}\ ,\ j,k \in I :$$

For each $a \in M$ and $j,k \in I$ one has $s_j(a) \neq 0 \neq s_k(a)$. Hence, $g_{kj}(a) \in \mathbb{C}$ is well-defined with $s_j(a) = g_{kj}(a)s_k(a)$.

Because if the importance of this simple identity, we give another description of $g_{kj} : U_{jk} := U_j \cap U_k \to \mathbb{C}$: The composition

$$\psi_k \circ \psi_j^{-1} : U_{jk} \times \mathbb{C} \to U_{jk} \times \mathbb{C}$$

where $\psi_j^{-1}(a,z) = zs_j(a)$ and $\psi_k(ws_k(a)) = (a,w)$ with $ws_k(a) = zs_j(a)$, acts as

$$(a,z) \mapsto \left(a, \frac{s_j(a)}{s_k(a)} z\right) = (a, g_{kj}(a).z)\ . \tag{19}$$

Proposition 3.5. *The transition functions* $(g_{jk})_{j,k\in I}$, $g_{jk} \in \mathcal{E}(U_{jk}, \mathbb{C}^\times)$, *satisfy the following* COCYCLE *condition ("Kozyklus-Bedingung"):*

$$\text{(C)} \qquad \begin{aligned} &g_{jj} = 1 \\ &g_{jk}g_{kj} = 1 \\ &g_{ij}g_{jk}g_{ki} = 1 \end{aligned}$$

Proof. Nearly trivial: g_{jj} comes by definition from $\psi_j \circ \psi_j^{-1} = \mathrm{id}|_{\{a\}\times\mathbb{C}}$. when $a \in U_j$, and the identity $\mathrm{id} : \mathbb{C} \to \mathbb{C}$ is given by the multiplication with $1 = g_{jj} :\ z \mapsto 1.z$. In the same way $g_{jk}g_{kj}$ comes from $\psi_j \circ \psi_k^{-1} \circ \psi_k \circ \psi_j^{-1} = \mathrm{id}|_{\{a\}\times\mathbb{C}}$, $a \in U_{jk}$, inducing $g_{jk}g_{kj} = 1$. And again, because of $\psi_i \circ \psi_j^{-1} \circ \psi_j \circ \psi_k^{-1} \circ \psi_k \circ \psi_k^{-1} = \mathrm{id}|_{\{a\}\times\mathbb{C}}$, $a \in U_{ijk} := U_i \cap U_j \cap U_k$, one obtains $g_{ij}g_{jk}g_{ki} = 1$. □

The transition functions describe how the various products $U_j \times \mathbb{C}$ glue together to form the total space[14].

The transition functions (g_{ij}) of a line bundle L describe the sections of that bundle, as explained in the following. Each section $s \in \Gamma(M,L)$ yields the local function

$$f_j := pr_2 \circ \psi_j \circ s|_{U_j},$$

[13] Here and in the following the case of an empty intersection $U_j \cap U_k = \emptyset$ contains no information, so it is, in general, correct.

[14] This feature holds for the more general cases of vector bundles of rank $k > 1$ and principal bundles as well

i.e. the diagram

$$\begin{array}{ccc} L|_{U_j} & \xrightarrow{\psi_j} & U_j \times \mathbb{C} \\ {\scriptstyle s|_{U_j}}\uparrow & & \downarrow{\scriptstyle pr_2} \\ U_j & \xrightarrow[f_j]{} & \mathbb{C} \end{array}$$

is commutative.

Lemma 3.6. *For all $j,k \in I$ the following equations hold:*

$$s|_{U_j} = f_j s_j \,, \quad on \ U_j \,,$$

$$f_k = g_{kj} f_j \,, \quad on \ U_{jk} \,.$$

Proof. For $a \in U_j$:

$$\begin{aligned} s(a) &= \psi_j^{-1} \circ \psi_j(s(a)) = \psi_j^{-1}(a, pr_2 \circ \psi_j \circ s(a)) \\ &= \psi_j^{-1}(a, f_j(a)) = f_j(a)\psi^{-1}(a,1) = f_j(a)s_j(a) \end{aligned}$$

which shows the first identity. On $U_{jk} \neq \emptyset$ we obtain

$$s|_{U_{jk}} = f_k s_k|_{U_{jk}} = f_j s_j|_{U_{jk}} = f_j g_{kj} s_k|_{U_{jk}},$$

hence $f_k = f_j g_{kj}$. □

The second condition is called SECTION CONDITION:

(S) $\quad f_k = g_{kj} f_j$

Proposition 3.7. *Let $\pi : L \to M$ be a line bundle over M with local trivializations $\psi_j : L|_{U_j} \to U_j \times \mathbb{C}, j \in I$, such that $(U_j)_{j \in I}$ is an open cover of M.*

1. *Then every global section $s \in \Gamma(M,L)$ defines a collection $f_j \in \mathcal{E}(U_j)$ of local functions with (S).*
2. *Conversely, every collection $(f_j)_{j\in I}$, $f_j \in \mathcal{E}(U_j)$, of local functions satisfying (S) yields a global section $s \in \Gamma(M,L)$ with $s|_{U_j} = f_j s_j$.*

Proof. The first statement has just been shown. The data (f_j) in the second statement have the property $f_j s_j|_{U_{jk}} = f_k s_k|_{U_{jk}}$ by (S) and thus define a global section s through

$$s(a) := f_j(a) s_j(a) \,, \, a \in U_j.$$

□

Observation 3.8. Note, that the result of Proposition 3.7 provides another interpretation of sections $s \in \Gamma(M, L)$ as generalized functions. A generalized function under this viewpoint is a collection of local functions (f_j) which satisfy $f_k = g_{kj} f_j$ on U_{jk} for all $j, k \in I$, i.e. it is a section. This generalization is adapted to our problem of not having a global potential for a given symplectic form, in general.

The transition functions of a line bundle determine the line bundle completely up to isomorphism: Similar to the reconstruction of a section from local functions (f_j) with condition (S) any collection of (g_{jk}) with condition (C) allows to construct a line bundle L with the (g_{jk}) as its transition functions:

Proposition 3.9. *Let $(U_j)_{j \in I}$ be an open cover of the manifold M, and let $g_{kj} \in \mathcal{E}(U_{jk}, \mathbb{C}^\times)$, $j, k \in I$, be a collection of functions forming a cocycle, i.e. such that (C) is satisfied. Then the data $(M, (U_j), (g_{jk}))$ induce a complex line bundle $\pi : L \to M$ over M with local trivializations*

$$\psi_j : L_{U_j} \to U_j \times \mathbb{C}$$

such that for $(a, z) \in U_{jk} \times \mathbb{C}$:

$$\psi_k \circ \psi_j^{-1}(a, z) = (a, g_{kj}(a).z).$$

Proof. On the disjoint union $R := \dot{\bigcup}_{j \in I} U_j \times \mathbb{C}$, which is an $(n+2)$-dimensional real manifold, we consider the equivalence relation

$$(a_j, z_j) \sim (a_k, z_k) \;:\Longleftrightarrow\; a_j = a_k \text{ and } z_j = g_{kj}(a).z_k, ,$$

where $(a_j, z_j) \in U_j \times \mathbb{C}, (a_k, z_k) \in U_k \times \mathbb{C}$.

The quotient manifold $L = R/\sim$ exists[15], see Definition A.10 ff. for the concept of a differentiable quotient manifold. The quotient map $\pi : L \to M$ is smooth, and the maps

$$\psi_j : L_{U_j} \to U_j \times \mathbb{C}, \; [(a, z)] \mapsto (a, z),$$

for $(a, z) \in U_j \times \mathbb{C}$ turn out to be local trivializations generating fibrewise the vector space structure on L_a through

$$pr_2 \circ \psi_j|_{L_a} : [(a, z)] \mapsto z.$$

The transition functions in this construction are the (g_{kj}). In fact, for $a = a_j = a_k \in U_{jk}$:

$$\psi_k \circ \psi_j^{-1}(a, z_j) = \psi_k\left([(a_j, z_j)]\right) = \psi_k\left([(a_k, g_{kj}(a).z_k)]\right) = (a, g_{kj}(a).z_k),$$

i.e. the (g_{kj}) are the transition functions of the constructed line bundle.

□

[15]One has to check that M is a Hausdorff space and that L is well-defined with the stated properties.

Thus, we see that a line bundle can essentially be recaptured by its transition functions.

With respect to this description of line bundles by cocycles, the trivial bundle is given by $U_1 = M, (U_j)_{j\in\{1\}}$ and $g_{11}(a) = 1$. But the data: $(U_j)_{j\in I}$ an open cover, $g_{jk} : U_{jk} \to \mathbb{C}$, $g_{jk}(a) = 1$, yield also a trivial bundle L, more precisely an isomorphism $\Theta : L \to M \times \mathbb{C}$ of line bundles.

In general, a homomorphisms Θ of line bundles has a description using the transition functions (i.e. the cocycles) of the two line bundles which are involved which is similar to the descriptions of sections:

Proposition 3.10. *For a homomorphism of line bundles $\Theta : L \to L'$, where L resp. L' have local trivializations ψ_j resp. ψ'_j with respect to an open cover $(U_j)_{j\in I}$*[16] *the local mappings $h_j : U_j \to \mathbb{C}$ given by*

$$h_j := pr_2 \circ \psi'_j \circ \Theta|_{L_{U_j}} \circ s_j$$

$$\begin{array}{ccccc} L_{U_j} & \xrightarrow[\Theta]{} & L'_{U_j} & \xrightarrow{\psi'_j} & U_j \times \mathbb{C} \\ {\scriptstyle s_j}\uparrow & & & & \downarrow{\scriptstyle pr_2} \\ U_j & & \xrightarrow[h_j]{\qquad\qquad} & & \mathbb{C} \end{array}$$

satisfy the following identity:

$$\boxed{h_k = \frac{g'_{kj}}{g_{kj}} h_j \,,}$$

$j, k \in I$. Conversely, (h_j) with this identity determines a homomorphism Θ by locally defining

$$\Theta|_{U_j}(\psi_j^{-1}(a, z)) := \psi_j'^{-1}(a, h_j(a).z).$$

Proof. The definition of h_j can also be read as follows:

$$(a, h_j(a).z) = \psi'_j \circ \Theta \circ \psi_j^{-1}(a, z) = \psi' \circ \Theta \circ z s_j(a)\,,\ (a, z) \in U_j \times \mathbb{C}.$$

In the original definition $h_j := pr_2 \circ \psi'_j \circ \Theta|_{L_{U_j}} \circ s_j$, on U_{jk} one can replace s_j by $g_{kj}s_k$ and ψ'_j by $g'_{jk}\psi'_k$. These replacements yield

$$h_j = pr_2 \circ g'_{jk}\psi'_k \circ \Theta|_{L_{U_{jk}}} \circ g_{kj}s_k = g'_{jk}g_{kj}pr_2 \circ \psi'_k \circ \Theta|_{L_{U_{jk}}} \circ s_k = g'_{jk}g_{kj}h_k$$

which is the required identity (note that g'_{jk} is the inverse of g'_{kj}). The converse is clear. □

[16] Given two line bundles there always exists such a cover.

Corollary 3.11. *Under the assumption of the preceding proposition the line bundles L and L' are isomorphic, if and only if there are non-vanishing functions $h_j : U_j \to \mathbb{C}$ with*

$$\text{(I)} \qquad \boxed{g'_{kj} = \frac{h_k}{h_j} g_{kj}}$$

on U_{jk} and for all $j, k \in I$. The isomorphism $\Theta : L \to L'$ given by (h_j) is locally defined as $\Theta(s_j(a)) = h_j(a)s'_j(a)$.

Observation 3.12. These results show that the isomorphism classes of line bundles on a manifold are essentially isomorphism classes of cocycles. Here, two cocycles g_{kj}, g'_{kj} are defined to be equivalent if there exists $h_j \in \mathcal{E}(U_j)$ satisfying (I). This resembles cohomology. We come back to this fact later.

But are there nontrivial line bundles at all? If not, the introduction of the notion of a line bundles would give very little sense.

Presumably, the case of the tangent bundle $T\mathbb{S}^2$ is known in the form of the "Hairy Ball Theorem" ("Satz vom Igel"): There is no non-vanishing (smooth) vector field on $\mathbb{S}^2$. Therefore, the tangent bundle cannot be trivial. It is easy to see that the tangent bundle is, in fact, a complex line bundle by observing that $\mathbb{S}^2$ has the interpretation of being the Riemann sphere, the complex projective space $\mathbb{P}^1(\mathbb{C})$.

The question of the existence of nontrivial line bundles has to do with the classification of all line bundles (determining the set of isomorphism classes) which could be achieved with the help of cocycles. But we want, first of all, to investigate all line bundles on a rather typical example, the complex projective space $\mathbb{P}^n(\mathbb{C})$ of complex dimension n.

3.3 The Tautological Line Bundle

The aim of this subsection is to present concrete line bundles and analyse their sections. This will be started for line bundles over the complex projective spaces where the tautological bundles are introduced and will be continued by determining all line bundles over the complex projective spaces.

The Case $\mathbb{P}^1(\mathbb{C})$

Example 3.13 (The projective line $\mathbb{P}^1(\mathbb{C})$)**.** Let $\mathbb{P}^1 := \mathbb{P}^1(\mathbb{C})$ be the Riemann sphere resp. the one dimensional projective space over $\mathbb{C}$: $\mathbb{P}^1$ is the space of lines in $\mathbb{C}^2$ through the origin $0 \in \mathbb{C}^2$. The best way to make this precise is to introduce the following equivalence relation in $\mathbb{C}^2 \setminus \{0\}$:

$$z \sim w \quad :\Longleftrightarrow \quad \exists\, \lambda \in \mathbb{C} \,:\, z = \lambda w.$$

Then $\mathbb{P}^1$ is the quotient $\mathbb{C}^2 \setminus \{0\}/\sim$ and is endowed with a natural projection

$$\gamma : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1 ,$$

mapping each line $\ell \in \mathbb{C}^2$ to its corresponding equivalence class $\ell \setminus \{0\}$ in $\mathbb{P}^1$ (see Definition A.10 ff. for the concept of a differentiable quotient).

The points of $\mathbb{P}^1$ are represented by the so called HOMOGENEOUS COORDINATES ("Homogene Koordinaten") induced from $\mathbb{C}^2 \setminus \{0\}$: For $z = (z_0, z_1) \in \mathbb{C}^2 \setminus \{0\}$ we set

$$(z_0 : z_1) := \gamma(z)$$

($= [z]$, the equivalence class of $z = (z_0, z_1)$). We conclude from $\gamma(\lambda z) = \lambda\gamma(z)$, for $\lambda \neq 0$, that the homogeneous coordinates fulfill

$$(z_0 : z_1) = (\lambda z_0 : \lambda z_1) , \quad \text{for } \lambda \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}.$$

$\mathbb{P}^1$ obtains its topological and complex manifold resp. differentiable structure as the quotient $\mathbb{C}^2 \setminus \{0\}/\sim$. Hence, $U \subset \mathbb{P}^1$ is open if and only if $\gamma^{-1}(U)$ is open, and a map $f : U \to \mathbb{C}$ is holomorphic (resp. smooth) if and only if $f \circ \gamma : \gamma^{-1}(U) \to \mathbb{C}$ is holomorphic (resp. smooth).

$\mathbb{P}^1$ can be covered by two holomorphic charts $\psi_j : U_j \to \mathbb{C}$, where $U_j := \{(z_0 : z_1) \mid z_j \neq 0\}$, $j \in \{0, 1\}$ and

$$\psi_0 : U_0 \to \mathbb{C}, (1 : z) \mapsto z \,,\; z \in \mathbb{C},$$
$$\psi_1 : U_1 \to \mathbb{C}, (w : 1) \mapsto w, w \in \mathbb{C}.$$

On the intersection $U_{01} = U_0 \cap U_1 = \{(z_0 : z_1) \mid z_0 \neq 0 \neq z_1\}$ we have for $w \in U_{01}$:

$$\psi_0 \circ \psi_1^{-1}(w) = \psi_0(w : 1) = \psi_0(1 : \frac{1}{w}) = \frac{1}{w} .$$

This is a holomorphic function on $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ with holomorphic inverse. Therefore, the complex structure on $\mathbb{P}^1(\mathbb{C})$ (and the differentiable structure as well) is given by these two charts.[17]

U_0 can be understood as the complex plane $\mathbb{C}$ with $(1 : 0)$ as 0, $(1 : z) \cong z$ as the coordinate and U_1 adds only the point $(0 : 1) = \infty$ to $U_0 \cong \mathbb{C}$, thus obtaining the sphere $\mathbb{S}^2 \cong U_0 \cup \{(0 : 1)\} \cong \mathbb{C} \cup \{\infty\}$.

Construction 3.14 (Tautological Bundle)**.** The product $\mathbb{P}^1 \times \mathbb{C}^2$ is a 3-dimensional complex manifold and a 6-dimensional real manifold. It also has the structure of the trivial complex vector bundle over $\mathbb{P}^1$ of rank 2. Define

$$\begin{aligned} T &:= \{(a, w) \in \mathbb{P}^1 \times \mathbb{C}^2 \mid \exists \lambda \in \mathbb{C} : w = (\lambda a_0, \lambda a_1) \text{ if } a = (a_0 : a_1)\} \\ &= \{(a, w) \in \mathbb{P}^1 \times \mathbb{C}^2 \mid w = 0 \text{ or } \gamma(w) = a\} \\ &= \bigcup_{a \in \mathbb{P}^1} \{a\} \times \big(\{0\} \cup \gamma^{-1}(a)\big) = \bigcup_{a \in \mathbb{P}^1} \{a\} \times \ell_a \end{aligned}$$

[17] One has to check that the quotient structure exists, which can be done, by proving that $\mathbb{P}^1(\mathbb{C})$ with the structure given by the two charts satisfies indeed the universal property of the quotient.

and $\pi := pr_1 : T \to \mathbb{P}^1, (a, w) \mapsto a$. Here, $\ell_a := \{0\} \cup \gamma^{-1}(a)$ is the complex line in $\mathbb{C}^2$ represented by $a \in \mathbb{P}^1$.

T is a complex submanifold of $\mathbb{P}^1 \times \mathbb{C}^2$ of dimension 2, since it also has the description

$$T = \{((a_0 : a_1), (w_0, w_1)) \in \mathbb{P}^1 \times \mathbb{C}^2 \mid a_0 w_1 - a_1 w_2 = 0\}.$$

And $\pi = pr_1|_T : T \to \mathbb{P}^1$ defines a smooth (even holomorphic) projection . For each $a \in \mathbb{P}^1$ the fibre $T_a = \pi^{-1}(a) = \{a\} \times \ell_a$ obtains its vector space structure from the vector subspace $\ell_a \subset \mathbb{C}^2$. Hence, the fibre $T_a = \{a\} \times \ell_a$ can be viewed to be precisely the line given by the equivalence class a. This property is the reason why T is called the Tautological Bundle.

Moreover, to see that T is indeed a line bundle, we consider, for $j = 0, 1$, the diffeomorphisms

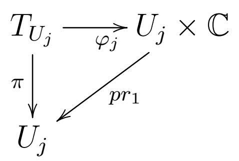
$$\varphi_j : T_{U_j} \to U_j \times \mathbb{C}\,,\ (a, (w_0, w_1)) \mapsto (a, w_j)$$

with inverses

$$\varphi_0^{-1} : ((a_0 : a_1), z)) \mapsto \left((a_0 : a_1), (z, z\frac{a_1}{a_0})\right)$$

$$\varphi_1^{-1} : ((a_0 : a_1), z)) \mapsto \left((a_0 : a_1), (z\frac{a_0}{a_1}, z)\right)$$

It is easy to see that the φ_j are local trivializations which respect the open U_j. The diagram



is commutative and the φ_j are linear in the fibres.

To calculate the transition functions with respect to φ_0, φ_1 we determine the action of $\varphi_1 \circ \varphi_0^{-1} : U_{01} \to U_{01}$ as

$$\varphi_1 \circ \varphi_0^{-1}((a_0 : a_1), z) = \varphi_1\left((a_0 : a_1), (z, z\frac{a_1}{a_0})\right) = \left((a_0 : a_1), (z\frac{a_1}{a_0})\right)$$

for $a = (a_0 : a_1) \in U_{01}$ and $z \in \mathbb{C}$. Hence, the corresponding transition function $g_{10} : U_{10} \to \mathbb{C}^\times$ (defined by $\varphi_1 \circ \varphi_0^{-1}(a, z) = (a, g_{10}(a).z)$, c.f. formula (19)) is simply

$$g_{10}(a) = \frac{a_1}{a_0}\,,\ \text{ i.e. }\ g_{10}(a)(z) = \frac{a_1}{a_0}.z.$$

Analogously,

$$g_{01}(a) = \frac{a_0}{a_1}\,.$$

Note that T is the tangent bundle with respect to the differentiable as well to the complex structure, i.e. $T \cong T\mathbb{P}^1$ as holomorphic line bundles, and the differentiable structure induced by $T \to \mathbb{P}^1$ (as a complex holomorphic line bundle over $\mathbb{P}^1$ just described) agrees with the differentiable structure of the tangent bundle $T\mathbb{S}^2$. In other words, the natural identification map $F : T \to T\mathbb{S}^2$ is a diffeomorphism.

Proposition 3.15. *The tautological bundle T over $\mathbb{P}^1$ has no holomorphic section other than the zero section: $\Gamma_{hol}(\mathbb{P}^1, T) = \{0\}$.*

Proof. Any holomorphic section $s : \mathbb{P}^1 \to T$ is given by holomorphic functions $f_j : U_j \to \mathbb{C}$ satisfying (S):

$$f_0(z) = g_{01}(z) f_1(z)\,,\ z = (z_0 : z_1) \in U_{01}\,,\ \text{ i.e. } f_0(z_0 : z_1) = \frac{z_0}{z_1} f_1(z_0 : z_1)$$

according to the calculation above. The two functions F_j on $\gamma^{-1}(U_j)$ given by $F_j(z) := \frac{1}{z_j} f_j(\gamma(z))\,,\ z = (z_0, z_1) \in \gamma^{-1}(U_j)$ are well-defined and they agree on $\gamma^{-1}(U_{01})$:

$$g_0(z) = \frac{1}{z_0} f_0\left(\gamma(z)\right) = \frac{1}{z_0}\frac{z_0}{z_1} f_1\left(\gamma(z)\right) = \frac{1}{z_1} f_1\left(\gamma(z)\right) = g_1(z).$$

As a consequence, the given holomorphic section s induces a holomorphic function F on $\mathbb{C}^2 \setminus \{0\}$ ($= F_1 = F_0$ on $\gamma^{-1}(U_{01})$) with the property

$$F(\lambda w) = \lambda^{-1} F(w)\,,\ \lambda \in \mathbb{C}^\times, w \in \mathbb{C}^2 \setminus \{0\}.$$

But such a holomorphic function $F : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C}$ is the zero function[18] and in turn the section s has to be zero. □

Corollary 3.16. *T is not isomorphic to the trivial bundle $\mathbb{P}^1 \times \mathbb{C}$ as a holomorphic bundle.*

This result is no surprise regarding the Hairy Ball Theorem.

THE CASE $\mathbb{P}^n(\mathbb{C})$

The whole consideration of the above example can be generalized to the n-dimensional projective space $\mathbb{P}^n := \mathbb{P}^n(\mathbb{C})$, the space of complex lines through 0 in $\mathbb{C}^{n+1}$. We define

$$\mathbb{P}^n := \mathbb{P}^n(\mathbb{C}) := \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\sim$$

with respect to the equivalence relation

$$w \sim z \ :\Longleftrightarrow\ \exists\, \lambda : w = \lambda z,$$

[18] One has to show that F defines a holomorphic function on all of $\mathbb{C}^2$, since there exist no isolated singularities for holomorphic functions in more than one variable, cf. Appendix, Corollary B.12.

and obtain the projection

$$\gamma : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n\,,\ \gamma(z) = [z] = (z_0 : z_1 : \ldots : z_n),$$

where $z = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$.

$\mathbb{P}^n$ obtains its topological, differentiable and complex structure as the quotient $(\mathbb{C}^{n+1} \setminus \{0\})/\sim$. Convenient holomorphic charts are the following: $\varphi_j : U_j \to \mathbb{C}^n$ on $U_j := \{(z_0 : z_1 : \ldots : z_n) \in \mathbb{P}^n \mid z_j \neq 0\}$ defined by

$$\varphi_j(z_0 : z_1, : \ldots : z_n) := \frac{1}{z_j}(z_0, z_1, \ldots \hat{z_j}, \ldots, z_n),$$

$j = 0, 1, \ldots n$, with biholomorphic $\varphi_j \circ \varphi_k^{-1} : U_{jk} \to U_{jk}$[19].

Construction 3.17 (Tautological Bundle)**.** As before, we start with the trivial complex vector bundle $\mathbb{P}^n \times \mathbb{C}^{n+1}$ over $\mathbb{P}^n$ and define

$$\begin{aligned} T &:= \{(z, w) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid \exists \lambda \in \mathbb{C} : w = \lambda(z_0, \ldots, z_n) \text{ if } z = (z_0 : \ldots : z_n)\} \\ &= \bigcup_{z \in \mathbb{P}^n} \{z\} \times \left(\{0\} \cup \gamma^{-1}(z)\right) = \bigcup_{z \in \mathbb{P}^1} \{z\} \times \ell_z \\ &= \{(z, w) \in \mathbb{P}^n \times \mathbb{C}^{n+1} \mid z_i w_j - z_j w_i = 0 \text{ for } i, j = 0, 1, \ldots n\} \end{aligned}$$

with projection $\pi = pr_1$. $T \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$ is a complex submanifold of dimension $n+1$ and it is the total space of a differentiable and holomorphic line bundle $\pi : T \to \mathbb{P}^n$, the tautological bundle over $\mathbb{P}^n$. T is not trivializable as a holomorphic bundle and also not as a differentiable line bundle.

Local trivializations are given by

$$\psi_j : T_{U_j} \to U_j \times \mathbb{C}\,,\ (z, w) \mapsto (z, w_j)$$

with the corresponding transition functions

$$g_{jk}(z) = \frac{z_j}{z_k}\,,\ z = (z_0 : z_1 : \ldots : z_n) \in U_{jk} = \{z \in \mathbb{P}^n \mid z_j \neq 0 \neq z_k\}.$$

In fact,

$$\psi_j \circ \psi_k^{-1}(z, \lambda) = \psi_j\left(z, \frac{\lambda}{z_k}(z_0, \ldots, z_n)\right) = (z, \frac{z_j}{z_k}\lambda)\,,$$

i.e.

$$g_{jk}(z) = \frac{z_j}{z_k}.$$

[19] Exercise: Check!

Construction 3.18 (Hyperplane Bundle and More)**.** In order to generate further line bundles on $\mathbb{P}^n$ we modify the transition functions to

$$g^m_{jk}(z_0 : \ldots : z_n) := \left(\frac{z_k}{z_j}\right)^m \ , \ z \in U_{jk},$$

where $m \in \mathbb{Z}$. $g^m_{jk} : U_{jk} \to \mathbb{C}$ is holomorphic, in particular smooth. Furthermore, for each $m \in \mathbb{Z}$ the collection $(g^m_{jk} \mid i, j \in I)$ satisfies the cocycle condition (C). Therefore, by Proposition 3.9 the cocycle (g^m_{jk}) determines a smooth line bundle, which will be denoted by $H(m)$. A slight modification of Proposition 3.9 yields, that $H(m)$ is a holomorphic line bundle.

Note that our tautological bundle is $T = H(-1)$. Obviously, $H(0)$ is the trivial bundle. $H := H(1)$ is called the HYPERPLANE BUNDLE, it is dual to $T = H(-1)$. H can alternatively defined as the space of all hyperplanes in $\mathbb{C}^{n+1}$, the n-dimensional complex vector subspaces of $\mathbb{C}^{n+1}$.

As before, one can show

Proposition 3.19. *$H(m)$ is not trivial as a holomorphic line bundle over $\mathbb{P}^n$ for $m \in \mathbb{Z}, m \neq 0$.*

See below, Corollary 3.23.

In order to determine the sections of each of the line bundles $H(m) \to \mathbb{P}^n$ over $\mathbb{P}^n$ we use the section condition (S) which has to be satisfied for the local functions which represent a given section.

We introduce for each $m \in \mathbb{Z}$ the m-homogeneous functions $\mathcal{E}_m(V)$ on a saturated[20] open subset $V = \gamma^{-1}(\gamma(V)) \subset \mathbb{C}^{n+1}$ as follows

$$\mathcal{E}_m(V) := \{F \in \mathcal{E}(V, \mathbb{C}) \mid \forall\, \lambda \in \mathbb{C}^\times\, \forall\, z \in V : F(\lambda z) = \lambda^m F(z)\}.$$

Now, let $U \subset \mathbb{P}^n$ be open and $V := \gamma^{-1}(U)$: Every section $s : U \to H(m)$ determines a function $\tilde{s} = F_s \in \mathcal{E}_m(V)$ in the following way: With respect to the open cover $(U_j)_{j=0,\ldots n}$ and to the transition functions

$$g_{jk}(z) = \left(\frac{z_k}{z_j}\right)^m \ , \ z = (z_0 : \ldots : z_n)\,,\ z_j \neq 0 \neq z_k\,,$$

the given section $s \in \Gamma(U, H(m))$ determines $f_j \in \mathcal{E}(U \cap U_{jk})$ such that (according to (S))

$$f_j = g_{jk} f_k \ \text{ on } \ U_{jk}\,.$$

Recall $s|_{U \cap U_j} = f_j s_j$ where $s_j(a) = \psi_j^{-1}(a, 1)$. We define

$$F_j(w) := w_j^m f_j(\gamma(w))\,,\ w \in \gamma^{-1}(U \cap U_j) = V \cap \gamma^{-1}(U_j)\,.$$

[20] V is saturated iff $\lambda V = V$ for all complex numbers $\lambda \in \mathbb{C}$.

For all $w \in \gamma^{-1}(U \cap U_{jk})$ we obtain

$$F_j(w) = w_j^m g_{jk}(\gamma(w)) f_k(\gamma(w)) = w_j^m \left(\frac{w_k}{w_j}\right)^m f_k(\gamma(w)) = F_k(w).$$

As a consequence, F_j and F_k agree on $V \cap \gamma^{-1}(U_{jk})$ and

$$F_s(w) := F_j(w) \text{ for } w \in V \cap \gamma^{-1}(U_j)$$

is a well-defined (smooth) function $\tilde{s} = F_s \in \mathcal{E}(V)$. Morover, it is m-homogeneous:

$$F_s(\lambda w) = (\lambda w_j)^m f_j(\gamma(w)) = \lambda^m w_j^m f_j(\gamma(w)) = \lambda^m g_s(w)\,,\ (w, \lambda) \in V \cap \gamma^{-1}(U_j) \times \mathbb{C}.$$

Therefore, $\tilde{s} = F_s \in \mathcal{E}_m(V)$. The induced map

$$\tilde{}: \Gamma(U, H(m)) \to \mathcal{E}_m(V)\,,\ s \mapsto \tilde{s}\,,$$

is linear over $\mathbb{C}$ (it is even $\mathcal{E}(U)$-linear) and injective. We have shown the first part of the following result:

Theorem 3.20. *For every open $U \subset \mathbb{P}^n$ and $V := \gamma^{-1}(U)$,*

$$\tilde{}: \Gamma(U, H(m)) \to \mathcal{E}_m(V)\,,$$

is an isomorphism.

Proof. It remains to prove that $\tilde{}$ is surjective. Let $F \in \mathcal{E}_m(V)$ und set for $z \in U \cap U_j$

$$f_j(z) := w_j^{-m} F(w)\,, \quad \text{if } \gamma(w) = z\,.$$

In case of $w' = \lambda w$, $\lambda \in \mathbb{C}^\times$, we have

$$(w_j')^{-m} g(w') = \lambda^{-m} w_j^{-m} \lambda^m g(w) = w_j^{-m} g(w)\,.$$

Therefore, $f_j \in \mathcal{E}(U \cap U_j)$ is well-defined. Moreover (f_j) satisfies (S):

$$f_j(z) = w_j^{-m} g(w) = \left(\frac{w_k}{w_j}\right)^m w_k^{-m} g(z) = \left(\frac{w_k}{w_j}\right)^m f_k(z)\,,\ z \in U \cap U_{jk}.$$

As a consequence, (f_j) defines a section $s \in \Gamma(U, H(m))$ such that $\tilde{s} = g$. □

We are also interested in determining the holomorphic sections of the bundles $H(m)$. In general, a line bundle over a complex manifold M (i.e. a manifold with enough holomorphically compatible charts) which has enough local trivializations which are biholomorphic (so that the transition functions are holomorphic) is called a Holomorphic Line Bundle. However, when it is clear from the context, one simply denotes it as a line bundle, as before. Let us introduce the following notations:

- $\mathcal{O}(U)$ is the $\mathbb{C}$-algebra of holomorphic functions $f : U \to \mathbb{C}$ on an open subset $U \subset M$ of a complex manifold M.
- $\Gamma_{\text{hol}}(U, L)$ is the $\mathcal{O}(U)$-module of holomorphic sections $s : U \to L$ of a (holomorphic) line bundle L on U.
- $\mathcal{O}_m(V)$ is the $\mathcal{O}(U)$-module of the m-homogeneous holomorphic functions on a saturated $V \subset \mathbb{C}^{n+1}$.
- $\mathbb{C}_m [z_0, z_1, \dots, z_n]$ is the $\mathbb{C}$-vector space of m-homogeneous complex polynomials in $n + 1$-variables for $m \geq 0$. $\mathbb{C}_m [z_0, z_1, \dots, z_n]$ is generated over $\mathbb{C}$ by the monomials $z_{i_1} z_{i_2} \dots z_{i_m}$, $i_j \in \{0, 1, \dots, n\}$ of degree m.

The arguments for the result of the preceding proposition also yield:

Corollary 3.21. *For every open $U \subset \mathbb{P}^n$ and $V := \gamma^{-1}(U)$,*

$$\tilde{\ }: \Gamma_{\text{hol}}(U, H(m)) \to \mathcal{O}_m(V)\,,$$

is an isomorphism.

With this result it is possible to determine all holomorphic sections in $H(m)$ rather explicitly:

Proposition 3.22.

$$\Gamma_{hol}(\mathbb{P}^n(\mathbb{C}), H(m)) \cong \{0\} \qquad \textit{for } m < 0\,,\ m \in \mathbb{Z}\,, \tag{20}$$

$$\Gamma_{hol}(\mathbb{P}^n(\mathbb{C}), H(m)) \cong \mathbb{C}_m [z_0, z_1, \dots, z_n] \qquad \textit{for } m \geq 0\,,\ m \in \mathbb{Z}\,. \tag{21}$$

Proof. For $m \geq 1$ one has to check that

$$\mathcal{O}_m(\mathbb{C}^{n+1} \setminus \{0\}) \cong \mathbb{C}_m [z_0, z_1, \dots, z_n]\,.$$

Since each $g \in \mathcal{O}(\mathbb{C}^{n+1} \setminus \{0\})$ has a unique extension to all of $\mathbb{C}^{n+1}$ as a holomorphic function (for $n \geq 1$ there are no isolated singularities for holomorphic functions of $n + 1$ variables, see Corollary B.12 in the Appendix) it is enough to observe that m-homogeneous holomorphic functions $g : \mathbb{C}^{n+1} \to \mathbb{C}$ are m-homogeneous polynomials.

For $m = 0$ the m-homogeneous functions are the constants. The case $m < 0$ is left as an exercise. □

As immediate consequences we obtain the following results:

Corollary 3.23.

- *The $\mathbb{C}$-vector space $\Gamma_{hol}(\mathbb{P}^n(\mathbb{C}), H(m))$ of holomorphic sections of the holomorphic line bundle $H(m)$ is finite dimensional for all $m \in \mathbb{Z}$.*[21]

[21] In general, the space of holomorphic sections of a holomorphic vector bundle over a compact complex manifold is finite dimensional.

- *All $H(m)$, $m \neq 0$, are nontrivial as holomorphic line bundles.*
- *The bundles $H(m)$ and $H(k)$, $m \neq k$, are not isomorphic as holomorphic line bundles.*

Proof. The first statement follows from the fact $\dim_{\mathbb{C}} \mathbb{C}_m[z_0, z_1, \ldots, z_n] < \infty$. For the second, observe that none of the above sections can be non-vanishing and apply the holomorphic version of Proposition 3.4. For the third: If $H(m)$ and $H(k)$ were isomorphic then, according to Corollary 3.11, there would exist $h_j \in \mathcal{O}(U_j)$ such that $g_{ij}^m = \frac{h_k}{h_j} g_{ij}^k$ on U_{ij} i.e.

$$\left(\frac{z_j}{z_i}\right)^m = \frac{h_j}{h_i}\left(\frac{z_j}{z_i}\right)^k, \quad \text{or} \quad \left(\frac{z_j}{z_i}\right)^{m-k} = \frac{h_j}{h_i},$$

which would imply that $H(m-k)$ is trivial in contradiction to the second statement. □

3.4 Classification

We conclude this chapter with a short digression of how cohomology is used to classify all line bundles on a given manifold M.

Observe, that two cocycles $(g_{jk}), (h_{jk})$ with respect to an open cover $(U_j)_{j \in I}$ which describe complex line bundles can be multiplied to yield another cocycle

$$f_{jk} := g_{jk} h_{jk} \in \mathcal{E}(U_{jk}, \mathbb{C}^\times).$$

In this way, a composition on the set

$$\mathrm{Pic}_{\mathrm{diff}}(M)$$

of isomorphism classes of line bundles on M is defined[22]. It is denoted $[L_g] * [L_h]$ when L_g resp. L_h is the line bundle given by $g = (g_{jk})$, resp. $h = (h_{jk})$, and $[L_g]$ is the isomorphism class of the bundles L_h. The composition is associative and commutative with the class of the trivial bundle as unit. The inverse of a class $[L] \in \mathrm{Pic}_{\mathrm{diff}}(M)$, where $g = (g_{jk})$ generates $L = L_g$ is the class generated by the so-called Dual Line Bundle $L^\vee$ associated to the cocycle (g_{jk}^{-1}).

In this way $\mathrm{Pic}_{\mathrm{diff}}(M)$ is endowed with the structure of an abelian group. It is called the (differentiable) Picard Group.

This composition can alternatively be described by the tensor product $L \otimes L'$ of line bundles L and L'[23], by

$$([L], [L']) \mapsto [L \otimes L'] = [L] * [L'].$$

[22] If necessary one has to refine the open cover (U_j).

[23] The tensor product of vector bundles is explained in the Appendix E.

The Picard group $\mathrm{Pic}_{\mathrm{diff}}(M)$ can be identified with the first (sheaf) cohomology group

$$\mathrm{Pic}_{\mathrm{diff}}(M) \cong \check{H}^1(M, \mathcal{E}^\times)$$

with respect to the sheaf $\mathcal{E}^\times$ of germs of nowhere vanishing (smooth) functions. (See Section E.3 for a short introduction to sheaf cohomology.) In fact, the equivalence relation for transition functions g_{ij}, g'_{ij} representing line bundles L, L' coincides with the equivalence relation of the Čech cocycles $(g_{ij}), (g'_{ij})$ as is explained in Example E.16!

Furthermore, $\check{H}^1(M, \mathcal{E}^\times)$ is isomorphic to $\check{H}^2(M, \mathbb{Z})$. This can be shown using the connecting homomorphism $\delta : \check{H}^1(M, \mathcal{E}^\times) \to \check{H}^2(M, \mathbb{Z})$, induced by the short exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{E} \xrightarrow{e} \mathcal{E}^\times \longrightarrow 0$$

with $e(\phi) := \exp(2\pi i \phi)$, $\phi \in \mathcal{E}$.

Note, that $c_1(L) = \delta[L]$ is called the Chern class of the line bundle L:

Since it is known that $\check{H}^2(\mathbb{P}^n(\mathbb{C}), \mathbb{Z}) \cong \mathbb{Z}$ (cf. [**?**]), we conclude

$$\mathrm{Pic}_{\mathrm{diff}}(\mathbb{P}^n) \cong \mathbb{Z}.$$

In the case of a complex manifold M the "holomorphic" Picard group $\mathrm{Pic}(M)$ is introduced analoguously:

$$\mathrm{Pic}(M) := \{[L] \mid L \text{ a holomorphic line bundle}\},$$

where $[L]$ is the class of holomorphic line bundles which are (holomorphically) isomorphic to L. The multiplication in the group is again given by

$$\big([(g_{jk})], \big[(g'_{jk})\big]\big) \longmapsto \big[(g_{jk}g'_{jk})\big] \quad \text{resp.}$$

$$([L], [L']) \longmapsto [L \otimes L'] .$$

Similar to the differentiable case, the group $\mathrm{Pic}(M)$ is essentially $\check{H}^1(M, \mathcal{O}^\times)$ where $\mathcal{O}^\times$ is the sheaf of nowhere vanishing holomorphic functions on M.

In particular, in the case of complex projective space $\mathbb{P}^n(\mathbb{C})$ the multiplication in $\mathrm{Pic}(\mathbb{P}^n(\mathbb{C}))$ of the classes $[H(m)]$ takes the form $[H(m)] * [H(k)] = [H(m+k)] = [H(m) \otimes H(k)]$ and $[H(m)] * [H(-m)] = [H(0)] = \text{unit}$. As a consequence, the results in 3.23 imply, that the $[H(m)], m \in \mathbb{Z}$, provide a subgroup $\{[H(m)] \mid m \in \mathbb{Z}\} \subset \mathrm{Pic}(\mathbb{P}^n(\mathbb{C}))$ of the Picard group $\mathrm{Pic}(\mathbb{P}^n(\mathbb{C}))$ isomorphic to $\mathbb{Z}$.

Moreover, by elementary cohomological methods it can be shown that this subgroup is the full Picard group, i.e. $\mathrm{Pic}(\mathbb{P}^n(\mathbb{C})) \cong \{[H(m)] \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$. As a consequence,

$$\mathrm{Pic}_{\mathrm{diff}}(\mathbb{P}^n) \cong \mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z} \cong \{[H(m)] \mid m \in \mathbb{Z}\}. \tag{22}$$

Summary:

In this chapter the basic notions and properties of line bundles and its cocycles are presented in order to be able to develop the geometry of line bundles which encompasses connections, parallel transport, curvature and Hermitian structure, the subject of the next four chapters. In addition, the example of the tautological line bundle T over the complex projective space $\mathbb{P}^n(\mathbb{C})$ is studied in detail, thereby applying the cocycle representation in order to determine the space of holomorphic sections $\Gamma_{\mathrm{hol}}(\mathbb{P}^n(\mathbb{C})), L$ for the line bundle $L = T$ and its companions $L = T^{\otimes m}$ $(= H(-m))$ for all $m \in \mathbb{Z}$.

There is a general aspect present in the programme of Geometric Quantization: Whenever holomorphic structure is available, i.e. when M can be viewed as to be a complex manifold, it is worthwhile to investigate the holomorphic case besides the smooth case, since, in general, for holomorphic functions there are strong results available for applications in Geometric Quantization. Note, that a symplectic manifold is not far from being a complex manifold, as well.

Because of the need of holomorphic functions in Geometric Quantization we present basic notions and results of complex analysis in the Appendix.

4 Connections

The concept of a connection on a line bundle and - more generally - on a vector bundle is fundamental for the geometry of such bundles: A connection is the basic geometric structure on a bundle. Moreover, connections on line bundles are needed in the program of Geometric Quantization.

A connection on a vector bundle induces a connection form on the corresponding frame bundle, - which is $L^\times$ in the case of a line bundle L. Conversely, a connection form on a principal fibre bundle P over M with structure group G induces a connection on all associated vector bundles $E_\rho = P \times_\rho \mathbb{C}^r$, where $\rho : G \to \mathrm{GL}(r, \mathbb{C})$ is a representation of Lie groups. In this chapter we focus on line bundles and present several equivalent descriptions of the notion of a connection. The general case of vector bundles and principal fibre bundles is treated in the Appendix D.

In the "ansatz" of geometric quantization presented in Chapter 2 together with our understanding of sections in a line bundle we are guided to replace $\mathcal{E}(M)$ with the space $\Gamma(M, L)$ of sections of a line bundle L. This leads immediately to the question of what the directional derivatives should be in the generalized context of line bundles. Geometry gives the answer: the Lie derivatives L_X on $\mathcal{E}(M)$ are replaced by covariant derivatives (also called connections) ∇_X for vector fields $X \in \mathfrak{V}(M)$.

4.1 Local Connection Form

Definition 4.1. A Connection ("Zusammenhang") on a line bundle $\pi : L \to M$ over a manifold M is a collection of maps

$$\nabla : \mathfrak{V}(U) \to \mathrm{End}_{\mathbb{C}}(\Gamma(U, L)) \,, \;\; X \mapsto \nabla_X \,,$$

indexed[24] by the open subsets $U \subset M$, which are compatible with restrictions to open subsets $V \subset U$[25], such that the following properties are satisfied:

(K1) ∇ is an $\mathcal{E}(U)$ module homomorphism with respect to X, i.e.

$$\nabla_{fX+Y} = f\nabla_X + \nabla_Y, \;\; \forall X, Y \in \mathfrak{V}(U), \forall f \in \mathcal{E}(U);$$

(K2) $\nabla_X \in \mathrm{End}_{\mathbb{C}}(\Gamma(U, L))$ for $X \in \mathfrak{V}(U)$ with the derivative-like property:

$$\nabla_X(fs) = (L_X f)s + f\nabla_X s \,, \quad \text{for all} \quad s \in \Gamma(U, L) \,,\; f \in \mathcal{E}(U) \,,\; X \in \mathfrak{V}(U).$$

[24] We omit the index "U" attached to ∇_X to make formulas not too complicated.

[25] That ∇_X has to be compatible with restrictions means essentially the following if V is an open subset of U: For $X \in \mathfrak{V}(U)$ and $s \in \Gamma(U, L)$ the restrictions satisfy $(\nabla_X(s))|_V = \nabla_{X|_V}(s|_V)$

A connection in the form above is also called a KOSZUL CONNECTION. The operator ∇_X is called COVARIANT DERIVATIVE ("Kovariante Ableitung").

THE CASE OF A TRIVIAL BUNDLE:

We observe immediately that on the trivial bundle

$$L = M \times \mathbb{C},$$

where we have $\Gamma(U, L) \cong \mathcal{E}(U)$ for open $U \subset M$, the Lie derivative $L_X : \mathcal{E}(U) \to \mathcal{E}(U)$ is an example of a connection, since $L_{X+Y} = L_X + L_Y$ as well as $L_{fX} = fL_X$, and $L_X(fg) = (L_X f)g + f(L_X g)$.

Are there more connections on $M \times \mathbb{C}$? How do they look?

Every section $s \in \Gamma(M, M \times \mathbb{C})$ has the form $s(a) = (a, B(a)), a \in M$, with a uniquely defined function $B \in \mathcal{E}(M)$. With respect to the previously defined section $s_1 \in \Gamma(M, L)$, $s_1(a) = (a, 1), a \in M$, we have $s = Bs_1$. A given connection ∇ defines, in particular, a map

$$B : \mathfrak{V}(M) \to \mathcal{E}(M)$$

by the unique function $B(X) \in \mathcal{E}(M)$ with $\nabla_X s_1 = B(X)s_1$, or $\nabla_X s_1(a) = (a, B(X)(a))$, $a \in M$. By (K1) B is $\mathcal{E}(M)$-linear:

$$B(fX+Y)s_1 = \nabla_{fX+Y}s_1 = f\nabla_X s_1 + \nabla_Y s_1 = fB(X)s_1 + B(Y)s_1 = (fB(X)+B(Y))s_1.$$

Therefore, B is a 1-form $B \in \mathcal{A}^1(M)$.

Lemma 4.2. *Every connection ∇ on $M \times \mathbb{C} = L \to M$ is of the form:*

$$\begin{aligned} \nabla_X s &= (L_X f + 2\pi i A(X) f)\, s_1, \quad \textit{if } s = f s_1|_U, f \in \mathcal{E}(U) \\ &= \left(\frac{L_X f}{f} + 2\pi i A(X) \right) s, \end{aligned}$$

where $s = fs_1|_U, f \in \mathcal{E}(M)$ and $X \in \mathfrak{V}(M)$, and where $A \in \mathcal{A}^1(M)$ is a one form[26].

Conversely, for each $A \in \mathcal{A}^1(M)$ the above formula defines a connection on L.

Proof. $\nabla_X s = \nabla_X(fs_1) = (L_X f)s_1 + fB(X)s_1$ according to (K2), and with $A = \frac{1}{2\pi i}B$ we get the corresponding form.

Conversely, for any $A \in \mathcal{A}^1(M)$:

$$\nabla_X s := (L_X f + 2\pi i A(X) f)\, s_1, \quad s = fs_1 \in \Gamma(U, L), X \in \mathfrak{V}(U),$$

[26]The choice of the factor $2\pi i$ in the formula fits to several expressions in the analysis of connections and their curvatures. It can be replaced by other constants.

defines a connection on L. (K1) is evidently satisfied. For (K2), take $g \in \mathcal{E}(U)$. Then for $s = fs_1, f \in \mathcal{E}(U)$:

$$\begin{aligned}\nabla_X(gs) &= \nabla_X(gf)s_1 = (L_X(gf) + 2\pi i A(X)(gf))s_1 \\ &= ((L_X g)f + gL_X f + gA(X)f)s_1 = (L_X g)s + g\nabla_X s.\end{aligned}$$

□

Note, that

$$\Gamma(U, T^*M \otimes L) \cong (\mathfrak{V}(U))^* \otimes \Gamma(U, L) \cong \mathrm{Hom}_{\mathcal{E}(M)}(\mathfrak{V}(U), \Gamma(U, L))$$

And a connection according to Definition 4.1 can also be described by a $\mathbb{C}$-linear map

$$\nabla : \Gamma(U, L) \to \Gamma(U, T^*M \otimes L)$$

with the property

$$\nabla(fs) = df \otimes s + f\nabla s, \quad s \in \Gamma(U, L), f \in \mathcal{E}(U)$$

and compatible with restrictions.

Local Connection Form:

We now want to describe an arbitrary connection on a non-trivial line-bundle $L \to M$ locally by using Lemma 4.2. Let (U_j) be an open cover of M with local trivializations $\psi_j : L_{U_j} \to U_j \times \mathbb{C}$ and corresponding transition functions $g_{jk} \in \mathcal{E}(U_{jk}, \mathbb{C}^\times)$. For each $U_j \times \mathbb{C}$ there exists $A_j \in \mathcal{A}^1(U_j)$ such that

$$\nabla_X|_{U_j}(fs_j) = (L_X f + 2\pi i A_j(X)f)s_j\,, \tag{23}$$

where $s_j(a) = \psi_j^{-1}(a, 1)$, as before.

We want to discover the interrelations of A_j and A_k on an intersection $U_{jk} = U_j \cap U_k$. Given a section $s \in \Gamma(U, L)$, we describe it locally by $s|_{U\cap U_j} = f_j s_j$, $s|_{U\cap U_k} = f_k s_k$, with suitable $f_j \in \mathcal{E}(U \cap U_j)$, $f_k \in \mathcal{E}(U \cap U_k)$. We obtain

$$s|_{U\cap U_{jk}} = f_j s_j = f_k s_k, \text{ on } U_{jk} \cap U$$

Consequently, on U_{jk} the local expressions for $\nabla_X s$ fulfill

$$\nabla_X s = (L_X f_j + 2\pi i A_j(X) f_j)\, s_j = (L_X f_k + 2\pi i A_k(X) f_k)\, s_k$$

We insert $s_j = g_{kj} s_k, f_j = g_{jk} f_k$, and obtain

$$\begin{aligned}(L_X f_k + 2\pi i A_k(X) f_k)\, s_k &= (L_X(g_{jk} f_k) + 2\pi i A_j(X) g_{jk} f_k)\, g_{kj} s_k \\ &= ((L_X g_{jk}) f_k + g_{jk} L_X f_k + 2\pi i A_j(X) g_{jk} f_k)\, g_{kj} s_k \\ &= (L_X f_k + 2\pi i A_j(X) f_k)\, s_k + (L_X g_{jk}) f_k g_{kj} s_k\end{aligned}$$

Hence,

$$2\pi i A_k(X) = 2\pi i A_j(X) + dg_{jk}(X)g_{kj} = 2\pi i A_j(X) + \frac{dg_{jk}(X)}{g_{jk}}\,,$$

which implies

$$(Z) \qquad \boxed{A_k = A_j + \frac{1}{2\pi i}\frac{dg_{jk}}{g_{jk}}} \tag{24}$$

on U_{jk}.

The A_j are the Local Connection Forms of the connection ∇. We have shown the first part of the following result:

Proposition 4.3. *Let (U_j) be an open cover of M such that the line bundle $L \to M$ of M has trivializations over U_j with transition functions $g_{jk} \in \mathcal{E}(U_{jk}, \mathbb{C}^\times)$. Any connection ∇ on L determines uniquely a collection (A_j) of 1-forms $A_j \in \mathcal{A}^1(U_j)$ with (Z). Conversely, (A_j) with (Z) induces a connection on L.*

Proof. Indeed, given $s \in \Gamma(U, L)$ with $s|_{U\cap U_j} = f_j s_j$, $f_j \in \mathcal{E}(U \cap U_j)$, by

$$\nabla_X f_j s_j := (L_X f_j + 2\pi i A_j(X) f_j) s_j$$

a connection is defined. □

The local 1-forms A_j are called the Local Connection Forms ("Zusammenhangsform") of the connection ∇.

4.2 Global Connection Form

In the next step, we want to understand how every line bundle connection is induced by a global connection on the corresponding frame bundle $L^\times \subset L$.

Let $\pi : L \to M$ be a line bundle. The Frame Bundle ("Reperbündel") $L^\times$ belonging to L is the bundle

$$L^\times := \{p \in L \mid p \neq 0_a, a = \pi(p)\} = L \setminus \{0_a \in L_a \mid a \in M\} = L \setminus z(M)\,,$$

where $z : M \to L$, $a \to 0_a$, is the zero section. $L^\times$ is a principal fibre bundle with structure group $\mathbb{C}^\times$, the multiplicative group of nonzero complex numbers (cf. Appendix, Construction D.4). The projection

$$\pi : L^\times \to M,$$

is the restriction of the projection $\pi : L \to M$. Moreover, the right action of $\mathbb{C}^\times$ is

$$\Psi : L^\times \times \mathbb{C}^\times \to L^\times, (p, z) \mapsto pz = \Psi(p, z),$$

where $pz = \Psi(p, z)$ is simply the multiplication in the fibre $L_a, a = \pi(p)$. The frame bundle $L^\times$ has the local trivializations

$$\psi_j : L^\times_{U_j} \to U_j \times \mathbb{C}^\times,$$

the restrictions of the trivializations $\psi_j : L_{U_j} \to U_j \times \mathbb{C}$ of the original line bundle $L \to M$. ψ_j respects the right action (the multiplication) of $\mathbb{C}^\times$ on $L^\times$: $\psi_j(pz) = \psi_j(p)z$ for all $(p, z) \in L^\times \times \mathbb{C}^\times$.

Let us consider a connection ∇ on L given by the one-forms $A_j \in \mathcal{A}^1(U_j)$ satisfying (Z) with respect to a suitable open cover (U_j) and corresponding transition functions g_{ij}, according to Proposition 4.3. Then the forms A_j can be lifted to $L^\times_{U_j}$ as the pullbacks by π (c.f. Appendix, Definition A.33). We obtain

$$\pi^*(A_j) \in \mathcal{A}^1(L^\times_{U_j}), j \in I.$$

By Proposition 4.3 the local forms A_j satisfy (Z), hence

$$\pi^* A_k = \pi^* A_j + \pi^* \left(\frac{1}{2\pi i} \frac{dg_{jk}}{g_{jk}} \right).$$

Moreover, one can show the following lemma:

Lemma 4.4.

$$\psi_j^* \left(\frac{dz}{z} \right) - \psi_k^* \left(\frac{dz}{z} \right) = \pi^* \left(\frac{dg_{jk}}{g_{jk}} \right)$$

Here, $\frac{dz}{z}$ is an abbreviation of $pr_2^*(\frac{dz}{z})$ on U_j resp. U_k, or simply $\frac{1}{z}dz$ on $U_j \times \mathbb{C}^\times$.

Proof. Exercise! □

As a consequence of these technical results, the two expressions

$$\pi^* A_j + \frac{1}{2\pi i} \psi_j^* \left(\frac{dz}{z} \right) \quad , \quad \pi^* A_k + \frac{1}{2\pi i} \psi_k^* \left(\frac{dz}{z} \right)$$

agree on $L^\times_{U_{jk}}$ and define a global 1-form on α on $\mathcal{A}^1(L^\times)$ by:

$$\alpha|_{L^\times_{U_j}} := \pi^* A_j + \frac{1}{2\pi i} \psi_j^* \left(\frac{dz}{z} \right), j \in I. \tag{25}$$

This 1-form $\alpha \in \mathcal{A}(L^\times)$ is the Global Connection Form of the connection! It is independent of the choice of local forms.

In order to investigate the main properties of the global connection form α let us introduce the fundamental field induced by the action of $\mathbb{C}^\times$ on $L^\times$. This is a special case of the notion of a fundamental field on a principal fibre bundle as defined in D.11.

Definition 4.5. For $\xi \in \mathbb{C} = \operatorname{Lie}\mathbb{C}^\times$ define the FUNDAMENTAL FIELD $Y_\xi : L^\times \to TL^\times$ by:

$$Y_\xi(p) := \frac{d}{dt}\left(p \exp(2\pi i \xi t)\right)|_{t=0} = [p\exp(2\pi i\xi t)]_p \in T_p L^\times .$$

It is easy to see that $T\pi(Y_\xi) = 0$ and $Y_\xi(p)$ is a tangent vector along the fibre $L_a^\times$ for $p \in L^\times$ with $a = \pi(p)$: $Y_\xi(p) \in T_p L_a^\times$. More is true: The fundamental vector fields generate the so called VERTICAL BUNDLE $V := \operatorname{Ker} T\pi$. Since the map $\mathbb{C} \to T_p L_a^\times$, $\xi \mapsto Y_\xi(p)$ is a linear isomorphism ($Y_\xi(p) \neq 0$ for $\xi \neq 0$!), we have

$$V_p = T_p L_a^\times = \{Y_\xi(p) \mid \xi \in \mathbb{C}\}. \tag{26}$$

In particular, a vertical vector field X, i.e. $X \in \mathfrak{V}(L^\times)$, $X(p) \in V_p$ for all $p \in L^\times$, can be described by $X(p) = Y_{\xi(p)}$ where $\xi(p) = \sigma_p(X(p))$ with $\sigma_p : V_p \to \mathbb{C}$ the inverse of $\xi \mapsto Y_\xi(p)$. Note, that Y_ξ is not the same as $Y_{\xi(p)}$, the above defined fundamental field Y_ξ is $Y_{\xi(p)}$ with constant map $\xi(p) = \xi$!

Lemma 4.6. *The global connection form $\alpha \in \mathcal{A}(L^\times)$ of a connection ∇ on L satisfies*

(I1) $\alpha\left(Y_\xi\right) = \xi$ *for all* $\xi \in \mathbb{C}$,

(I2) $\Psi_c^* \alpha = \alpha$ *for all* $c \in \mathbb{C}^\times$.

Proof. Let $p \in L_{U_j}^\times$ with $\psi_j(p)) = (a, z) \in U_j \times \mathbb{C}^\times$. Then $\pi(p\exp(2\pi i\xi t)) = \pi(p)$, hence $\pi^* A_j(Y_\xi) = A_j(0) = 0$. Furthermore,

$$\psi_j^*\left(dz\right)\left(Y_\xi\right)(p) = dz[(a, z\exp(2\pi i\xi t))]_{(a,z)} = dz[z\exp(2\pi i\xi t)]_z = 2\pi i\xi z\,,$$

hence

$$\frac{1}{2\pi i}\psi_j^*\left(\frac{dz}{z}\right)\left(Y_\xi\right) = \xi.$$

Using the definition of the global connection form α (cf. (25)), we deduce condition (I1).

To show (I2) let $X = [p(t)]_p$ be a tangent vector at $p = p(0) \in L^\times|_{U_j}$. By definition we have

$$\Psi_c^*\alpha(X)(p) = \alpha([p(t)c]_{pc}) = A_j([\pi(p(t)c)]_{\pi(pc)}) + \frac{1}{2\pi i}\left(\frac{dz}{z}\right)\left([\psi_j(p(t)c))]_{\psi(pc)}\right). \tag{27}$$

Because of $\pi(p(t)c) = \pi(p(t))$, the first term in (27) satisfies

$$A_j\left([\pi(p(t)c)]_{\pi(pc)}\right) = A_j\left([\pi(p(t))]_{\pi(p)}\right) .$$

For the second term of (27) let

$$\psi_j(p(t)) = (a(t), z(t)) \;\text{ with }\; \psi_j(p) = (a(0), z(0)) =: (a, z)\,.$$

Then $\psi_j(p(t)c) = (a(t), z(t)c)$ implies $dz([\psi_j(p(t)c))]_{\psi_j(pc)}) = dz([(a(t), z(t)c)]_{(a,zc)}) = \dot{z}(0)c$ and $dz\left([\psi_j(p(t))]_{\psi_j(p)}\right) = dz([(a(t), z(t))]_{(a,z)}) = \dot{z}(0)$. Hence, the second term of (27) is also independent of c:

$$\frac{1}{2\pi i}\left(\frac{dz}{z}\right)\left([\psi_j(p(t)c))]_{\psi(pc)}\right) = \frac{1}{2\pi i}\frac{\dot{z}(0)c}{z(0)c} = \frac{1}{2\pi i}\frac{\dot{z}(0)}{z(0)} = \frac{1}{2\pi i}\left(\frac{dz}{z}\right)\left([\psi_j(p(t))]_{\psi_j(p)}\right)$$

This proves (I2). □

Proposition 4.7. *Let $\alpha \in \mathcal{A}^1(L^\times)$ be the global connection form of a connection, defined as above. Then α satisfies the conditions* (I1), (I2), *and the covariant derivative has the form*

$$\boxed{\nabla_X s = 2\pi i s^*\alpha(X)s,} \tag{28}$$

for $X \in \mathfrak{V}(U)$ and $s \in \Gamma(U, L^\times)$, where $U \subset M$ is open in M.

Conversely, each $\alpha \in \mathcal{A}^1(L^\times)$ with (I1) *and* (I2) *defines a connection on L by* (28).

Proof. We confirm the formula $\nabla_X s = 2\pi i s^*\alpha(X)s$ and leave the rest as an exercise: We can restrict the consideration to the case $s \in \Gamma(U_j, L)$. We have $s = fs_j$, where $s_j(a) := \psi_j^{-1}(a, 1)$ and $f \in \mathcal{E}(U_j)$. We know from Proposition 4.3

$$\nabla_X s = (L_X f + 2\pi i A_j(X)f)\, s_j,$$

where A_j is the local connection form. Now,

$$s^*\alpha = s^*\pi^*A_j + s^*\frac{1}{2\pi i}\psi_j^*\left(\frac{dz}{z}\right) = (\pi \circ s)^*A_j + (\psi_j \circ s)^*\left(\frac{1}{2\pi i}\frac{dz}{z}\right).$$

We know $\pi \circ s = \mathrm{id}_{U_j}$ and $\psi_j \circ s(a) = (a, f(a)), a \in U_j$. We assume U_j to be a coordinate neighbourhood, so that a vector field $X \in \mathfrak{V}(U_j)$ has the form $X : U_j \to U_j \times \mathbb{R}^n, a \longmapsto (a, V(a)), V(a) \in \mathbb{R}^n$. We conclude

$$\left((\psi_j \circ s)^*\left(\frac{dz}{z}\right)\right)(X)(a) = \frac{dz}{f(a)}\left(T(\psi_j \circ s).V(a)\right).$$

The derivative of $\psi_j \circ s$ is

$$T(\psi_j \circ s) = \begin{pmatrix} E_n \\ \partial_1 f, \ldots \partial_n f \end{pmatrix},$$

with the Jacobi matrix as a block matrix (E_n is the identity matrix in $\mathbb{R}^{n\times n}$). As a consequence,

$$T(\psi_j \circ s)\, V(a) = \partial_j f(a)V^j(a) = L_X f(a). \tag{29}$$

Therefore

$$2\pi i s^*\alpha(X) = 2\pi i \left(A_j(X) + \frac{1}{2\pi i}\frac{L_X f}{f} \right), \text{ i.e}$$

$$2\pi i s^*\alpha(X) f s_j = \left(2\pi i A_j(X) f + L_X f\right) s_j,$$

and finally,

$$2\pi i s^*\alpha(X) s = \nabla_X s.$$

□

4.3 Horizontal Bundle

There are more possibilities to describe a connection on a line bundle $\pi : L \to M$. With the global connection form α one defines the HORIZONTAL BUNDLE $H := \operatorname{Ker}\alpha$ over $L^\times$. H obtains its manifold structure from the inclusion $H \subset TL^\times$ and its linear structure from the inclusion $H_p \subset T_pL^\times$, $p \in L^\times$. Because of property (I1), $\alpha(Y_\xi) = \xi$, $\xi \in \mathbb{C}$, the linear map $\alpha_p : T_pL^\times \to \mathbb{C}$ is surjective, so that $\operatorname{Ker}\alpha_p = H_p$ is a n-dimensional subspace of the tangent space $T_pL^\times$ of real dimension $n+2$. Consequently, H is a real vector bundle of rank n.

Proposition 4.8. *For a global connection form α on $L^\times$ the horizontal bundle $H = \operatorname{Ker}\alpha$ satisfies*

(H1) $H \oplus V = TL^\times$ *(H has the vertical bundle as a complement),*

(H2) $T\Psi_c(H_p) = H_{pc}$ *(H is invariant).*

Conversely, such a horizontal bundle defines a connection on L.

Proof. The property $\alpha(Y_\xi) = \xi$ (I1) implies $H_p \cap V_p = \{0\}$ since V_p is generated by the $Y_\xi(p)$, $\xi \in \mathbb{C}$. Therefore, $H_p \oplus V_p = T_pL^\times$, which is (H1). To prove (H2) it is enough to show $T\Psi_c(H_p) \subset H_{pc}$. But for $X \in H_p$ the condition (I2) implies $\Psi_c^*\alpha(X) = \alpha(X) = 0$, hence $\alpha(T\Psi_c(X)) = 0$, i.e. $T\Psi_c(X) \in H_{pc}$.

The converse follows from the next proposition. □

For a subbundle H with (H1) and (H2) let $v : TL^\times \to TL^\times$ the projection with $\operatorname{Ker} v = H$ and $\operatorname{Im} v = V$. Then $T\Psi_c \circ v = v \circ T\Psi_c$ can be deduced from (H2).

Proposition 4.9. *Let $v : TL^\times \to TL^\times$ a homomorphism of vector bundles with*

(V1) $v \circ v = v$ *(v is a projection) and* $\operatorname{Im} v = V$

(V2) $T\Psi_c \circ v = v \circ T\Psi_c$ *(v is equivariant)*

Then $\alpha = \sigma \circ v$ defines a connection form.

Proof. (I1) holds immediately because of $v(Y_\xi) = Y_\xi$ by (V1): $\alpha(Y_\xi) = \sigma Y_\xi = \xi$ (recall that $\sigma_p : V_p \to \mathbb{C}$ is the inverse of the map $\xi \mapsto [p\exp(2\pi i\xi t] = Y_\xi(p))$. (I2) is clear for $X \in \operatorname{Ker} v_p$: $\Psi_c^*\alpha(X) = \sigma \circ v \circ T\Psi_c(X) = \sigma \circ T\Psi_c \circ v(X) = 0 = \alpha(X)$. And for Y_ξ we obtain

$$\Psi_c^*\alpha(Y_\xi)(p) = (\sigma_{pc} \circ v_{pc} \circ T_p\Psi_c)(Y_\xi)(p) = (\sigma_{pc} \circ v_{pc}(Y_\xi)(pc) = \alpha(Y_\xi)(pc),$$

because of $T_p\Psi_c)(Y_\xi)(p) = Y_\xi(pc)$, which confirms (I2). □

For the formulation of the quantum operator in the context of half-density quantization and half-form quantization we need another equivalent description of a connection, which uses the frame bundle $L^\times$ of L and where the operation ∇_X on sections in L is essentially reduced to a Lie derivative on functions on $L^\times$.

We begin with the fact, that the sections of L can be described by invariant functions on $L^\times$. Every section $s \in \Gamma(M, L)$ of the line bundle induces a unique function $s^\sharp \in \mathcal{E}(L^\times)$ which satisfies

$$s(a) = ps^\sharp(p)\,,\ p \in \pi^{-1}(a)\,,$$

for $p \in L^\times$.

Lemma 4.10. $s^\sharp : L^\times \to \mathbb{C}$ *is a well-defined smooth function with* $s^\sharp \circ s = 1$. *Furthermore,*

$$s^\sharp(pc) = s^\sharp(p)c^{-1}$$

for all $p \in L^\times$ *and* $c \in \mathbb{C}^\times$. *With the notation*

$$\mathcal{E}_{-1}(L^\times) := \{g \in \mathcal{E}(L^\times) \mid g(pc) = c^{-1}g(p) = g(p)c^{-1} \text{ for all } (p,c) \in L^\times \times \mathbb{C}^\times\}$$

the map

$$^\sharp : \Gamma(M, L) \to \mathcal{E}_{-1}(L^\times)\,,\ s \mapsto s^\sharp\,,$$

is an isomorphism of $\mathcal{E}(M)$*-modules.*

Proof. Of course, $s^\sharp$ is well-defined and smooth, which can be seen using a local trivialization. The identity $s^\sharp \circ s = 1$ follows directly from $s(a) = s(a)s^\sharp(s(a))$. Because of $s(a) = ps^\sharp(p) = pcs^\sharp(pc)$ the invariance $s^\sharp(pc) = c^{-1}s^\sharp(p)$ holds. It is easy to see, that $^\sharp$ is $\mathcal{E}(M)$-linear and injective. Finally, this map is surjective: For $g \in \mathcal{E}_{-1}(M)$ the definition $s(a) := pg(p)$, $p \in \pi^{-1}(a)$, yields a section s of L with $s^\sharp = g$. □

The action of the fundamental field Y_c on the invariant functions $g \in \mathcal{E}_{-1}(L^\times)$ is simply multiplication:

Lemma 4.11. $L_{Y_c}g = -2\pi i g$ *for* $g \in \mathcal{E}_{-1}(L^\times)$.

Proof.

$$\begin{aligned} L_{Y_c} g(p) =& \frac{d}{dt} g(e^{2\pi i c t} p)|_{t=0} \\ =& \frac{d}{dt} e^{-2\pi i c t} g(p)|_{t=0} \\ =& -2\pi i c g(p) \end{aligned}$$

□

Now, let a connection on L be given by the global connection form α and its horizontal bundle $H \subset TL^\times$. Since for $p \in L_a^\times$, $a \in M$, the restriction

$$T_p\pi|_{H_p} : H_p \to T_a M$$

of $T_p\pi$ is bijective, each $X \in T_a M$ has a unique lift $X^\sharp \in H_p$ such that $T_p\pi(X^\sharp) = X$.

Definition 4.12. $X^\sharp \in H_p$ is called the HORIZONTAL LIFT of $X \in T_a M$.

The horizontal lift $X^\sharp$ is determined by the two conditions $T_p\pi(X^\sharp) = X$ and $\alpha(X^\sharp) = 0$, in other words: $\{X^\sharp\} = (T_p\pi)^{-1}(X) \cap \operatorname{Ker} \alpha_p$.

The following result shows that the connection ∇_X can be defined as a Lie derivative on the level of $L^\times$.

Proposition 4.13. *Let ∇ be a connection on L. For sections $s \in \Gamma(M, L)$ and vector fields $X \in \mathfrak{V}(M)$ the following formula holds*

$$\boxed{L_{X^\sharp} s^\sharp = (\nabla_X s)^\sharp.}$$

Proof. First of all, we show that $X^\sharp(p) = T_a s(X(a)) - Y_{s^*\alpha(X)}(s(a))$, $\pi(p) = a$, by establishing the above two conditions. Because of $\alpha_p(T_a s(X(a)) = (s^*\alpha_p)(X(a))$, we have

$$\alpha(T_a s(X(a)) - Y_{s^*\alpha(X)}(s(a))) = s^*\alpha(X(a)) - s^*\alpha(X(a)) = 0\,.$$

Moreover, $T_p\pi(Y_c) = 0$, which implies

$$T_p\pi(T_a s(X(a)) - Y_{s^*\alpha(X)}(s(a))) = T_p\pi T_a s(X(a)) = X(a).$$

We now determine $L_{Ts(X)} s^\sharp$ and $L_{Y_{s^*\alpha(X)}} s^\sharp$ to obtain $L_{X^\sharp} s^\sharp$. Let $\gamma(t)$ represent $X(a)$, i.e. $X(a) = [\gamma(t)]$, where $\gamma(0) = a$.

$$L_{Ts(X)} s^\sharp(p) = \frac{d}{dt} s^\sharp(s(\gamma(t)))|_{t=0} = 0\,,$$

since $s^\sharp \circ s = 1$. The outcome of the other term is

$$L_{Y_{s^*\alpha(X)}} s^\sharp = -2\pi i s^*\alpha(X) s^\sharp\,,$$

according to Lemma 4.11.

Altogether, $L_{X^\sharp} s^\sharp = 2\pi i s^* \alpha(X) s^\sharp$ and the last term is $(\nabla_X s)^\sharp$ since $2\pi i s^* \alpha(X) s = \nabla_X s$ saccording to Proposition 4.7 □

The last result gives another formulation of the concept of a connection:

Proposition 4.14. *The horizontal lifts define a bundle homomorphism*

$$\Gamma : TM \times_M L^\times \to TL^\times \ , \ (X,p) \mapsto X^\sharp \, ,$$

on the fibre product $TM \times_M L^\times$ *(cf. Definition A.8), which is equivariant, i.e. such that*

$$\Gamma(X, pc) = T\Psi_c(\Gamma(X,p)).$$

Conversely, an eqivariant bundle homomorphism $\Gamma : M \times_M L^\times \to TL^\times$ *defines a connection.*

Remark 4.15. The horizontal lift $\Gamma : M \times_M L^\times \to TL^\times$ is sometimes called an Ehresmann connection. And this expression is also used for the definition of a connection on a bundle in terms of the (equivariant) horizontal bundle H or the (equivariant) projection v.

We collect the results on the 6 different ways to introduce a connection on a line bundle:

Proposition 4.16. *A connection on a line bundle* $L \to M$ *is given by one of the following five equivalent data:*

1. *A* Covariant Derivative $\nabla_X : \Gamma(U,L) \to \Gamma(U,L)$ *satisfying* (K1) *and* (K2) *of Definition 4.1 (c.f. D.5 for the vector bundle case).*

2. *A collection of 1-forms, the* Local Connection Forms*,* $(A_j)_{j\in I}, A_j \in \mathcal{A}^1(U_j)$*, satisfying* (Z)*, with respect to an open cover* $(U_j)_{j\in I}$*, where* L_{U_j} *has local trivializations* ψ_j*, see Proposition 4.3 and D.14.*

3. *A* Global Connection Form α *on the frame bundle* $L^\times$ *with the two properties* (I1), (I2) *in Lemma 4.6 and D.18.*

4. *A vector bundle* H*, the so called* Horizontal Bundle $H \subset TL^\times$ *with* (H1) *(i.e.* H *is a complement to the vertical bundle* V*) and* (H2) *(i.e.* H *is equivariant), see Proposition 4.8 and Proposition D.15.*

5. *A vector bundle homomorphism* $v : TL^\times \to TL^\times$ *with* (V1) *(i.e.* v *is a projection onto the* Vertical Bundle *and* (V2) *(i.e.* v *is equivariant), see Proposition 4.9 and Proposition D.16.*

6. *An equivariant lifting* $\Gamma : TM \times_M L^\times \to TL^\times$*.*

To finish the chapter, we illustrate the 6 equivalent data on the trivial line bundle $U \times \mathbb{C}$ in a local situation, i.e. U is an open subset of $\mathbb{R}^n$.

1. ∇ is given by $A = A_i dq^i \in \mathcal{A}^1(U)$, $A_i \in \mathcal{E}(U, \mathbb{C})$ as a local connection form, in such a way that ∇_X for $X = X^i \partial_i$ is

$$\begin{aligned} \nabla_X f s_1 &= (L_X f + 2\pi i A(X) f)\, s_1 \\ &= \left(X^i \partial_i f + 2\pi i A_i X^i f\right) s_1 \end{aligned}$$

   or

$$\nabla_X(a, f(a)) = \left(a, X^i\left(\partial_i f + 2\pi i A_i f\right)\right)$$

   for $f \in \mathcal{E}(U)$, where $s_1(a) = (a, 1), a \in U$.

2. This is essentially the same, since $U_j = U$.

3. The global connection form α on $L^\times \cong U \times \mathbb{C}^\times$ is

$$\alpha = A + \frac{1}{2\pi i}\frac{dz}{z} = A_i dq^i + \frac{1}{2\pi i}\frac{dz}{z}$$

4. The horizontal subspace $H_p \subset T_p L^\times \cong \mathbb{R}^n \times \mathbb{C}$ in $p = (a, z) \in L^\times$, is given by $A_i \in \mathcal{E}(U)$ as

$$H_p = \left\{(X, \xi) \in \mathbb{R}^n \times \mathbb{C} \mid 2\pi i A_i(a) X^i + \frac{\xi}{z} = 0\right\}.$$

5. This is essentially the same as in 4., but now H being defined by the corresponding projection $v : TL^\times \to TL^\times$ onto the vertical bundle V: In $p = (a, z) \in L^\times$ given by

$$v_p(X, \xi) = (0, \xi + 2\pi i A_i(a) X^i z)\,,$$

   with $p = (a, z)$.

6. With $TU \times_U L^\times \cong U \times \mathbb{R}^n \times \mathbb{C}^\times$ and $TL^\times \cong (U \times \mathbb{C}^\times) \times (\mathbb{R}^n \times \mathbb{C})$ the lift $\Gamma : U \times \mathbb{R}^n \times \mathbb{C}^\times \longrightarrow (U \times \mathbb{C}^\times) \times (\mathbb{R}^n \times \mathbb{C})$ takes the form

$$\Gamma(X, z) = (X, -2\pi i A_j X^j z)\,,$$

   where we drop the base points $a, p = (a, z)$. The equation has the meaning that for a given local connection potential $A = A_i dq^i$ the lift Γ is of the described form. It also has the meaning, that an equivariant lifting Γ determines A.

In order to recognize the geometric nature of the notion of connection, we present – in the Appendix – the concept of a connection on a principal fibre bundle and its associated vector bundles, and we summarize some properties on principal connections. In this way, we can regard connections on a line bundle with its principal fibre bundle

$L^\times$ in the framework of general connections. Some parts become more complicated, but others look simpler in the general case.

Summary: The description of a connection on a line bundle is now complete. The geometric nature of this concept has not been emphasized so far except for the decomposition of the tangent bundle of the frame bundle $L^\times$ of a line bundle L into its vertical and horizontal subbundles. The geometry of connections on a line bundle is the subject of the next chapters where we investigate parallel transport, curvature and Hermitian structure.

5 Parallel Transport and Curvature

We introduce and study parallel transport (also called horizontal transport) along a curve $\gamma : I \to M$ in the base manifold M induced by a connection ∇ on a line bundle $\pi : L \to M$. Furthermore, in order to determine whether the parallel transport is independent of the curves, horizontal sections and the curvature of (L, ∇) are investigated. An important result is the integrality condition for a 2-form on M to be the curvature of a line bundle on M which is needed for the program of Geometric Quantization.

5.1 Parallel Transport

Definition 5.1. Let $\pi : L \to M$ be a line bundle with connection ∇, and let $p \in L^\times \subset L$ be a point in the frame bundle of L with base point $a = \pi(p)$.

1. A Horizontal (or Parallel) Lift ("horizontale Liftung")[27] of a tangent vector $X \in T_aM$ at p is a tangent vector $X^\sharp \in T_pL$ with:

   (a) $T_p\pi(X^\sharp) = X$ ($X^\sharp$ is a Lift)

   (b) $X^\sharp \in H_p$ ($X^\sharp$ is Horizontal)

2. Let γ be a (smooth) curve $\gamma : I \to M$ in M (where $I \subset \mathbb{R}$ is an open interval). A horizontal lift of γ (through $p_0 \in L_{\gamma(0)}$) is a smooth curve $\lambda : I \to L$ with $\gamma(t_0) = p_0$, $t_0 \in I$, such that

   (a) $\gamma = \pi \circ \lambda$ (λ is a lift of γ through p_0), and

   (b) $\dot\lambda(t) \in H_{\lambda(t)}$ for all $t \in I$ (the lifted curve λ is horizontal)[28].

Remark 5.2. Recall, that a horizontal lift $X^\sharp$ of X exists, since $T_p\pi : T_pL^\times \to T_aM$ is surjective, and it is unique, since $T_p\pi|_{H_p} : H_p \to T_aM$ is an isomorphism. Also, if λ is a horizontal lift of γ then each $\dot\lambda(t)$ is the horizontal lift of $\dot\gamma(t)$. As a result, the horizontal lift of a curve is unique if it exists (and it exists as we will see soon in Proposition 5.3).

In order to explain the definition, the notion of horizontal subbundle $H \subset TL^\times$ induced by the connection ∇ on L will be recalled (c.f. Section 4.3): For a point $a \in M$ there exists a trivialization

$$\psi : L_U \to U \times L$$

[27] known already from Definition 4.12

[28] A remark on the notation $\dot\lambda(t)$ or $\dot\gamma(t)$ seems to be appropriate: $\dot\gamma(t)$ is the tangent vector at the point $\gamma(t) = a \in M$ given by the curve $s \mapsto \gamma(t+s)$, i.e. $\dot\gamma(t) = [\gamma(t+s)]_a \in T_aM$. Also, with $1 \in T_tI \cong \mathbb{R}$ we can write $\dot\gamma(t) = T_t\gamma(1) \in T_aM$.

of the line bundle $L_U = \pi^{-1}(U) \to U$ over an open neighbourhood U of a. With respect to this trivialization the connection ∇ has the form

$$\nabla_X f s_1 = (L_X f + 2\pi i A(X) f) s_1, \quad f \in \mathcal{E}(U), X \in \mathfrak{V}(U),$$

where $s_1(a) := \psi^{-1}(a, 1)$ as before, and $A \in \mathcal{A}^1(U)$ is a one-form, the local connection form (or local gauge potential), uniquely defined by ∇: $A(X) \in \mathcal{E}(U)$ is the function with $\nabla_X s_1 = 2\pi i A(X) s_1$. The horizontal space $H_p \subset T_p L^\times$, $\psi(p) = (a, z) \in U \times \mathbb{C}^\times$, is now given by

$$H_p := \left\{ Y = T_{(a,z)}\psi^{-1}(X, \zeta) \in T_p L^\times \mid (X, \zeta) \in T_a U \times T_z \mathbb{C}^\times : 2\pi i A(X) + \frac{\zeta}{z} = 0 \right\}. \quad (30)$$

This digression shows again that every $X \in T_a M$ has a unique horizontal lift $X^\sharp \in T_p M$ through a point $p = \psi^{-1}(a, z) \in L_a^\times$: $X^\sharp = T_{(a,z)}\psi^{-1}(X, -2\pi i A(X) z)$. Moreover, the map $\Gamma_p : T_a M \to H_p$, $X \mapsto X^\sharp$ is an isomorphism.

The collection of these lifts Γ_p yield the Ehresmann connection $\Gamma : TM \times_M L^\times \to TL^\times$ introduced in Proposition 4.14.

If $q^1, \ldots, q^n$ are local coordinates in U around a with $A = A_k dq^k$, then the

$$Y_k = T_{\psi(p)}\psi^{-1} (\partial_k, -2\pi i z A_k)$$

$k = 1, \ldots, n$ span the horizontal subspace H_p, so they provide a vector space basis of H_p.

Proposition 5.3. *Let ∇ be a connection on the line bundle $L \to M$, and let $\gamma : I \to M$ be a (smooth) curve with $\gamma(t_0) = a$. For every point $p \in L_a^\times$ there exists a uniquely defined horizontal lift $\hat{\gamma} : I \to L^\times$ through $p : \hat{\gamma}(t_0) = p$. In particular, the horizontal lift of the curve γ through pc is the curve $\hat{\gamma}c$ for $c \in \mathbb{C}^\times$.*

Proof. Let $\psi(p) = (a, z)$. In the above local situation one looks for a curve $\zeta : I \to \mathbb{C}^\times$, $\zeta(t_0) = z$ such that $\hat{\gamma} := \psi^{-1}(\gamma, \zeta)$ is a lift of γ with $\hat{\gamma}(t_0) = \psi^{-1}(\gamma(t_0), \zeta(t_0)) = p$. In order that this lift $\hat{\gamma}$ through p is, moreover, horizontal, the two curves γ, ζ have to satisfy

$$2\pi i A(\dot{\gamma}(t)) + \frac{\dot{\zeta}(t)}{\zeta(t)} = 0,$$

according to (30), which amounts to the differential equation

$$\dot{\zeta}(t) = -2\pi i A(\dot{\gamma}(t)) \zeta(t)$$

And this differential equation has a unique solution on all of I with $\zeta(t_0) \in \mathbb{C}^\times$ (initial value problem for linear ordinary equations). The solution can be written in the form

$$\zeta(t) = c \exp\left(-2\pi i \int_{t_0}^{t} A(\dot{\gamma}(\tau)) d\tau \right),$$

with $c = \zeta(t_0)$. □

Corollary 5.4. *From the proof of the proposition we obtain the following characterization for a lift to be horizontal in a local situation: A lift $\psi^{-1}(\gamma,\zeta)$ of γ is horizontal if and only if*

$$\boxed{\dot{\zeta}(t) + 2\pi i A(\dot{\gamma}(t))\zeta(t) = 0,}$$

for $t \in I$, which is - in slightly greater detail $\dot{\zeta}(t) + 2\pi i A_j(\gamma(t))(\dot{\gamma}^j(t))\zeta(t) = 0$, or, in a very short form:

$$\boxed{\nabla_{\dot{\gamma}}\dot{\zeta} = 0\,.}$$

Observation 5.5. This result allows it to extend the horizontal lifting through all points of the fibre L_a, i.e. also through $p \in L \setminus L^{\times}$. However, if the horizontal lift $\hat{\gamma}$ of a curve γ once is zero ($\hat{\gamma}(t_1) = 0_{\gamma(t_1)}$) over a point $b = \gamma(t_1)$ it vanishes completely: $\hat{\gamma}(t) = 0_{\gamma(t)}$ for all $t \in I$.

Remark 5.6. Note, that the definitions and results extend directly to connections on a vector bundle E of rank k. Such a connection ∇ is locally given by

$$\nabla_X f = L_X f + A(X)f, \quad X \in \mathfrak{V}(U), f \in \mathcal{E}\,(U, \mathbb{K}^r)$$

where $A \in \mathcal{A}^1\,(U, \operatorname{End}\mathbb{K}^r)$ is a $\mathfrak{g} = \operatorname{End}\mathbb{K}^r$-valued 1-form. Hence a horizontal lift of a curve $\gamma : I \to M, \gamma(t_0) = a$, looks locally like $\hat{\gamma} = \psi^{-1}(\gamma, \eta)$, with $\eta \in \mathcal{E}(I, \mathbb{K}^r)$ and

$$\dot{\eta} + A(\dot{\gamma})\eta = 0\,.$$

Proposition 5.3 leads to the concept of ”parallel transport” or ”horizontal transport”.

Definition 5.7. With the notation of the last proposition and the choice of another $t_1 \in I$ let $\hat{\gamma} = \hat{\gamma}_p$ be the horizontal lift of γ through $\hat{\gamma}\,(t_0) = p \in L_a^{\times}$. Then the map

$$p \mapsto \hat{\gamma}_p\,(t_1)\,, L^{\times}_{\gamma(t_0)} \to L^{\times}_{\gamma(t_1)},$$

is bijective: According to the last statement in Proposition 5.3 pc is mapped to $\hat{\gamma}_p\,(t_1)\,c$. The mapping can be continued to all of $L_{\gamma(t_0)}, L_{\gamma(t_1)}$ by $0_{\gamma(t_0)} \mapsto 0_{\gamma(t_1)}$, or using the fact, mentioned above in Observation 5.5, that the zero lift $\nu(t) := 0_{\gamma(t)}$ over γ can also be viewed as a horizontal lift of γ. In this way one obtains an isomorphism (of $\mathbb{C}$-vector spaces) $L_{\gamma(t_0)} \to L_{\gamma(t_1}$. This isomorphism is called Parallel Transport Along γ (also called horizontal transport along γ) and will be denoted by

$$\mathbb{P}^{\gamma}_{t_1,t_0} : L_{\gamma(t_0)} \to L_{\gamma(t_1)}.$$

The parallel transport $\mathbb{P}^{\gamma}_{t_1,t_0}$ (also called parallel operator) describes a linear shift of vectors over $\gamma(t_0)$ to those over $\gamma(t_1)$. This shift depends in general on the curve from $\gamma(t_0)$ to $\gamma(t_1)$ (see below).

The parallel operators $\mathbb{P}^{\gamma}_{s,t}$ have many natural properties like

$$\mathbb{P}^{\gamma}_{s,t} \circ \mathbb{P}^{\gamma}_{t,s} = \mathrm{id}_{L_{\gamma(s)}} \quad \text{and} \quad \mathbb{P}^{\gamma}_{r,s} \circ \mathbb{P}^{\gamma}_{s,t} = \mathbb{P}^{\gamma}_{r,t} \quad \text{for } r,s,t \in I \tag{31}$$

One can reconstruct the connection ∇ from the family $\left(\mathbb{P}^{\gamma}_{t_0,t_1}\right)_{\gamma,t_0,t_1}$ (c.f. Proposition 12.3).

5.2 Horizontal Section

Definition 5.8. A section $s \in \Gamma(U, L^{\times})$ over an open subset $U \subset M$ is called HORIZONTAL if

$$T_a s\,(T_a M) \subset H_{s(a)}$$

holds for all $a \in U$.

In case of a horizontal section $s \in \Gamma(U, L^{\times})$ one even has $T_a s\,(T_a M) = H_{s(a)}$, and $T_a s$ is the inverse of the restriction $T_{s(a)}\pi\big|_{H_{s(a)}} : H_{s(a)} \to T_a M$ for all a $\in U$.

The inclusion $T_a s\,(T_a M) \subset H_{s(a)}$ for a horizontal section s over U implies that for each curve $\gamma : I \to U, \gamma(0) = a$, the composition $\lambda := s \circ \gamma$ satisfies

$$\dot{\lambda} = T_{\gamma(t)} s(\dot{\gamma}(t)) \in H_{s(\gamma(t))}\,,$$

i.e. $\lambda = s \circ \gamma$ is a horizontal lift of γ. Hence, with the notation $s \circ \gamma = \psi^{-1}(\gamma, \zeta)$ in a local trivialization $\psi : L^{\times}|_{U'} \to U' \times \mathbb{C}^{\times}$ (where $U' \subset U$ is an open subset of U), the second component $\zeta(t) = pr_2 \psi(s \circ \gamma(t))$ satisfies

$$\dot{\zeta} + 2\pi i A(\dot{\gamma})\zeta = 0,$$

and we conclude that $\nabla_X s = 0$ for all $X \in \mathfrak{V}(U)$: When $X(a) = [\gamma(t)]_a$ we obtain, using $s(\gamma(t)) = \zeta(t) s_1(\gamma(t)) = \psi^{-1}(\gamma(t), \zeta(t))$, that

$$\begin{aligned}
\nabla_X s(a) &= \nabla_X \zeta s_1(\gamma)|_{(t_0)} \\
&= (L_X \zeta + 2\pi i A(X))\,\zeta)|_{t_0} s_1(a) \\
&= \left(\dot{\zeta}(t_0) + 2\pi i A(\dot{\gamma}(t_0))\zeta(t_0)\right) s_1(a) \\
&= 0\, s_1(a) = 0\,.
\end{aligned}$$

We have essentially shown:

Proposition 5.9. *Let $L \to M$ be a line bundle with connection ∇. Then $s \in \Gamma(U, L^{\times})$ is a horizontal section if and only if $\nabla_X s = 0$ for all $X \in \mathfrak{V}(U)$.*

Examples 5.10.

1. ln the trivial case $L = M \times \mathbb{C}$, let A be the zero potential $A = 0$, i.e. $\nabla_X f s_1 = L_X f s_1$. In this case: $s = f s_1$ is horizontal if and only if f (and hence s) is locally constant.

2. Again in the trivial case $L = M \times \mathbb{C}$ with $M = \mathbb{R}^2$ let us consider the connection given by the non-zero local connection form $A = q^2 dq^1 - q^1 dq^2$. (Note, that $dA \neq 0$, i.e. the connection has non-zero curvature, see below.) We show that for this connection there is no horizontal section. If $s(a) = (a, f(a)), a \in U$, would be a horizontal section on $U \subset M$ open, $U \neq \emptyset$, with $f(a_0) \neq 0$ at one point $a_0 \in U$ we can assume $f(a) \neq 0$ throughout U (by possibly taking a smaller neighbourhood of a_0). Then Proposition 5.9 implies $\nabla_X s = 0$, i.e. $L_X f + 2\pi i A(X) f = 0$. Hence,
$$\begin{aligned}\frac{\partial f}{\partial q^1} + 2\pi i A_1 f &= \frac{\partial f}{\partial q^1} + 2\pi i q^2 = 0\\ \frac{\partial f}{\partial q^2} + 2\pi i A_2 f &= \frac{\partial f}{\partial q^2} - 2\pi i q^1 = 0\end{aligned}$$
and this leads to the contradiction
$$-2\pi i = +\frac{\partial^2 f}{\partial q^1 \partial q^2} = 2\pi i.$$

One can prove the following direct relation between ∇ and the corresponding parallel transport:

Proposition 5.11. *For every curve $\gamma : I \to U$ and every section $s \in \Gamma(U, L)$*
$$\nabla_X s(a) = \lim_{t \to 0} \frac{1}{t} \left(\mathbb{P}^\gamma_{t_0, t_0+t}(s \circ \gamma(t_0 + t)) - s(a) \right) \tag{32}$$
where γ represents $X(a)$: $X = \dot\gamma(t_0) = [\gamma]_a$, $\gamma(t_0) = a$.

This result leads to an interesting GEOMETRIC INTERPRETATION of the covariant derivative: The covariant derivative ∇_X measures along the curve γ to what extent a given section s of the line bundle deviates infinitesimally from being horizontal.

The connection can be reconstructed knowing the action of sufficiently many of its parallel operators:

Proposition 5.12. *Let $\pi : L \to M$ be a line bundle over M. Let $(\mathbb{P}^\gamma_{s,t})$ a collection of linear operators $L_{\gamma(t)} \to L_{\gamma(s)}$ assigned to all curves $\gamma : I \to M$ and $s, t \in I$. Assume that* (31) *is satisfied, that $\mathbb{P}^\gamma_{s,t}$ depends differentiably on t and that the $\mathbb{P}^\gamma_{s,t}$ do not depend on the parametrization of the curves γ. Then* (32) *defines a connection with the given $\mathbb{P}^\gamma_{s,t}$ as parallel operators.*

This construction is not needed in the sequel. The rather involved proof will not presented.

This result yields another way to characterize the notion of a connection. In [Poo81] a comprehensive study of the various appearances of the concept of connections on general bundles can be found.

Under which conditions does there exist a horizontal section, at least locally? We have seen, that in case of a horizontal section $s \in \Gamma(U, L^\times)$ for each curve γ in U its horizontal lift through $s(\gamma(t_0))$ has the form $s \circ \gamma$. Consequently, for any two points $a, b \in U$ and any curve γ in U with $\gamma(t_0) = a$, $\gamma(t_1) = b$, parallel transport of $p = s(a) = s(\gamma(t_0)) \in L_a$ to L_b along γ is $s \circ \gamma(t_1) = s(b) : \mathbb{P}^\gamma_{t_1,t_0}(s(a)) = s(b)$ independently of γ (as long as the curves stay in U). (For $p' \in L_a^\times, p' = pc$, with $c \in \mathbb{C}^\times$, and $s'(a) = s(a)c$ is a horizontal section transporting p' to $s(b)c$, again independently of the curve.) We have shown one direction of the following equivalence.

Proposition 5.13. *Let $L \to M$ be a line bundle with connection ∇ and $U \subset M$ open. Then U admits a horizontal section $s \in \Gamma(U, L^\times)$ if and only if the parallel transport from a point $a \in U$ to $b \in U$ is independent of the curves in U connecting a and b.*

Proof. Assume that parallel transport is independent of the curves. Without loss of generality we assume furthermore, that U is connected. We obtain to each $a \in U$ and $p \in L_a^\times$ a unique horizontal section $s : U \to L^\times$ with $s(a) = p$ by the following prescription: $s(b) := \mathbb{P}^\gamma_{t_1,t_0}(p)$, where γ is a curve $\gamma : I \to U$ with $\gamma(t_0) = a$ and $\gamma(t_1) = b : s(b)$ is well-defined since the value does not depend on γ, s is smooth since all the curves γ are smooth, and s is horizontal, since, by definition $s \circ \gamma(t)$ is the horizontal lift of γ. □

The question of whether or not parallel transport is independent of the curve connecting the points in M is essentially related to the notion of curvature which is the subject of the next section.

5.3 Curvature

We continue the discussion under which conditions on a given line bundle with connection (L, ∇) there exist horizontal sections

$$s : U \to L^\times$$

over an open subset U of M.

Assume, that U is a coordinate neighbourhood with coordinates $q^1, \cdots q^n$, i.e. we have a diffeomorphism $\varphi = (q^1, \ldots q^n) : U \to V \subset \mathbb{R}^n$ with $\varphi = (q^1, \ldots, q^n)$. Let $a \in U$ with $\varphi(a) = 0$ and assume $V = \varphi(U) = I_1 \times \ldots \times I_n$ where $I_k =]-r, r[$ are equal open intervals around 0. Then the vector fields $X_k = \partial_k \in \mathfrak{V}(U)$ yield a basis of the $\mathcal{E}(U)$-module of vector fields $\mathfrak{V}(U)$. We want to construct a horizontal section

$s \in \Gamma(U, L^{\times})$, and we see, that to check whether s is horizontal it us enough to show $\nabla_{X_k} s = 0$ for $k = 1, \ldots, n$.

We start with the curve $\gamma_1(t) = \varphi^{-1}(te_1)$ through a representing $X_1 = \partial_1, t \in I_1$ and choose a point $p_0 \in L_a^{\times}$. There exists the horizontal lift (c.f. Proposition 5.3) $\hat{\gamma}_1$ of γ_1 through p_0. In particular, $\gamma_1(0) = a$ and $\hat{\gamma}_1(0) = p_0$. We set $s(\gamma_1(t)) := \hat{\gamma}_1(t)$, $t \in I_1$ with $s(a) = p_0$. For each $t_1 \in I_1$ the curve $\gamma_2^{t_1}(t) := \varphi^{-1}(t_1 e_1 + te_2), t \in I_2 \subset \mathbb{R}$, has again a horizontal lift $\hat{\gamma}_2^{t_1}(t)$ through $\hat{\gamma}_1(t)$. We set $s\left(\gamma_2^{t_1}(t_2)\right) := \hat{\gamma}_2^{t_1}(t_2)$. In the same way one can proceed with $3, \ldots, n$. But let us stick to the case $n = 2$. (The case $n > 2$ is completely analogous.) Then s as above defines a section $s : U = \varphi^{-1}(I_1 \times I_2) \to L^{\times}$, $s(\varphi^{-1}(t_1 e_1 + t_2 e_2)) = \hat{\gamma}_2^{t_1}(t_2)$.

Let us check whether s is a horizontal section. s is horizontal if and only if $\nabla_X s = 0$ for all vector fields $X \in \mathfrak{V}(U)$, i.e. if and only if $\nabla_{X_i} s = 0$ for $i = 1, 2$ in our special situation. Now, $\nabla_{X_2} s = 0$ is evident by the definition of s, since $t \mapsto s\left(\gamma_2^{t_1}(t)\right) = \hat{\gamma}_2^{t_1}(t)$ is horizontal for each $t_1 \in I_1$.

If now ∇_{X_1} and ∇_{X_2} commute, we have $\nabla_{X_2}\nabla_{X_1} s = \nabla_{X_1}\nabla_{X_2} s = 0$ (because of $\nabla_2 s = 0$) and it follows that (for fixed t_1)

$$\eta(t) = \nabla_{X_1} s(\varphi^{-1}(t_1, t))$$

is a horizontal lift of $\gamma_2(t_1, t)$ with $\eta(0) = 0$. ($\eta(0) = \nabla_{X_1}\hat{\gamma}_1(t) = 0$ since $\hat{\gamma}_1$ is a horizontal lift of γ_1). η is horizontal because of $\nabla_{X_2}\eta = 0$. Eventually, since the horizontal lift is unique, we have $\eta(t) = 0$ and:

$$\nabla_{X_1} s(\varphi^{-1}(t_1, t)) = \eta(t) = 0\,,$$

which implies $\nabla_{X_1} s = 0$

We have shown that 3. implies 2. of the following theorem:

Theorem 5.14. *For a line bundle $L \to M$ with connection ∇ the following properties are equivalent for each open $U \subset M$:*

1. *Parallel transport over U is independent of the curves.*
2. *There exists a horizontal section $s \in \Gamma(U, L^{\times})$.*
3. $[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = 0$ *for $X, Y \in \mathfrak{V}(U)$.*

Proof. That the first and second conditions are equivalent is the content of 5.13. That 3. implies 2. has been shown just before this proposition. To prove the converse, i.e. 2. implies 3., let $s \in \Gamma(U, L^{\times})$ be a horizontal section in a neighbourhood of a. Any other section $U \to L$ has the form fs with $f \in \mathcal{E}(U)$. Now, by (K2),

$$\nabla_X(fs) = (L_X f)s + f\nabla_X s = (L_X f)s\,,$$

since $\nabla_X s = 0$ for a horizontal section. Hence,

$$[\nabla_X, \nabla_Y](fs) = ([L_X, L_Y]f)s = (L_{[X,Y]}f)s = \nabla_{[X,Y]}(fs),$$

which had to be proven. □

Definition 5.15 (Curvature)**.** Let $\pi : L \to M$ a line bundle with connection ∇ :

1. The Curvature Operator is, for $U \subset M$ open:

$$\begin{aligned} F = F_\nabla : \mathfrak{V}(U) \times \mathfrak{V}(U) &\longrightarrow \operatorname{End}\Gamma(U, L) \\ (X, Y) &\longmapsto \frac{1}{2\pi i}\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) \end{aligned}$$

2. The Curvature Form $\Omega = \operatorname{Curv}(L, \nabla) \in \mathcal{A}^2(M)$ is defined as follows: If $(U_j)_{j\in I}$ is an open cover of M with trivializations $\psi_j : L_{U_j} \to U_j \times \mathbb{C}$ and local connection forms $A_j \in \mathcal{A}^1(U_j)$ for ∇ then

$$\Omega|_{U_j} := dA_j, \quad j \in I$$

The last expression is well-defined, since we know from Proposition 4.3 that:

$$(Z) \qquad A_k = A_j + \frac{1}{2\pi i}\frac{dg_{jk}}{g_{jk}}$$

on $U_{jk} = U_j \cap U_k$ i.e. $dA_j = dA_k$.

F_∇ can also interpreted as an operator with values in the endomorphism bundle $\operatorname{End} L$ or even in $\mathcal{E}(M)$ by the first result in the following proposition which also treats the interrelations between Ω, the global connection form α and sections $s \in \Gamma(U, L^\times$:

Proposition 5.16. *For a connection ∇ on a line bundle L we have*

1. $F_\nabla(X, Y) = \Omega(X, Y)$ *for* $X, Y \in \mathfrak{V}(M)$*, in the sense of: For* $s \in \Gamma(U, L)$*:* $F_\nabla(X, Y)s = \Omega(X, Y)s$.
2. $\pi^*\Omega = d\alpha$*, where* $\alpha \in \mathcal{A}^1(L^\times)$ *is the global connection form of* ∇ *on* $L^\times$.
3. $\mathrm{s}^*d\alpha = \Omega|_U$ *for any section* $s \in \Gamma(U, L^\times)$.

Proof. 1. First of all, for each section $s \in \Gamma(U, L)$ there exists a function $\beta(X, Y) \in \mathcal{E}(U)$ such that $F_\nabla(X, Y)s = \beta(X, Y)s$. It is easy to show that $\beta(X, Y)$ is independent of s (because $F_\nabla(X, Y)$ is linear over $\mathcal{E}(U)$) and it is bilinear over $\mathcal{E}(U)$ and alternating. Hence, it is a 2-form. The main point of the statement is, that this form is the curvature form Ω.

This can be checked by showing it over each U_j, i.e. we need to show it only for trivial bundles with $\nabla_X f s_1 = (L_X f + 2\pi i A(X) f)\, s_1$, $s_1(a) = (a,1)$ and $s = fs_1 = (a, f(a))$, where $A \in \mathcal{A}^1(U)$ is the local gauge potential:

$$\begin{aligned}[\nabla_X, \nabla_Y]\, f s_1 =& (L_X L_Y f - L_Y L_X f + 2\pi i\,(A(X)L_Y f - A(Y)L_X f) \\ &+2\pi i L_X(A(Y)f) - 2\pi i L_Y(A(X)f))\, s_1 \\ =& \left(L_{[X,Y]}f + 2\pi i\,(L_X A(Y) - L_Y A(X))\, f\right) s_1\end{aligned}$$

$$\nabla_{[X,Y]} f s_1 = \left(L_{[X,Y]}f + 2\pi i A([X,Y]) f\right) s_1$$

Therefore

$$\begin{aligned}&\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) f s_1 = 2\pi i \overbrace{(L_X A(Y) - L_Y A(X) - A([X,Y]))}^{dA(X,Y)} f s_1 \\ &F_\nabla(X,Y)s = dA\,(X,Y)\, s = \Omega(X,Y)s \\ &\Rightarrow F_\nabla = \Omega.\end{aligned}$$

2. From Section 4.2 we know

$$\alpha|_{L^\times_{U_j}} = \pi^* A_j + \frac{1}{2\pi i}\psi_j^*\left(\frac{dz}{z}\right), j \in I,$$

hence,

$$d\alpha|_{L^\times_{U_j}} = \pi^* dA_j = \pi^*\left(\Omega|_{U_j}\right), j \in I.$$

3. The same relation between α and A_j yields for $s \in \Gamma\,(U, L)$

$$s^*\alpha = s^*\pi^* A_j + \frac{1}{2\pi i} s^*\psi_j^*\left(\frac{dz}{z}\right) = A_j + \frac{1}{2\pi i}\,(\psi_j \circ s)^*\left(\frac{dz}{z}\right)$$

on $U \cap U_j$ and

$$s^* d\alpha = dA_j = \Omega \text{ on } U \cap U_j.$$

□

5.4 Integrality Condition

In the following we want to show how the parallel transport for a line bundle with connection (L, ∇) can be expressed by a suitable integral over the curvature form $\Omega = \mathrm{Curv}(L, \nabla)$.

Let $\mathcal{L}(a)$ be the set of all loops (closed smooth curves[29]), $\gamma : [t_0, t_1] \to M$, which start and end in a fixed point a $\in M$: $\gamma(t_0) = \gamma(t_1) = a$. Then the parallel transport

$$\mathbb{P}^\gamma_{t_1,t_0} : L_a = L_{\gamma(t_0)} \to L_a = L_{\gamma(t_1)}\,,\, \gamma\,(t_0) = \gamma\,(t_1) = a,$$

[29] one can allow also continuous curves which are piecewise smooth

is an isomorphism of 1-dimensional complex vector spaces. Therefore, it is determined by a complex number $Q(\gamma) \in \mathbb{C}^\times \cong \mathrm{GL}(1,\mathbb{C})$:

$$\mathbb{P}^\gamma_{t_1,t_0} = \mathbb{P}^\gamma : L_a \to L_a\,,\ p \mapsto Q(\gamma)\,p = p\,Q(\gamma)\,.$$

Proposition 5.17. *Let (L,∇) be a line bundle with connection and $\Omega = \mathrm{Curv}(L,\nabla)$ its curvature.*

1. *Let $\gamma \in \mathcal{L}(a)$ be a closed curve being contained in a coordinate neighbourhood U of a. The parallel transport $\mathbb{P}^\gamma : L_a \to L_a$ along γ is given by*

$$Q(\gamma) = \exp\left(-2\pi i \int_\gamma A\right) = \exp\left(-2\pi i \int_{t_0}^{t_1} A(\dot\gamma(s))ds\right)\,.$$

   with A a local connection form of ∇.

2. *Let $S \subset M$ be an oriented compact surface in M with boundary ∂S parametrized by $\gamma \in \mathcal{L}(a)$. The parallel transport $\mathbb{P}^\gamma : L_a \to L_a$ along γ is given by*

$$Q(\gamma) = \exp\left(-2\pi i \int_S \Omega\right)\,.$$

Proof. Ad 1.: We can assume $L_U = U \times \mathbb{C}$. The horizontal lift of $\gamma(t) \in M$ has the form

$$\hat\gamma(t) = (\gamma(t),\zeta(t)),\ t \in [t_0,t_1]\,,$$

with $\zeta(t) = \zeta(t_0)\rho(t) \in \mathbb{C}$, where $\rho(t) := \zeta(t)\zeta(t_0)^{-1}$. By definition: $Q(\gamma) = \rho(t_1)$. According to Proposition 5.4 we have

$$\dot\zeta + 2\pi i A(\dot\gamma)\zeta = 0\,,\quad \text{and consequently}\quad \dot\rho + 2\pi i A(\dot\gamma)\rho = 0\,,$$

where A is a local connection form ∇. Hence, $\rho(t)$ can be expressed as the INTEGRAL

$$\rho(t) = \exp\left(-2\pi i \int_{t_0}^{t} A(\dot\gamma(s))ds\right)\,.$$

In particular,

$$\rho(t_1) = \exp\left(-2\pi i \int_{t_0}^{t_1} A(\dot\gamma(s))ds\right)\,.$$

Ad. 2: It is enough to show the result locally, i.e. we can assume the line bundle to be trivial: $L = M \times \mathbb{C}$. This is evident if the surface is contained in one of the open subsets U_j where there is the trivialization $\psi_j : L_{U_j} \to U_j \times \mathbb{C}$. Otherwise S will be contained in finitely many of the U_j's, since S is compact, and the surface will be cut into finitely many surfaces with boundary, each of them contained in one U_k[30]

[30]Exercise! Give a detailed proof of the cutting and gluing.

The integral along the full curve is

$$\int_{t_0}^{t_1} A(\dot{\gamma}(s))ds = \int_\gamma A = \int_{\partial S} A = \int_S dA = \int_S \Omega$$

by Stokes' theorem. Therefore,

$$Q(\gamma) = \rho(t_1) = \exp\left(-2\pi i \int_\gamma A\right) = \exp\left(-2\pi i \int_S \Omega\right).$$

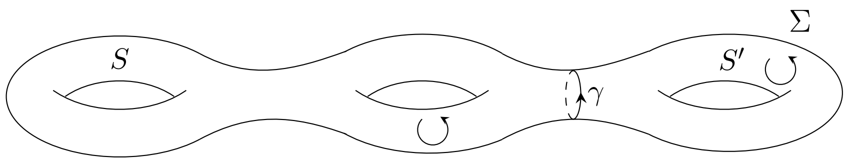
□

Instead of restricting to smooth curves one often uses the more general class of piecewise smooth curves with the same results for lifting to horizontal curves and for parallel transport.

The last result in the preceding proposition leads to an INTEGRALITY CONDITION for the curvature $\Omega = \mathrm{Curv}(L, \nabla) \in \mathcal{A}^2(M)$ of the line bundle (L, ∇) with connection, which is of a topological nature and which is of great importance from the point of view of quantization.

Let us explain this in some detail:

Let $\Sigma \subset M$ be an oriented, compact surface smoothly embedded into M. Assume, moreover, that Σ is closed, i.e. Σ has empty boundary. Then Σ is a 2-dimensional oriented and compact submanifold of M (see the example in the illustration below). We can find a simple closed smooth curve γ dividing the surface Σ into two parts S, S' such that S is an oriented compact surface with boundary ∂S parametrized by γ, and S' is another oriented compact surface with boundary $\partial S' = \partial S$ parametrized by γ^- (i.e. the curve γ paramatrized in the opposite direction). We have $S \cup S' = \Sigma$, and $S \cap S' = \partial S = \partial S' = |\gamma|$ (as sets without orientation), where $|\gamma| := \{\gamma(t) \mid t \in I\}$ is the support of γ. For example, as in the following illustration:



Closed oriented surface Σ divided into 2 oriented surfaces S, S' with boundary $|\gamma|$

Let $a \in \partial S$ be the initial and end point of γ. And suppose that Σ is contained in a an open $U_j \subset \Sigma$ with trivialization $\psi_j : L_{U_j} \to U_j \times \mathbb{C}$. Then the parallel transport along γ is given by the number

$$Q = \exp\left(-2\pi i \int_\gamma A_j\right) = \exp\left(-2\pi i \int_S \Omega\right)$$

and the parallel transport along γ^-, where $\gamma^-(t) := \gamma(t_1 - t + t_0))$, $t \in [t_0, t_1]$,, is given correspondingly by:

$$Q^- = \exp\left(-2\pi i \int_{\gamma^-} A_j\right) = \exp\left(-2\pi i \int_{S'} \Omega\right)$$

The formulas

$$Q = \exp\left(-2\pi i \int_S \Omega\right), Q^- = \exp\left(-2\pi i \int_{S'} \Omega\right)$$

hold true for general compact and closed oriented Σ in M which are not necessarily contained in a U_j by cutting Σ into suitable pieces, such that the pieces are in suitable U_j's.

Since Q^- is the inverse of Q (because Q^- describes the parallel transport in the opposite direction) we have

$$\begin{aligned} 1 = Q^- Q &= \exp\left(-2\pi i \int_{S'} \Omega\right) \exp\left(-2\pi i \int_S \Omega\right) \\ &= \exp\left(-2\pi i \left(\int_S \Omega + \int_{S'} \Omega\right)\right) = \exp\left(-2\pi i \int_\Sigma \Omega\right) \end{aligned}$$

As a consequence,

$$\int_\Sigma \Omega \in \mathbb{Z},$$

which is the Integrality Condition ("Ganzheitsbedingung").

Proposition 5.18. *Let (L, ∇) be a line bundle with connection. Then the curvature $\Omega = \mathrm{Curv}(L, \nabla)$ satisfies the following integrality condition:*

$$\text{(G)} \qquad \boxed{\int_\Sigma \Omega \in \mathbb{Z}}$$

for every oriented closed compact surface $\Sigma \subset M$ in M.

Another description of the integrality is given by cohomology:

Proposition 5.19. *A closed two form $\Omega \in \mathcal{A}^2(M)$ on a manifold M satisfies the above integrality condition* (G) *if and only if the deRham cohomology class*[31] $[\Omega] \in H^2_{dR}(M, \mathbb{C})$ *is in the image of $\iota^2 : H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{C})$:*

$$\text{(D)} \qquad \boxed{[\Omega] \in \mathrm{Im}\, \iota^2}$$

[31] see Section E.2 in the Appendix

The homomorphism ι^2 is induced as part of the long exact sequence coming from the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \to 1,$$

where $\iota(n) = n \in \mathbb{C}, n \in \mathbb{Z}$, is the inclusion and $\exp(z) = e^{2\pi i z}, z \in \mathbb{C}$. In this situation $H^*(M, G)$ is any of the cohomology theories like singular cohomology or Čech cohomology, which are equivalent on a paracompact manifold M.

The condition (D) can be understood without general knowledge of cohomology by using Čech cohomology and its equivalence with deRham cohomology as presented in Chapter E of the Appendix. In particular, there is a natural isomorphism of the groups $\check{H}^2(M,\mathbb{C}) \cong H^2_{dR}(M,\mathbb{C})$ and $\iota^2 : \check{H}^2(M,\mathbb{Z}) \to \check{H}^2(M,\mathbb{C})$ is directly induced by the inclusion homomorphism $\iota : \mathbb{Z} \to \mathbb{C}$.

With this information the condition (D) for a two form Ω can be reformulated in the following way within deRham cohomology: Recall (see Section E.1), that a cohomology class in $H^2_{dR}(M,\mathbb{C})$ is given by a 2-cochain $c = (c_{ijk})$ with respect to an open covering (U_j) of M, where the coefficients c_{ijk} are in $\mathbb{C}$: $Z = [c_{ijk}]$. In particular, a closed two form Ω defines the following class $[\Omega]$ (see Remark E.10) if U_j, U_{jk}, U_{ijk} are contractible: Because of $d\Omega = 0$ there are one forms α_j on U_j with $d\alpha_j = \Omega|_{U_j}$. Since $d(\alpha_k - \alpha_j) = 0$, there are functions $f_{jk} \in \mathcal{E}(U_j \cap U_k)$ with $df_{jk} = \alpha_j - \alpha_k$. Now, $c_{ijk} = f_{ij} + f_{jk} + f_{ki}$ is constant on $U_{ijk} = U_i \cap U_j \cap U_k$ because of $d(f_{ij} + f_{jk} + f_{ki}) = 0$. The class of Ω is $[\Omega)] = [c_{ijk}]$.

Definition 5.20. The 2-form $\Omega \in \mathcal{A}^2(M)$ is called ENTIRE if it fulfills condition (E):

(E) The $c_{ijk} \in \mathbb{C}$ can be chosen to be entire, i.e. $c_{ijk} \in \mathbb{Z}$.

The above discussion amounts to

Proposition 5.21. *In more concrete terms, condition* (D) *for a 2-form* Ω *is equivalent to condition* (E).

The proof of the equivalence of (G) and (E), a purely topological result attributed to A. Weil, will not be discussed here. But we come back to the integrality condition and prove parts of the equivalence in Chapter 8 on Integrality.

Summary: This chapter's study of parallel transport and curvature of connections ∇ on line bundles culminates in the integrality condition which is the basis of the concepts of a QUANTIZABLE MANIFOLD and PREQUANTUM LINE BUNDLE in Chapter 7. The impact of the integrality condition and the question of existence and uniqueness of prequantum bundles will be investigated in Chapter 8.

6 Hermitian and Holomorphic Line Bundles

The Hilbert space we eventually want to determine for our program of Geometric Quantization will be a generated by a subspace of the space of sections $\Gamma(M, L)$ of a line bundle. In order to obtain on such a subspace a Hermitian scalar product we need the notion of a Hermitian structure on the line bundle in question which we now introduce and study in the first part of the chapter. In the second part we investigate another additional structure on a line bundle, the holomorphic structure, and we consider the question how the connection, the Hermitian structure and the holomorphic structure fit together. Of course, the notion of a holomorphic structure on a complex line bundle gives only sense if the base manifold is a complex manifold, i.e. has a holomorphic structure.

6.1 Hermitian Line Bundles

As before, M is a smooth manifold and $\pi : L \to M$ denotes a complex line bundle over M.

Definition 6.1. A HERMITIAN LINE BUNDLE is a line bundle $\pi : L \to M$, for which every fibre $L_a \in M$ has a Hermitian metric or Hermitian form depending smoothly on the points $p \in L$. The Hermitian metric will be given by a map

$$H : \bigcup_{a \in M} L_a \times L_a \to \mathbb{C}^{32}$$

such that $H|_{L_a \times L_a} : L_a \times L_a \to \mathbb{C}$ is a Hermitian scalar product[33] for all $a \in M$. And the smoothness condition simply means that H is smooth. We denote:

$$H(p, p') =: \langle p, p' \rangle, \quad p, p' \in L.$$

Remark 6.2. Such a Hermitian metric H on L will induce a smooth function $h : L^times \to \mathbb{R}_+$ by $h(p) := H(p, p)\,,\, p \in L^\times$ with $h(\lambda p) = |\lambda|^2 h(p)$ for all $\lambda \in \mathbb{C}\times$ and $p \in L\times$. Conversely, h with these properties defines a Hermitian metric by: $H(zp_a, wp_a) := \bar{z}wh(p_a)\,$, where $z, w \in \mathbb{C}$ and $p_a \in L_a^\times$.

Example 6.3. In case of the trivial line bundle $L = M \times \mathbb{C}$ we obtain a natural Hermitian metric H_0 by defining

$$H_0((a, z), (a, w)) = \langle (a, z), (a, w) \rangle := \bar{z}w\,,$$

$z, w \in \mathbb{C}, a \in M$. H_0 is called the CONSTANT HERMITIAN METRIC.

[32] which is the same as a map $H : L \times_M L \to \mathbb{C}$ on the fibre product $L \times_M L$, see A.8 for the definition of a fibre product

[33] The properties of a Hermitian metric are recalled in Appendix F, (90) ff.

We see that any other Hermitian metric on $L = M \times \mathbb{C}$ is given by a smooth $h : M \to \mathbb{R}$, $h(a) > 0$, for all $a \in M$ by $H(p, p') := h(a) \langle p, p' \rangle = h(a) H_0(p, p')$, i.e.

$$H((a, z), (a, w)) = h(a) \bar{z} w.$$

Lemma 6.4. *A Hermitian line bundle (L, H), whose underlying line bundle is the trivial line bundle $M \times \mathbb{C}$ is isomorphic to the trivial line bundle with constant Hermitian metric H_0.*

Proof. The general case is of the form

$$H((a, z), (a, w)) = h(a) \bar{z} w,$$

and $\Phi : M \times \mathbb{C} \to M \times \mathbb{C}, (a, z) \mapsto (a, \sqrt[2]{h(a)} z)$, defines an isomorphism of Hermitian line bundles $(M \times L, H), (M \times L, H_0)$: Φ is an isomorphism of line bundles with

$$H(p, p') = H_0(\Phi(p)), \Phi(p')), p, p' \in M \times L.$$

□

Using the local existence of a Hermitian metric, we can conclude that on every line bundle over a paracompact manifold there exists a Hermitian metric H such that it turns our line bundle into a Hermitian line bundle. This can be proven in the same way as the proof of existence of connection, c.f. Proposition D.19.

To a Hermitian line bundle (L, H) one associates the Circle Bundle $L^1 \to M$, where

$$L^1 := \{p \in L : H(p, p) = 1\}$$

This is a principal fibre bundle with the circle group $\mathrm{U}(1) \cong \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ as its structure group. Conversely, if $P \to M$ is a principal fibre bundle with structure group $\mathbb{S}^1$ and $\rho : \mathbb{S}^1 \to \mathbb{C}^\times = GL(1, \mathbb{C})$ is the natural representation $\rho(z) = z : \mathbb{C} \to \mathbb{C}$, w $\mapsto$ zw, then the associated vector bundle (see Section D.5 for the concept of associated bundle) $L = P \times_\rho \mathbb{C}$ is a line bundle, where $\mathbb{S}^1$ acts on the fibers of $P \times \mathbb{C}$ by scalar multiplication. The Hermitian metric H on L is then given by:

$$H([x, z], [y, w]) := \bar{z} w$$

where $x, y \in P, z, w \in \mathbb{C}$.

Proposition 6.5. *The group of isomorphism classes of Hermitian line bundles (L, H) over M is isomorphic to $\check{H}^1(M, \mathrm{U}(1))$.*

Here, $\check{H}^1(M, \mathrm{U}(1))$ is Čech cohomology with respect to the $\mathrm{U}(1)$-valued locally constant functions on M (see Chapter E).

We now study line bundles on which there exists a connection together with a Hermitian metric.

Definition 6.6. Given a Hermitian line bundle (L, H) over M, a connection ∇ on L is called COMPATIBLE with H if for all sections $s, t \in \Gamma(U, L)$ and all vector fields $X \in \mathfrak{V}(U), U \subset M$ open, we have

$$L_X\langle s,t\rangle = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle$$

(recall $\langle s,t\rangle = H(s,t)$). Such a connection is also called a HERMITIAN CONNECTION.

The notion of a Hermitian connection is similar to the notion of a Levi-Civita connection in Riemann Geometry: A connection ∇ on the tangent bundle of a semi-Riemannian manifold M with Riemannian metric g is a Levi-Civita connection if it respects the metric g:

$$L_Z g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)\,,$$

$X, Y, Z \in \mathfrak{V}(M)$.

Proposition 6.7. *A connection ∇ on L is compatible with a Hermitian metric H if and only if the local gauge potentials $(A_j)_{j\in I}$ with respect to local trivializations $\psi_j : L_{U_j} \to U_j \times \mathbb{C}$, (where the U_j cover M: $\bigcup_j U_j = M$) can be chosen to be real one-forms $A_j \in \mathcal{A}^1(U_j, \mathbb{R})$.*

Proof. A collection of trivializations $\psi_j : L_{U_j} \to U_j \times \mathbb{C}$ can be chosen in such a way that ψ_j is an isomorphism $(L_{U_j}, H|_{U_j}) \to (U_j \times \mathbb{C}, H_0)$ of Hermitian line bundles with respect to the constant Hermitian metric H_0 on $U_j \times \mathbb{C}$, see Lemma 6.4. Now, $s, t \in \Gamma(U_j, L)$ have the form

$$s = f s_j, t = g s_j, \quad g, f \in \mathcal{E}(U_j)\,,$$

where $s_j(a) = \psi_j^{-1}(a,1)$. Hence, we have for geneeral local sections s, t at $a \in U_j$: $\langle s,t\rangle(a) = \langle (a, f(a)), (a, g(a))\rangle_0 = \bar{f}(a) g(a)$, It follows:

$$\begin{aligned} L_X\langle s,t\rangle &= \left(L_X \bar{f}\right) g + \bar{f} L_X g. \\ \langle \nabla_X s, t\rangle &= \langle (a, L_X f(a) + 2\pi i A_j(X) f(a))\,, (a, g(a)) \\ &= \left(\overline{L_X f} - 2\pi i \bar{A}_j(\bar{X}) \bar{f}(a)\right) g(a) \\ \langle s, \nabla_X t\rangle &= \bar{f}(a) \left(L_X g(a) + 2\pi i A_j(X) g(a)\right). \end{aligned}$$

Compatibility is therefore equivalent to:

$$\left(L_X \bar{f}\right) g + \bar{f} L_X g = \overline{L_X f} g + \bar{f} L_X g - 2\pi i \left(\bar{A}_j(X) - A_j(X)\right) \bar{f} g\,.$$

for all $f, g \in \mathcal{E}(U_j)$. If we restrict this equation to real vector fields X and evaluate it for all f, g the equation amounts to:

$$0 = \bar{A}_j(X) - A_j(X).$$

Hence, A_j is a real form. The converse can be read off the above formulas. □

Another characterization is the following:

Proposition 6.8. *A connection ∇ on L is compatible with a Hermitian metric H if and only if for all non-zero sections $s \in \Gamma(U, L^\times), U \subset M$ open, we have:*

$$d(H(s,s)) = 2H(s,s)\mathrm{Re}\left(\frac{\nabla s}{s}\right),$$

where $\frac{\nabla s}{s}$ denotes the one-form $\beta \in \mathcal{A}^1(U,\mathbb{C})$ given by

$$\nabla_X s = \beta(X)s\,,\ X \in \mathfrak{V}(U).$$

The compatibility is therefore also equivalent to:

Corollary 6.9. *If $s \in \Gamma(U, L^\times)$ is of length 1, i.e. $H(s,s) = 1$, then $\frac{\nabla s}{s}$ is purely imaginary.*

Concerning the existence of compatible connections, we conclude:

Corollary 6.10. *Let (L, H) be a Hermitian line bundle. Then there exists a compatible connection ∇. Given such a compatible connection ∇, the set of all connections compatible with H is the affine space*

$$\nabla + \mathcal{A}^1(M,\mathbb{R}).$$

Proof. In the existence discussion in Proposition D.19, one only has to make sure to choose the one-forms on the trivializations $L_{U_j} \to U_j \times \mathbb{C}$ as real one-forms. The second statement follows from the description of all connections on L as the affine space $\nabla + \mathcal{A}^1(M,\mathbb{C})$ (c.f. Proposition D.20). □

Remark 6.11.

1. A given connection ∇ on L, on the other hand, may not be compatible to any Hermitian metric H on L[34].

2. The curvature $\Omega = \mathrm{Curv}(\nabla, L)$ of a connection on L compatible with a given Hermitian metric is always a real two-form $\Omega \in \mathcal{A}^2(M,\mathbb{R})$.

6.2 Holomorphic Case

In this section M is supposed to be a complex manifold. For a complex line bundle $\pi : L \to M$ the following additional structures will be considered:

1. connections ∇ on L,

[34] Exercise: Give an example!

2. Hermitian metric H on L,
3. holomorphic structure on L.

For holomorphic functions and complex manifolds, in particular for holomorphic vector bundles, we refer to Chapter B. Recall, that a complex line bundle $\pi : L \to M$ over a complex manifold M is a HOLOMORPHIC LINE BUNDLE if L is a complex manifold, $\pi : L \to M$ is a holomorphic map and there exists an open cover $(U_j)_{j \in I}$ of M with trivializations

$$\psi_j : L_{U_j} \to U_j \times \mathbb{C},$$

which are holomorphic maps.

Similar to our results on general complex line bundles over a manifold, the holomorphic line bundles are given by transition functions $(g_{jk})_{j,k \in I}$, but now they are holomorphic functions:

$$g_{jk} : U_{jk} \to \mathbb{C}^\times, \text{ or } g_{jk} \in \mathcal{O}^\times(U_{jk}).$$

The group of isomorphism classes of holomorphic line bundles over the complex manifold is $H^1(M, \mathcal{O}^\times)$.

A section $s : U \to L$ of a holomorphic line bundle L over an open subset $U \subset M$ is called a HOLOMORPHIC SECTION if s is a holomorphic map. $\Gamma_{\text{hol}}(U, L) \subset \Gamma(U, L)$ denotes the subspace of holomorphic sections

$$\Gamma_{\text{hol}}(U, L) := \{s \in \Gamma(U, L) \mid s \text{ holomorphic}\}.$$

$\Gamma_{\text{hol}}(U, L)$ is a complex vector space and a module over the ring $\mathcal{O}(U)$ of holomorphic functions on $U \subset M$.

Definition 6.12. Let $\pi : L \to M$ be a holomorphic line bundle with connection ∇ on L.

1° ∇ is a HOLOMORPHIC CONNECTION if for any holomorphic section $s \in \Gamma_{hol}(U, L)$ the map

$$X \mapsto \nabla_X s, X \in \mathfrak{V}(U) \text{ with } X \text{ holomorphic},$$

is a holomorphic one-form (with values in $\Gamma(U, L)$), i.e. in local holomorphic coordinates $\varphi = (z^1, \dots z^n) : U \to V \subset \mathbb{C}^n$

$$\nabla s = f_j dz^j s$$

with holomorphic $f_j : U \to \mathbb{C}$. Hence, $\nabla_X s = f_j X^j s$ for holomorphic vector fields $X = X^j \frac{\partial}{\partial z_j}$.

2° ∇ is COMPATIBLE with the holomorphic structure on L if for any local holomorphic section $s \in \Gamma_{\text{hol}}(U, L)$, the one-form

$$X \mapsto \nabla_X s$$

is of pure type $(1,0)$, i.e. in local holomorphic coordinates

$$\nabla s = f_j dz^j s,$$

where $f_j \in \mathcal{E}(U)$.

Proposition 6.13. *Every holomorphic line bundle L over the complex manifold M admits a connection ∇ compatible with the holomorphic structure on the line bundle L.*

Proposition 6.14. *Let L be a holomorphic line bundle and let ∇ be a connection on L, which is compatible with the holomorphic structure on L. Then the curvature* $\mathrm{Curv}(\nabla, L) = \Omega$ *has components only of type* $(2,0)$ *and* $(1,1)$, *i.e. in local coordinates* $\varphi = (z^1, \dots, z^n) : U \to V \subset \mathbb{C}^n$:

$$\Omega = \omega_{jk} dz^j \wedge dz^k + \rho_{jk} dz^j \wedge d\bar{z}^k\,, \;\; \omega_{jk}, \rho_{jk} \in \mathcal{E}(U).$$

Without proof state we the following result:

Proposition 6.15. *Let $L \to M$ be a holomorphic line bundle, which is also endowed with a Hermitian metric H. Then there exists a unique connection ∇ , which is compatible both with the holomorphic structure and the Hermitian structure. This connection is called the* Chern connection *of the Hermitian holomorphic line bundle. The curvature is of type* $(1,1)$.

Example 6.16. The tautological line bundle

$$T = H(-1) \longrightarrow \mathbb{P}_n(\mathbb{C})$$

is a holomorphic line bundle (c.f. Construction 3.17 for the definition). Given an open subset $U \subset \mathbb{P}^n(\mathbb{C})$ a local section $s \in \Gamma_{hol}(U,T)$ yields for every $a \in U$ a point $s(a) \in T \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$, $s(a) = (a, z_0(a), \dots, z_n(a)) \in \mathbb{P}^n \times \mathbb{C}^{n+1}$, and the value $s(a)$ describes the point $a \in \mathbb{P}^n$ in homogeneous coordinates: $a = (z_0(a) : \dots : z_n(a))$. A natural connection is defined by:

$$\nabla s := \frac{\bar{z}_j dz^j}{\sum |z^j|^2} s\,,$$

where $z_j = z^j$ for notational reasons.

∇ is compatible with the holomorphic structure of the line bundle (but it is not a holomorphic connection!): Indeed, for a local holomorphic section $s(a) = (a, z_0(a), \dots, z_n(a))$, $a \in U$, the connection is

$$\nabla s := \sum_j \frac{\bar{z}_j}{\sum_k |z^k|^2} dz^j s = \sum_j f_j dz^j s\,,$$

with $\sum f_j dz^j \in \mathcal{A}^{(1,0)}(U)$ as required.

∇ is also compatible with the Hermitian metric

$$H(s, s') = \sum_{j=0}^{n} \bar{z}^j z'^j.$$

H is induced from the standard Hermitian metric on $\mathbb{C}^{n+1}$. To check the compatibility:

$$dH(s, s) = z_j d\bar{z}^j + \bar{z}_j dz^j = 2(p_j dq^j + q^j dp_j)$$

and

$$\mathrm{Re}\left(\bar{z}_j dz^j\right) = p_j dq^j + q^j dp_j$$

lead to

$$dH(s, s) = 2H(s, s)\mathrm{Re}\left(\frac{\bar{z}_j dz^j}{\sum |z^j|^2}\right),$$

which is the criterium of Proposition 6.8.

Proposition 6.17. *Let $L \to M$ be a smooth complex line bundle over the complex manifold M equipped with a connection whose curvature is purely of type $(1, 1)$. Then there exists a unique holomorphic structure on L for which a local section s of L is holomorphic if and only if $X \mapsto \nabla_X s$ is a one-form of type $(1, 0)$.*

Proofs can be found in Brylinski [Bry93], for example.

7 Prequantization

In the preceding chapters we have collected all the ingredients which we need in order to continue the discussion started in Section 2 about obtaining a canonical quantization by using derivatives on functions or covariant derivatives on sections of line bundles. We now can describe the process of prequantization properly.

7.1 Quantizable Phase Space

Definition 7.1. A symplectic manifold (M, ω) is said to be QUANTIZABLE if there exists a complex line bundle $\pi : L \to M$ with connection ∇ such that $\mathrm{Curv}(L, \nabla) = \omega$.

Definition 7.2. A PREQUANTUM LINE BUNDLE (L, ∇, H) on a symplectic manifold (M, ω) is a Hermitian line bundle (L, H) together with a compatible connection ∇ such that $\mathrm{Curv}(L, \nabla) = \omega$.

Evidently, when (L, H, ∇) is a prequantum bundle, the base space (M, ω) has to be quantizable. Conversely, on a quantizable symplectic manifold there always exist prequantum bundles: since the connection ∇ with $\mathrm{Curv}(L, \nabla) = \omega$ can be chosen to be real, we can find with the help of a partition of unity a Hermitian metric H such that ∇ is compatible with respect to H.

We have seen that for a symplectic manifold (M, ω) the condition to be quantizable is a topological condition on M and ω: The cohomology class induced by ω has to be an entire cohomology class (condition (E)) or, equivalently, ω has to satisfy integrality condition (G) (see Section 5.4), i.e.:

$$\int_S \omega \in \mathbb{Z}$$

for all compact, oriented and closed surfaces $S \subset M$.

We come back to these conditions in the subsequent chapter. In particular, we will construct a line bundle L with connection ∇ such that $\mathrm{Curv}(L, \nabla) = \omega$ when ω fulfills (E). And we discuss the uniqueness of this construction.

Before we come to these matters, in this chapter we want to present examples and we describe the prequantization process.

Example 7.3 (Simple Phase Space)**.** Let $M = T^*Q$ be a cotangent bundle for an open subset $Q \subset \mathbb{R}^n$ with the standard symplectic form $\omega = -d\lambda = dq^j \wedge dp_j$, and its symplectic potential $A = -\lambda = -p_j dq^j$. The trivial line bundle $L = M \times \mathbb{C}$ with the connection

$$\nabla_X f s_1 = (L_X f + 2\pi i A(X) f) s_1,$$

$s_1(a) = (a, 1)$, has as its curvature

$$dA = \omega$$

Since for every compact, oriented and closed surface $S \subset T^*Q$ one has

$$\int_S \omega = \int_{\partial S} A = 0$$

by Stokes' theorem ($\partial S = \varnothing$), the symplectic manifold (T^*Q, ω) is quantizable.

Remark 7.4. In the same way, any symplectic manifold (M, ω) for which ω is exact, i.e. $\omega = d\alpha$, for a suitable $\alpha \in \mathcal{A}^1(M)$, is quantizable.

Proposition 7.5 (Twisted Case)**.** *Let $M = T^*Q$ be a cotangent bundle for a manifold Q with the twisted symplectic form*

$$\omega_F := \omega_0 + \tau^*(F)\,,$$

where $\omega_0 = -d\lambda$ is the standard symplectic form on the momentum phase space $M = T^*Q$ and $F \in \mathcal{A}^2(Q)$ is a closed 2-form on the configuration space Q (see Subsection 1.3.4). In general, F will not be exact and so ω_F will not be exact. But the following result holds true: (T^*Q, ω_F) is quantizable if and only if $\int_S F = 0$ for all surfaces $S \subset Q$, i.e. F satisfies the integrality condition on Q.

Proof. If T^*Q is quantizable, the integrality condition $\int_S \omega_F = 0$ implies that, in particular, for all surfaces $S \subset Q \subset T^*Q$ we have $\int_S \omega_F = \int_S F = 0$. Conversely, if Q satisfies the integrality condition with respect to F there exists a line bundle L_Q on Q with connection ∇_Q such that $\mathrm{Curv}(L_Q, \nabla_Q) = F$. On T^*Q we have the trivial line bundle L with connection ∇ with $\mathrm{Curv}(L, \nabla) = \omega_0$ locally. Now; $L \otimes \tau^* L_Q$ with $\nabla \otimes \tau^+ \nabla_Q$ is a prequantum bundle since

$$\mathrm{Curv}(L \otimes \tau^* L_Q, \nabla \otimes \tau^* \nabla_Q = \mathrm{Curv}(L, \nabla) + \mathrm{Curv}(\tau^* L_Q, \tau^* \nabla_Q) = \omega_0 + \tau^* F\,.$$

□

Example 7.6. Let M be the two sphere $M = \mathbb{S}^2$ of radius 1 with the symplectic form $\omega_C = C\mathrm{vol}$, for some constant $C \in \mathbb{R} \setminus \{0\}$, where vol is the standard volume form on $\mathbb{S}^2$ ($= \sin\theta d\theta \wedge d\phi$ in polar coordinates). Since $\int_{\mathbb{S}^2} \omega_C = C\mathrm{vol}(\mathbb{S}^2) = C4\pi$ the symplectic manifold $(\mathbb{S}^2, \omega)$ is quantizable in the sense of Definition 7.1 if and only if

$$4\pi C = \mathbb{Z} \setminus \{0\},$$

i.e. $C = \frac{1}{4\pi} N, N \in \mathbb{Z} \setminus \{0\}$.

Example 7.7. The hydrogen atom. We recall from Example 1.3.3:

The classical system is given by the manifold $M = T^*(\mathbb{R}^3 \setminus 0) \cong (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3$ with standard symplectic form $\omega = d(-\lambda) = \sum_j q^j \wedge p_j$ and hamiltonian

$$H(q,p) = \frac{1}{2m}|p|^2 - \frac{k}{|q|},$$

where $m, k \in \mathbb{R}, m > 0, k > 0$. Since $dH \neq 0$, the "energy surface" for $E \in (-\infty, 0)$

$$\Sigma_E := H^{-1}(E) \subset M$$

is a 5-dimensional submanifold of M (hypersurface in M).

Identifying points $x, y \in \Sigma_E$ on a joint orbit leads to the orbit space

$$M_E := \Sigma_E / \sim$$

as a quotient. Here, the equivalence relation $\sim$ is given by

$$x \sim y \iff \exists \text{ solution of } \dot{\gamma} = X_H(\gamma) \text{ with } \gamma(0) = x \text{ and } \gamma(t) = y$$

for a suitable t in the domain of definition of γ.

The orbit space M_E has the structure of a quotient manifold for which the natural quotient map $\pi : \Sigma \to M_E$ is a submersion. In order to show this, we consider the map

$$\Psi : \Sigma_E \to \mathbb{S}^2(mk) \times \mathbb{S}^2(mk)$$

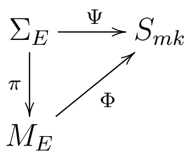
into the product of two spheres of radius $r = mk$ given by

$$\Psi(a) = (\rho I(a) + R(a), \rho I(a) - R(a))$$

with $a = (q, p) \in \Sigma_E$ and where

$$\begin{aligned} I(a) &= q \times p \text{ angular momentum} \\ R(a) &= I(a) \times p \text{ Runge-Lenz vector} \\ \rho &= \sqrt{-2mE} \end{aligned}$$

The map Ψ is constant on the orbits, since the observables I and R are constants of motion. Moreover, the fibres $\Psi^{-1}(s)$, $s \in S_{mk} := \mathbb{S}^2(mk) \times \mathbb{S}^2(mk)$, are the orbits in Σ_E. Hence, there is a unique bijection $\Phi : M_E \to S_{mk}$ such that $\Psi = \Phi \circ \pi$: The following diagram is commutative.



Of course, Ψ is smooth, and it can be shown that Ψ has maximal rank in all points of Σ_E. Hence, $\Psi \to S_{mk}$ is a differential quotient according to Proposition A.29. As a consequence, the differentiable structure on M_E induced by the bijection $\Phi : M_E \to S_{mk}$ makes $\pi : \Sigma \to M_E$ to a differentiable quotient.

Quantizing the classical hydrogen atom with energy E is therefore equivalent to quantizing the symplectic manifold

$$(\mathbb{S}^2(mk) \times \mathbb{S}^2(mk), \omega'_E)$$

It can be shown that ω'_E, induced on S_{mk} from $\omega_E := \omega|_{\Sigma_E}$, has the form

$$\omega'_E = \frac{1}{2\rho}\left(\frac{dx_1 \wedge dx_2}{x_3} + \frac{dy_1 \wedge dy_2}{y_3}\right), x_3 \neq 0 \neq y_3,$$

where $(x_1, x_2, x_3) : \mathbb{S}^2(mk) \to \mathbb{R}^3$ are the standard coordinates of $\mathbb{R}^3$ and similarly (y_1, y_2, y_3) for the second sphere $\mathbb{S}^2(mk)$.
Because of

$$\int_{\mathbb{S}^2(r)} r\frac{dx_1 \wedge dx_2}{x_3} = 4\pi(r)^2$$

we obtain for $S := \mathbb{S}^2(mk) \times \{y\} \subset \mathbb{S}^2(mk) \times \mathbb{S}^2(mk)$ the quantization condition

$$\int_S \omega'_E = \frac{1}{2\rho}\int_S \frac{dx_1 \wedge dx_2}{x_3} = \frac{4\pi}{2\rho}mk = N \in \mathbb{Z}\,!$$

As a consequence, (M, ω_E) is quantizable only if $2\pi\frac{mk}{\rho} = N \in \mathbb{Z}$, i.e. if $4\pi^2\frac{(mk)^2}{-2mE} = N^2$. Hence, $-2mE = \frac{1}{N^2}4\pi^2m^2k^2$ and we conclude that only for the energy values

$$\boxed{E_N = -\frac{2\pi^2}{N^2}mk^2}$$

the symplectic manifolds $(\mathbb{S}^2(mk) \times \mathbb{S}^2(mk), \omega'_E)$ are quantizable, i.e. (M_{E_N}, ω'_{E_N}) is quantizable. We know this from experimental physics, this is the well known Balmer Series!

7.2 Prequantum Operator

We are now in the position to define the prequantum operator of geometric quantization on the space of sections of a prequantum line bundle, thus generalising the preliminary ansatz in Section 2.2.

Theorem 7.8. *Let (L, ∇, H) be a prequantum line bundle over the symplectic manifold (M, ω). Then the $\mathbb{C}$-linear map $q : \mathcal{E}(M, \mathbb{C}) \to \mathrm{End}_{\mathbb{C}}(\Gamma(M, L))$*

$$\boxed{q(F) := -\frac{i}{2\pi}\nabla_{X_F} + F}$$

satisfies the Dirac conditions:

(D1) $q(1) = \mathrm{id}_{\Gamma(M,L)}$,

(D2) $[q(F), q(G)] = \frac{i}{2\pi}q(\{F, G\})$ *for all* $F, G \in \mathcal{E}(M, \mathbb{C})$.

$q(F)$ is called the PREQUANTUM OPERATOR.

Proof. Evidently, q is $\mathbb{C}$-linear, and $q(1) = \mathrm{id}$. To show (D2) we start with the quantizibilty condition of the symplectic manifold (M, ω):

$$\frac{1}{2\pi i}\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) = \mathrm{Curv}(\nabla, L)(X, Y) = \omega(X, Y).$$

Applied to $X = X_F, Y = Y_G$ and using $[X_F, X_G] = -X_{\{F,G\}}$ this reads:

$$[\nabla_{X_F}, \nabla_{X_G}] = 2\pi i \{F, G\} - \nabla_{X_{\{F,G\}}}.$$

Hence,

$$\begin{aligned}
[q(F), q(G)] &= \left[-\frac{i}{2\pi}\nabla_{X_F} + F, -\frac{i}{2\pi}\nabla_{X_G} + G\right] \\
&= \left(\frac{i}{2\pi}\right)^2 [\nabla_{X_F}, \nabla_{X_G}] - \frac{i}{2\pi}F\nabla_{X_G} + \frac{i}{2\pi}\nabla_{X_G} \circ F + \frac{i}{2\pi}G\nabla_{X_F} - \frac{i}{2\pi}\nabla_{X_F} \circ G \\
&= \left(\frac{i}{2\pi}\right)^2 \left(-\nabla_{X_{\{F,G\}}} + 2\pi i \{F, G\}\right) - \frac{i}{2\pi}\left(L_{X_F}G - L_{X_G}F\right) \\
&= \frac{i}{2\pi}\left(-\frac{i}{2\pi}\nabla_{X_{\{F,G\}}} - \{F, G\}\right) + 2\frac{i}{2\pi}\{F, G\} \\
&= \frac{i}{2\pi} q\left(\{F, G\}\right),
\end{aligned}$$

where we have used among others the identities: $\nabla_{X_F} \circ G = (L_{X_F}G) + G\nabla_{X_F}$ and $L_{X_F}G = \{G, F\}$. □

Remark 7.9. One might not be content with the factor $\frac{i}{2\pi}$ in front of $q(\{F, G\})$ (in (D2)), preferring $\frac{1}{2\pi i}$ or $\frac{\hbar}{i}$, where $\hbar := \frac{h}{2\pi}$ for some $h > 0$. This can be achieved by changing the quantization condition $\mathrm{Curv}(L, \nabla) = \omega$ to a new condition $\mathrm{Curv}(L, \nabla) = -\omega$ or $\mathrm{Curv}(L, \nabla) = k\omega$ for a suitable $k \in \mathbb{C}$.

The special outcome in D2 in our case is reflected by the choices (see Subsection 18.3 for sign conventions in the literature):

1. X_H defined by $i_{X_H}\omega := dH$ (and not $-dH$),
2. $\{f, g\} = \omega(X_f, X_g)$ (and not $-\omega(X_f, X_g)$),
3. $\mathrm{Curv}(L, \nabla) = \frac{1}{2\pi i}\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right)$ and not $\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right)$,
4. $k = 1$.

In general, to obtain an arbitrary c in (D2) we have to choose in our conventions 1.-3. $k = \frac{i}{2\pi}$ in the quantization condition of Definition 7.1: $\mathrm{Curv}(L, \nabla) = k\omega$.

In particular, for $c = \frac{\hbar}{i} = \frac{h}{2\pi i}$ the quantization condition reads $\mathrm{Curv}(L, \nabla) = -\frac{1}{h}\omega$.

Before we discuss how to obtain a suitable representation space, the Hilbert space on which the prequantum operators act, we want to describe an alternative way to introduce the prequantum operator which is more in the spirit of the description of a connection on a line bundle L given in Proposition 4.13 and where the connection is induced by a Lie derivative on invariant functions on $L^\times$.

Let $F \in \mathcal{E}(M)$ a classical real observable. The hamiltonian vector field $X_F \in \mathfrak{V}(M)$ has a unique horizontal lift $X_F^\sharp = (X_F)^\sharp \in \Gamma(L^\times, H) \subset \mathfrak{V}(L^\times)$ (cf. Definition 4.12). Adding the vertical field $-Y_{F\circ\pi} \in \Gamma(L^\times, V)$, where Y_c is the fundamental field we obtain

$$Z_F := X_F^\sharp - Y_{F\circ\pi}\,,$$

a vector field on $L^\times$, which we call the natural lift of X_F.

The natural lift Z_F can also be defined by the following two conditions

$$T\pi \circ Z_F = X_F \circ \pi \;\text{ and }\; \alpha(Z_F) = F \circ \pi\,,$$

where α is the global connection form.

Proposition 7.10. *The prequantum operator $q(F)$ can be obtained by using the Lie derivative L_{Z_F} on $\mathcal{E}_{-1}(L^\times)$ in the following way: For $s \in \Gamma(M, L)$ the section $q(F)s \in \Gamma(M, L)$ is given by*

$$\boxed{(q(F)s)^\sharp = -\frac{i}{2\pi} L_{Z_F} s^\sharp.}$$

Moreover, Z_F preserves the global connection form α, i.e. $L_{Z_F}\alpha = 0$, and any real vector field $Z \in \Gamma(M, TL^\times)$ preserving α is of the form $Z = Z_F$ for a suitable $F \in \mathcal{E}(M)$.

Proof. We concentrate on the formula: We have $L_{X_F^\sharp} s^\sharp = (\nabla_{X_F} s)^\sharp$ according to Proposition 4.13 and we know $L_{Y_F} s^\sharp = -2\pi i F s^\sharp$ by Lemma 4.11. Therefore,

$$L_{Z_F} s^\sharp = L_{X_F^\sharp} s^\sharp - L_{Y_F} s^\sharp = (\nabla_{X_F} s)^\sharp - L_{Y_F} s^\sharp = (\nabla_{X_F} s + 2\pi i F s)^\sharp = 2\pi i (q(F)s)^\sharp,$$

which is essentially the formula. □

In order to understand the relationship between X_F and its natural lift Z_F we study their local flows.

Lemma 7.11. *Let $\Phi_t = \Phi_t^F : M_t \to M_{-t}$ be the flow of X_F and $\Phi_t^\sharp = \Phi_t^{F^\sharp} : L^\times{}_t \to L^\times{}_{-t}$ the flow of the natural lift Z_F of X_F. Then*

1. *$\pi \circ \Phi_t^\sharp = \Phi_t \circ \pi$ for all $t \in \mathbb{R}$. More explicitly, let M_t be the maximal domain on which the local flow Φ_t is defined (see Proposition A.20), and correspondingly, let $L^\times{}_t$ be the maximal domain for the local flow $\Phi_t^\sharp$, then $\pi^{-1}(M_t) = L^\times{}_t$ and the following diagram is commutative*

$$\begin{array}{ccc} L^\times{}_t & \xrightarrow{\Phi_t^\sharp} & L^\times{}_{-t} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M_t & \xrightarrow[\Phi_t]{} & M_{-t} \end{array}$$

2. *X_F is complete if and only if Z_F is complete.*

3. *$\Phi_t^\sharp$ commutes with the action of c.*

Proof. For $p \in L^\times$ with $\pi(p) = a$ the curve $\varphi(t) := \Phi_t(a)$ is the unique solution of

$$\frac{d}{dt}\varphi = X_F(\varphi)\,,\ \varphi(0) = a \in M\,, \tag{33}$$

and, correspondingly, the curve $\varphi^\sharp(t) := \Phi_t^\sharp(p)$ is the unique solution of

$$\frac{d}{dt}\varphi^\sharp = Z_F(\varphi^\sharp)\,,\ \varphi^\sharp(0) = p \in L^\times\,.$$

Because of

$$\frac{d}{dt}(\pi \circ \varphi^\sharp) = T\pi(\frac{d}{dt}\varphi^\sharp) = T\pi \circ Z_F(\varphi^\sharp) = X_F \circ \pi(\varphi^\sharp)$$

and $\pi \circ \varphi^\sharp(0) = a$ we deduce $\pi \circ \varphi^\sharp = \varphi$ by the uniqueness of the solutions of (33). This implies, that the maximal intervals on which the solutions φ and $\varphi^\sharp$ are defined, coincide. Hence, $\pi^{-1}(M_t) = L^\times{}_t$ for all $t \in \mathbb{R}$. Now 1. follows since $\pi \circ \Phi_t^\sharp(p) = \Phi_t \circ \pi(p)$ is nothing else then $\pi \circ \varphi^\sharp(p) = \varphi(a)$.

Moreover, 2. holds, since the maximal intervals coincide.

Finally, 3. is again a consequence of the uniqueness of solutions of integral curves, here for the equation $\frac{d}{dt}\gamma = Z_F(\gamma)$: Set $\psi := \Psi_c\varphi^\sharp = \varphi^\sharp c$, where $\varphi^\sharp(t) = \Phi_t^\sharp(p)$. Then $\psi(0) = pc$ and

$$\frac{d}{dt}\psi = T\Psi_c(\frac{d}{dt}\varphi^\sharp) = T\Psi_c(Z_F(\varphi^\sharp)) = Z_F(\Psi_c(\varphi^\sharp)) = Z_F(\psi)\,,$$

and therefore, $\psi = \Phi_t^\sharp(pc)$ which implies $\Phi_t^\sharp(pc) = \Phi_t^\sharp(p)c$. We have used $T\Psi_c \circ Z_F = Z_F \circ \Psi_c$. □

The last property guarantees that for sections $s \in \Gamma(M_t, L)$ the function $s^\sharp \circ \Phi^\sharp_{-t} : L^\times{}_{-t} \to \mathbb{C}$ is invariant, i.e. in $\mathcal{E}_{-1}(L^\times{}_{-t})$. As a consequence there exists a unique $\rho_t^F(s) \in \Gamma(M_{-t}, L)$ satisfying

$$(\rho_t^F(s))^\sharp = s^\sharp \circ \Phi^\sharp_{-t}\,.$$

It is easy to show

Lemma 7.12. $\rho_t^F : \Gamma(M_t, L) \to \Gamma(M_{-t}, L)$ *is an isomorphism of* $\mathcal{E}(M)$*-modules. When* X_F *is complete, all spaces* M_t *coincide with* M *and the maps* $\rho_t^F : \Gamma(M, L) \to \Gamma(M, L)$ *satisfy*

$$\rho^F_{t+t'} = \rho^F_t \circ \rho^F_{t'} = \rho^F_{t'} \circ \rho^F_t$$

for all $t, t' \in \mathbb{R}$.

The last equality holds true in the general case for suitable restrictions of the ρ_t and if the $|t|, |t'|$ are small enough.

Proposition 7.13. *The infinitesimal generator of* (ρ_t^F) *is* $q(F)$*:*

$$\boxed{q(F)s = \frac{i}{2\pi}\frac{d}{dt}\rho_t^F(s)|_{t=0}\,.} \tag{34}$$

Proof. We use

$$\begin{aligned}\left(\frac{d}{dt}\rho_t^F(s)\right)^\sharp &= \frac{d}{dt}\left(s^\sharp \circ \Phi^\sharp_{-t}\right)\\ &= ds^\sharp\left(\frac{d}{dt}\Phi^\sharp_{-t}\right)\\ &= -ds^\sharp\left(Z_F\left(\Phi^\sharp_{-t}\right)\right)\end{aligned}$$

to obtain

$$\frac{i}{2\pi}\left(\frac{d}{dt}\rho_t^F(s)\right)^\sharp|_{t=0} = -\frac{i}{2\pi}ds^\sharp(Z_F) = -\frac{i}{2\pi}L_{Z_F}s^\sharp = (q(F)s)^\sharp\,,$$

where the last equality is the result of Proposition 7.10. □

Remark 7.14. The definition of the prequantum operator we gave in Theorem 7.8 is inspired by the considerations made in Chapter 2 and the properties of connections on line bundles. The description of $q(F)$ in Equation (34) can be used as a definition as well. This definition is more geometric in nature and it can be generalized to the half-density quantization and the half-form quantization.

7.3 Prequantum Hilbert space

In quantum models one wants to represent the observables as operators in a Hilbert space (see the Chapter F on Quantum Mechanics in the Appendix), in the so-called representation space of the model. It is not difficult in the Geometric Quantization program to replace the complex vector space $\Gamma(M,L)$, on which our prequantum operators $q(F)$ act, with a natural Hilbert space of sections: The symplectic form ω of a symplectic manifold (M,ω) always induces a volume form (the so-called Liouville measure)

$$\mathrm{vol} := (-1)^{\frac{1}{2}n(n-1)}\frac{1}{n!}\omega^n \in \mathcal{A}^{2n}(M),$$

where $\omega^n = \omega \wedge \omega \cdots \wedge \omega$ (n times) and $\dim M = 2n$. We obtain the Hilbert space $L^2(M,\mathrm{vol})$ by completing the prehilbert space $\{f \in \mathcal{E}(M,\mathbb{C}) \mid \int_M |f|^2 d\mathrm{vol} < \infty\}$. But we are interested in the corresponding Hilbert space of square integrable sections in the prequantum bundle.

Let (L,H) be a Hermitian line bundle over a symplectic manifold (M,ω). We define

$$H_{\mathrm{pre}} := \{s \in \Gamma(M,L) \mid \int_M |s|^2 d\mathrm{vol} < \infty\} \subset \Gamma(M,L)$$

the space of SQUARE INTEGRABLE smooth sections where $|s(a)|^2 = H(s(a),s(a))\,,\ a \in M$. This space is a subspace of $\Gamma(M;L)$ and it is a prehilbert space with respect to the scalar product

$$\langle s,t\rangle := \int_M H(s,t)d\mathrm{vol}\,,$$

$s,t \in \mathbb{H}_{\mathrm{pre}}$. The completion of H_{pre} with respect to the norm

$$\|s\| := \sqrt{\int_M |s|^2 d\mathrm{vol}}$$

is the Hilbert space $\mathbb{H} = \mathbb{H}(M,L)$ of square integrable sections This Hilbert space will be called the PREQUANTUM HILBERT SPACE.

The prequantum operator $q(F)$ is defined on the the space $\Gamma_0 = \Gamma_0(M,L) \subset \mathbb{H}$ of sections having compact support in M. And $q(F)(\Gamma_0) \subset \Gamma_0$. Therefore, the prequantum operator $q(F)$ induces in any case a linear operator in $\mathbb{H}$ with domain $D(q(F))$ containing Γ_0. Thus, $q(F)$ is densely defined. One can show, that for real F the prequantum operator $q(F)$ will be symmetric on Γ_0 in the same way as in Proposition 2.4. Thus:

Proposition 7.15. *For real $F \in \mathcal{E}(M,\mathbb{R})$, the prequantum operator $q(F)$ is a densely defined symmetric operator in the prequantum Hilbert space $\mathbb{H} = \mathbb{H}(M,L)$ of square integrable sections of L.*

However, given an observable $F \in \mathcal{E}(M)$ it is, in general, difficult, to decide whether the operator induced by $q(F)$ is essentially self-adjoint as an operator of $\mathbb{H}$. The problem is, to extend the symmetric and densely defined operator $q(F) : \Gamma_0 \to \Gamma_0$ to a unique self-adjoint operator on a suitable domain $D(q(F))$ with $\Gamma_0 \subset D(q(F)) \subset \mathbb{H}$.

For a the class of observables F with complete hamiltonian vector field X_F we have the following positive result.

Proposition 7.16. *Let (M, ω) be a quantizable symplectic manifold and let (L, ∇, H) be a prequantum bundle. For every $F \in \mathcal{E}(M, \mathbb{R})$ for which $X_F \in \mathfrak{V}(M)$ is a complete vector field on M the prequantum operator $q(F)$ is an essentially self-adjoint operator*[35] *in $\mathbb{H}(M, L)$.*

Proof. Let $Z_F \in \mathfrak{V}(L^\times)$ be the natural lift of X_F as before. Since X_F is complete by assumption, Z_F is complete as well, and the local flows are global flows. As a consequence ρ_t^F is defined on $\Gamma(M, L)$ and leads a one parameter group of linear maps. It is easy to see that $\rho_t^F(\Gamma_0) \subset \Gamma_0 \subset \mathbb{H}$, and that $\rho_t^F|_{\Gamma_0}$ is bounded as an operator from Γ_0 to $\mathbb{H}$, with a bounded continuation to all of $\mathbb{H}$. This operator, which will be denoted by ρ_t^F again, turns out to be unitary, so that $U_t := -2\pi\rho_t^F$ defines a one parameter group of unitary operators (U_t). According to the Theorem of Stone F.5 the infinitesimal generator

$$As = i \lim_{t \to 0} \frac{1}{t}(U_t s - s)\,,\; s \in D(A),$$

is a self-adjoint operator $A : D(A) \to \mathbb{H}$. From the preceding proposition we know

$$\left.\frac{d}{dt}\rho_t^F(s)\right|_{t=0} = -\frac{i}{2\pi}q(F)(s)$$

for $s \in \Gamma_0 \subset D(A)$. Hence, $q(F)$ has A a self-adjoint extension, and therefore is essentially self-adjoint. □

Observation 7.17. In the case of a compact symplectic manifold all $q(F)$ are essentially self-adjoint for $F \in \mathcal{E}(M, \mathbb{R})$.

Example 7.18 (Simple Phase Space)**.** Let us recall the example at the end of Chapter 2, Examples 2.6, i.e. the case $M = T^*Q$ of the momentum phase space, $Q \subset \mathbb{R}^n$ open (see also Example 7.3). This example explains why we have to introduce and study polarizations later. On M we have the standard symplectic structure given by the 2-form $\omega = \sum_j dq^j \wedge dp_j = d(-\lambda)$, $\lambda = p_j dq^j$ with respect to the canonical coordinates q^j, p_j on $T^*Q \subset \mathbb{R}^n \times \mathbb{R}^n$. Let $L = M \times \mathbb{C}$ the trivial line bundle with connection

$$\nabla_X f s_1 = (L_X f - 2\pi i \lambda(X) f) s_1\,,$$

[35] i.e. the closure of $q(F)$ is a self-adjoint operator, see Section F.2 in the Appendix F for the basics of self-adjoint operators on a Hilbert space

$f \in \mathcal{E}(M)$ and $s_1(a) := (q, 1)$ as before. Then $\mathrm{Curv}(L, \nabla) = \omega$. To determine the operators $q(F) = \frac{1}{2\pi i}\nabla_{X_F} + F$ for $F = p_j$,$F = q^j$, we use

$$X_F = \frac{\partial F}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial F}{\partial q^k}\frac{\partial}{\partial p_k}\,,$$

to obtain

$$X_{q^j} = -\frac{\partial}{\partial p_j} \quad \text{and} \quad X_{p_j} = \frac{\partial}{\partial q^j}\,.$$

Hence,

$$q\left(q^j\right) = -\frac{i}{2\pi}\nabla_{X_{q^j}} + q^j = -\frac{i}{2\pi}\left(-\frac{\partial}{\partial p_j} + 2\pi i\lambda\left(\frac{\partial}{\partial p_j}\right)\right) + q^j,$$

$$q\left(q^j\right) = \frac{i}{2\pi}\frac{\partial}{\partial p_j} + q^j =: Q^j. \quad \text{Analogously,}$$

$$q\left(p_j\right) = -\frac{i}{2\pi}\frac{\partial}{\partial q^j} =: P_j.$$

As a result:

$$\left[q\left(q^j\right), q\left(p_k\right)\right] = \left[Q^j, P_k\right] = \frac{i}{2\pi}\delta^j_k = \frac{i}{2\pi}q\left(\left\{q^j, p_k\right\}\right). \tag{35}$$

We see that the Dirac conditions are confirmed on the space $\Gamma(M, L) \cong \mathcal{E}(M)$ (which is a consequence of Proposition 7.8), and also on the space

$$\mathbb{H} = \mathbb{H}(M, L) = L^2(T^*Q)\,.$$

In particular, they are satisfied for the unbounded operators Q^j and P_k which are self-adjoint. That is, the canonical commutation relations (CCR) (cf. Definition F.38) are satisfied.

However, in comparison to the usual canonical quantization in this situation, we observe that too many variables are involved: the wave function $\psi \in \mathbb{H}$ should depend on n variables and not $2n$. In a more abstract wording, the representation of the Q^j, P_k is not irreducible. By definition, the representation of the (Lie algebra generated by) Q^j, P_k is called irreducible, if no proper closed subspace $\mathbb{H}_0$ of $\mathbb{H}$ is invariant under the action of the Q^j, P_k. But $\mathbb{H}_0 := \{f \in \mathbb{H} \mid f = g \circ \pi\}$ is invariant! And this $\mathbb{H}_0$ is a good candidate for a better representation space as we show in the following.

By replacing the Hilbert space $\mathbb{H}(M, L)$ by its closed subspace $\mathbb{H}_P(M, L)$ of all functions f of the form $f = g \circ \pi$, $g : Q \to \mathbb{C}$, for suitable g, i.e. $\mathbb{H}_P(M, L) = \mathbb{H}_0$, we arrive at a prequantization with the correct dependencies and moreover

$$Q^j := q^j \quad \text{and} \quad P_j := -\frac{i}{2\pi}\frac{\partial}{\partial q^j} \tag{36}$$

This quantization is called the Schrödinger Representation of the Q^j, P_k with the commutation rules

$$\left[P_j, Q^k\right] = -\frac{i}{2\pi}\delta^j_k\,.$$

The replacement of $\mathbb{H}(M, L)$ by $\mathbb{H}_P(M, L)$ is one of the possibilities of arriving at a correct representation space (space of wave functions). This procedure is not limited to the simple case $M = T^*Q$, $Q \subset \mathbb{R}^n$ open, but can be generalized to all symplectic manifolds by introducing the notion of a polarization. We study polarizations in Chapter 9 and present their applications to geometric quantization in the subsequent chapters.

Summary: The concept of prequantization is completed in this chapter as an important step towards Geometric Quantization. It yields results in great generality as well as in specific examples.

In the examples one sees that the condition for the existence of a prequantum bundle i.e. the integrality condition allows only discrete values: The underlying symplectic manifold is quantized. This is familiar from elementary quantum theory.

The general result is, that on the basis of a prequantum bundle on a symplectic manifold the prequantum operators $q(F)$ satisfy the Dirac conditions (D1) and (D2) and a natural Hilbert space is determined on which all prequantum operators are densely defined and symmetric. In many cases, namely if the hamiltonian vector field X_F is complete, they are even self-adjoint. In particular, the canonical commutations relations are rediscovered. The question of whether enough prequantum bundles exist and how many there are leads to a closer inverstigation of the integrality condition in the next chapter.

8 Integrality

We now study in detail the question of existence and uniqueness of prequantum line bundles on a given symplectic manifold (M, ω). In particular, we construct a prequantum bundle for a given symplectic manifold (M, ω) when ω is entire in the sense of (E) and we show that the equivalence classes of such prequantum bundles are in one-to-one corespondence to $\check{H}^1(M, \mathrm{U}(1))$. Finally, we bring this classification in connection with flat line bundles and there classification.

8.1 Existence

Recall, from Chapter 5, Definition 5.20, that condition (E) can be stated as follows:

A closed $\omega \in \mathcal{A}^2(M)$ is ENTIRE (or ω respectively its deRham class $[\omega]$ satisfies the condition (E)) if for an open cover $\mathfrak{U} = (U_j)_{j \in I}$ of M the Čech class $[\omega] \in \check{H}^2((U_j)_{j\in I}, \mathbb{C}) \cong \check{H}^2(M, \mathbb{C}) \cong H^2_{dR}(M, \mathbb{C})$ induced by the deRham cohomology class of ω contains a cocycle $z = (z_{ijk})$, i.e. $[\omega] = [z]$, with

$$\text{(E)} \qquad \boxed{z_{ijk} \in \mathbb{Z}}$$

for all $i, j, k \in I$ with $U_{ijk} \neq \varnothing$.

Proposition 8.1. *Let ω be a closed two form $\omega \in \mathcal{A}^2(M, \mathbb{C})$ satisfying the integrality condition* (E). *Then there exists a line bundle with connection ∇ such that* $\mathrm{Curv}(L, \nabla) = \omega$.

Proof. As before, we work with an open cover $\mathfrak{U} = (U_j)_{j \in I}$ of M where all intersections $U_{j_0 j_1 \dots j_p} = U_{j_0} \cap U_{j_1} \cap \dots \cap U_{j_p}$, $j_0, j_1, \dots j_p \in I$, are empty or they are contractible (e.g. diffeomorphic to convex open subsets of $\mathbb{R}^n$), so that one can apply the Lemma of Poincaré repeatedly.

We start the construction with the possible local connection forms $\alpha_j \in \mathcal{A}^1(U_j)$ without having determined the line bundle yet. Since ω is closed, there exist, in fact, $\alpha_j \in \mathcal{A}^1(U_j)$ so that $d\alpha_j = \omega$ by Poincaré's Lemma in U_j for each $j \in I$. On $U_{jk} \neq \varnothing$ the one-forms $\alpha_k - \alpha_j$ are closed:

$$d\alpha_k - d\alpha_j = \omega - \omega = 0.$$

Hence, there exist $f_{jk} \in \mathcal{E}(U_{jk})$ with $df_{jk} = \alpha_k - \alpha_j$ by Poincaré's Lemma. We can choose f_{jk} so that $f_{jk} + f_{kj} = 0$ for all $j, k \in I$. Because of

$$d(f_{jk} + f_{ki} + f_{ij}) = 0$$

we finally obtain constants $c_{ijk} \in \mathbb{C}$ defined by

$$c_{ijk} := f_{ij} + f_{jk} + f_{ki}, \quad \text{on} \;\; U_{ijk} \neq \varnothing.$$

(c_{ijk}) is a cocycle associated with ω, $(c_{ijk}) \in \check{Z}^2(\mathfrak{U}, \mathbb{C})$, where $c_{ijk} \in \mathbb{C}$, in general. (c_{ijk}) determines the Čech cohomology class

$$[(c_{ijk})] \in \check{H}^2(\mathfrak{U}, \mathbb{C}) \cong H^2_{dR}(M, \mathbb{C})$$

of the two form ω.

Now we need the integrality condition (E) in order to go on with the construction. (E) implies that there are entire numbers $z_{ijk} \in \mathbb{Z}$ which forms a cocycle (z_{ijk}) such that (z_{ijk}) is equivalent to the cocycle (c_{ijk}). That is, there exist $x_{ij} \in \mathbb{C}$ forming a cocycle $(x_{jk}) \in \check{Z}^1(\mathfrak{U}, \mathbb{C})$ such that

$$z_{ijk} = c_{ijk} + x_{ij} + x_{jk} + x_{ki} \in \mathbb{Z}, \;\; \text{if} \;\; U_{ijk} \neq \varnothing.$$

In the case of $U_{jk} \neq \varnothing$, we now set

$$g_{jk} := \exp(2\pi i f_{jk} + 2\pi i x_{jk})$$

and obtain $g_{jk} \in \mathcal{E}(U_{jk}, \mathbb{C}^\times)$. We immediately conclude that

$$g_{ij} g_{jk} g_{ki} = \exp\left(2\pi i (c_{ijk} + x_{ij} + x_{jk} + x_{ki})\right) = \exp\left(2\pi i (z_{ijk})\right) = 1$$

on U_{jk} (here, the integrality condition (E) is essential).

As a result, the smooth functions

$$g_{jk} : U_{jk} \to \mathbb{C}^\times \,, \;\; j, k \in I, \;\; U_{jk} \neq \varnothing$$

satisfy (C) and therefore, define a complex line bundle L over M according to Proposition 3.9.

The forms α_j define a connection over each U_j since the condition (Z) is fulfilled

$$\alpha_k - \alpha_j = df_{jk} = \frac{1}{2\pi i} \frac{dg_{jk}}{g_{jk}}.$$

Therefore, the α_j are local gauge potentials (local connection forms) of a connection ∇ on L. Because of $\omega|_{U_j} = d\alpha_j$ the curvature of ∇ is ω: $\mathrm{Curv}(L, \nabla) = \omega$. □

Observation 8.2. Proposition 8.1 is formulated for general non-degenerate closed two forms with (E).

In the real case, i.e. in case of a real form ω, it is clear that the α_j-s can be chosen to be real valued as well, and hence the constructed connection allows a compatible Hermitian structure.

Corollary 8.3. *Let* (M, ω) *be a symplectic manifold where* ω *satisfies* (E). *Then the line bundle with connection, which has been constructed in the proof of Proposition 8.1, has a compatible Hermitian structure* H. *As a result, according to Proposition 6.7, we obtain a prequantum line bundle* (L, ∇, H).

Remark 8.4. As we mentioned before, in Chapter 5, the integrality conditions (G) and (E) are equivalent according to A. Weil's Theorem. We have given a complete proof for the implication (E) $\implies$ (G): In fact, assuming (E) we obtain by Proposition 8.1 a line bundle L over M with connection ∇ such that $\mathrm{Curv}(L, \nabla) = \omega$. Now our previous result in Proposition 5.18 assures that ω satisfies (G), i.e. $\int_S \omega \in \mathbb{Z}$ for all oriented compact closed surfaces $S \subset M$.

The converse of the implication "ω satisfies (E)" $\implies$ "there exists a prequantum bundle with curvature ω" will be discussed at the end of the next section.

8.2 Uniqueness

After the question of existence, we now discuss the uniqueness of line bundles with connection with given curvature form $\omega \in \mathcal{A}^2(M)$. In other words:

What freedom do we have in constructing (L, ∇, H)? How many inequivalent prequantum line bundles exist on (M, ω)? Under which conditions is the prequantum bundle essentially unique?

In order to answer these questions, we look at the construction in the proof of Proposition 8.1 which establishes the existence of a line bundle with connection (L, ∇) with $\mathrm{Curv}(L, \nabla) = \omega$, and check step by step what freedom we have in the choice of the transition functions g_{jk} and the local connection forms α_j. Before we start this program, we make precise what the equivalence of line bundles with connections should be.

We will study pairs (L, ∇) of line bundles L with connection ∇.

Definition 8.5. Two line bundles with connection $(L, \nabla), (L', \nabla')$ will be called Equivalent (or isomorphic) (denoted by $(L, \nabla) \sim (L', \nabla')$), if there exists an isomorphism $F : L \to L'$ of line bundles such that for all local sections $s \in \Gamma(U, L)$:

$$F \circ (\nabla s) = \nabla'(F \circ s) .$$

In form of a commutative diagram:

$$\begin{array}{ccc} \Gamma(U, L) & \xrightarrow{F\circ} & \Gamma(U, L') \\ \nabla \downarrow & & \downarrow \nabla' \\ \Gamma(U, L) & \xrightarrow{F\circ} & \Gamma(U, L') \end{array}$$

Recall (cf. Corollary 3.11), that with respect to local trivializations ψ_j, ψ'_j for L, L' related to an open cover $\mathfrak{U} = (U_j)$ of M, an isomorphism $F : L \to L'$ is given by functions $h_j \in \mathcal{E}^\times(U_j)$ such that they satisfy condition (I), i.e.

$$g'_{jk} = g_{jk}\frac{h_j}{h_k}\,,\,,$$

for all $j, k \in I$ with respect to the respective transition functions g_{jk}, g'_{jk} of the line bundles L and L'.

Proposition 8.6. *Let* $(L, \nabla)\,, (L', \nabla')$ *line bundles with connection and assume an isomorphism* $F : L \to L'$ *is given by* (h_j)*. Then* $(L, \nabla) \sim (L', \nabla')$ *with this isomorphism* F *if and only if*

$$\alpha_j - \alpha'_j = \frac{1}{2\pi i}\frac{dh_j}{h_j}\,. \tag{37}$$

Proof. With respect to the sections $s_j(a) = (\psi_j)^{-1}(a,1)\,$, and $s'_j(a) = (\psi'_j)^{-1}(a,1)\,$, $a \in U_j$, the isomorphism is given by $F \circ s_j = h_j s'_j$. Using $\nabla'(Fs_j) = \nabla' h_j s'_j = (dh_j + 2\pi i\alpha'_j h_j)s'_j$ and $F(\nabla s_j) = F(2\pi i\alpha_j s_j) = 2\pi i\alpha_j h_j s'_j$ the equivalence of the pair implies

$$dh_j + 2\pi i\alpha'_j h_j = 2\pi i\alpha_j h_j\,,$$

and immediately the result (37). Conversely, if (37) holds then the calculation above shows that $F(\nabla s) = \nabla'(F(s))$ for local section, i.e. the pair is equivalent. □

Example 8.7. (Trivial Line Bundle) As an example, let be $L = L_0 = M \times \mathbb{C}$ the trivial line bundle. The trivial connection ∇_0 is $\nabla_0(fs_1) = dfs_1$, where as before, $s_1(a) = (a, 1)$. Hence ∇_0 could also be denoted by d.

1. A general connection ∇ on L_0 is of the form $\nabla(fs_1) = (df + 2\pi i\alpha f)s_1$, where $\alpha \in \mathcal{A}^1(M, \mathbb{C})$. and will be denoted by ∇_α.

2. A general connection ∇ with $\mathrm{Curv}(L_0, \nabla) = 0$ is of the form ∇_α with $d\alpha = 0$.

3. A general isomorphism $F : L_0 \to L_0$ is of the form $F(p) = h(a)p$ for $\pi(p) = a$ where $h(a) := pr_2(F(s_1(a)))\,$, $a \in M$. This can be confirmed using the general description of ismorphisms through (h_j) with the condition (I), i.e. $g_{jk} = g_{jk}\frac{h_j}{h_k}$, which implies that $h_j = h_k$ glue together to global functions h.

4. For an arbitrary function $g \in \mathcal{E}(M)$ the one form $\alpha = dg$ is closed, hence the corresponding connection yields a line bundle (L_0, ∇) with curvature $\mathrm{Curv}(L_0, \nabla_\alpha) = 0$, and all these zero curvature bundles (i.e. flat line bundles in the language of the next section) are equivalent to the trival bundle (L_0, d) with trivial connection. The equivalence is given by $h := \exp(-2\pi i g)$ as follows from

$$0 - \alpha = -dg = \frac{1}{2\pi i}\frac{dh}{h}\,.$$

5. The remaining closed one forms $\alpha \in \mathcal{A}^1(M,\mathbb{C})$ defining a connection ∇_α with curvature $\mathrm{Curv}(L_0,\nabla_\alpha)=0$ are the closed one forms which are not exact. This situation will be studied in detail in the next section.

   We see, that it is possible that line bundles L and L' are equivalent as line bundles, but not as line bundles with connection.

We now come the the uniqueness result:

Theorem 8.8. *Let $\omega \in \mathcal{A}^2(M)$ satisfy the integrality condition* (E). *Then the set of equivalence classes of line bundles with connection (L,∇) such that $Curv(L,\nabla)=\omega$ is in one-to-one correspondence with $\check{H}^1(M,\mathrm{U}(1))$, the first Čech cohomology group with values in the circle group $\mathrm{U}(1)\cong\mathbb{S}^1$.*

Proof. As mentioned before, we follow the construction of the possible pair (L,∇) with $\mathrm{Curv}(L,\nabla)=\omega$ presented in the proof of Proposition 8.1. The proof will be carried through in 3 steps. In the first step the possible changes in the α_j-s of the construction are considered, the second step treats the possible choices of the f_{jk} and the third and last step is devoted to the possible changes in the x_{jk} which makes the cocycle c_{ijk} induced by ω entire. In the first two variations we remain in the class given by the original construction, while the third step yields the isomorphism of the group $\check{H}^1(M,\mathrm{U}(1))$ with the set of classes of pairs (L,∇) with $\mathrm{Curv}(L,\nabla)=\omega$.

1. Step: First of all, in the construction of the proof of Proposition 8.1 we choose, using $d\omega=0$, a one form $\alpha_j \in \mathcal{A}^1(U_j)$[36] with $d\alpha_j=\omega|_{U_j}$. Any other one form $\alpha'_j \in \mathcal{A}^1(U_j)$ with $d\alpha'_j=\omega|_{U_j}$ satisfies $d(\alpha_j-\alpha'_j)=0$, and there are $g_j\in\mathcal{E}(U_j)$ with

$$\alpha_j=\alpha'_j+dg_j\,.$$

The $f_{jk}\in\mathcal{E}(U_{jk})$ with $\alpha_k-\alpha_j=df_{jk}$ in the above construction will be replaced by

$$f'_{jk}=f_{jk}+g_j-g_k\,,$$

and we obtain

$$df'_{jk}=d(f_{jk}+g_j-g_k)=\alpha_k-\alpha_j+dg_j-dg_k=\alpha'_k-\alpha'_j$$

The f'_{jk} lead to the same cocycle c_{ijk} as before:

$$\begin{aligned} c_{ijk} &= f_{ij}+f_{jk}+f_{ki}\\ &=(f_{ij}+g_i-g_j)+(f_{jk}+g_j-g_k)+(f_{ki}+g_k-g_i)\,.\\ &=f'_{ij}+f'_{jk}+f'_{ki}. \end{aligned}$$

[36] We have again an open cover (U_j) with all intersections diffeomorphic to contractible open subsets of $\mathbb{R}^n$

As a result, the corresponding transition functions g'_{jk} (instead of g_{jk} in the above construction) are

$$g'_{jk} := \exp\left(2\pi i\left(f_{jk} + g_j - g_k\right) + 2\pi i x_{jk}\right)$$

where the $x_{jk} \in \mathbb{R}$ make (c_{ijk}) entire: $z_{ijk} := c_{ijk} + x_{ij} + x_{jk} + x_{ki} \in \mathbb{Z}$ as before. In particular, $g'_{jk}g'_{ki}g'_{ij} = 1$ and

$$g'_{jk} = g_{jk}\frac{h_j}{h_k}\,,$$

where $h_j = \exp\left(2\pi i g_j\right)$.

As a consequence, the line bundle L' defined by the cocycle (g'_{jk}) is isomorphic to the line bundle L defined by the cocycle (g_{jk}) according to condition (I) in Corollary 3.11: the cocycles are equivalent. (They are also equivalent in the sense of Čech cohomology and provide a class in $\check{H}^1(\mathfrak{U}, \mathcal{E}^\times)$).

Moreover, if ∇ is the connection on L given by (α_j) and ∇' is the connection on L' given by (α'_j), then (L, ∇) is equivalent to (L', ∇') since

$$\alpha_j - \alpha'_j = dg_j = \frac{1}{2\pi i}\frac{dh_j}{h_j}\,.$$

2. Step: Secondly, fixing α_j with $d\alpha_j = \omega|_{U_j}$, we can replace each f_{jk} by $f'_{jk} := f_{jk} + b_{jk}$ where $b_{jk} \in \mathbb{C}$ are constants satisfying $b_{jk} + b_{kj} = 0$. There are no further possibilities in changing f_{jk} if one wants to follow the above construction. Then, we get a new cocycle $c'_{ijk} = c_{ijk} + b_{ij} + b_{jk} + b_{ki}$ representing ω and a new cocycle x'_{jk} in order to achieve that $z'_{ijk} = c'_{ijk} + x'_{ij} + x'_{jk} + x'_{ki}$ becomes entire. With the choice $x'_{jk} := x_{jk} - b_{jk}$ the cocycle given by $c'_{ijk} + x'_{ij} + x'_{jk} + x'_{ki}$ is indeed entire, since it agrees with z_{ijk}. Since $\exp(2\pi i(f'_{jk} + x'_{jk})) = \exp(2\pi i(f_{jk} + b_{jk} + x_{jk} - b_{jk})) = \exp(2\pi i(f_{jk} + xjk))$ the possible new line bundle with transition funcions $g'_{jk} = \exp(2\pi i(f'_{jk} + x'jk))$ is the same. So again we stay in the same equivalence class of line bundles with connection.

3. Step: Finally, we fix α_j and f_{jk}. We can replace the constants x_{jk} by $x_{jk} + y_{jk}$, where $y_{jk} \in \mathbb{R}$ with

$$y_{ij} + y_{jk} + y_{ki} \in \mathbb{Z}$$

for all $i, j, k \in I$, with $U_{ijk} \neq \varnothing$ and $y_{jk} + y_{kj} = 0$. To maintain the construction there are no other possibilities for changing (x_{jk}).

Let $t_{jk} := \exp(2\pi i y_{jk}) \in \mathrm{U}(1)$. In particular, $t_{jj} = \exp 0 = 1 = t_{jk}t_{kj}$. Then $t_{ij}t_{jk}t_{ki} = \exp(2\pi i(y_{ij} + y_{jk} + y_{ki})) = 1$ because of $y_{ij} + y_{jk} + y_{ki} \in \mathbb{Z}$, which implies that the collection $t = (t_{jk})$ is a cocycle in $\check{Z}^1(\mathfrak{U}, \mathrm{U}(1))$. We replace g_{jk} by

$$g^t_{jk} := \exp\left(2\pi i(f_{jk} + x_{jk} + y_{jk})\right) = g_{jk}t_{jk}\,,$$

in accordance to the construction. (g^t_{jk}) defines a line bundle L^t with connection ∇^t given by the same α_j. In fact, we have

$$\frac{1}{2\pi i}\frac{dg^t_{jk}}{g^t_{jk}} = \frac{1}{2\pi i}\frac{d(g_{jk}t_{jk})}{g_{jk}t_{jk}} = \frac{1}{2\pi i}\frac{dg_{jk}}{g_{jk}} = df_{jk} = \alpha_k - \alpha_j\,,$$

since t_{jk} is constant. Therefore, the α_j determine a connection ∇^t on L^t.

The cocycle $t = (t_{jk})$ induces a cohomology class

$$[\,t\,] = [(t_{jk})] \in \check{H}^1(\mathfrak{U}, \mathrm{U}(1)) = \check{H}^1(M, \mathrm{U}(1))\,.$$

Let $t' = (t'_{jk})$ be another cocycle $t' \in \check{Z}^1(\mathfrak{U}, \mathrm{U}(1))$.

Claim: $(L^t, \nabla^t) \sim (L^{t'}, \nabla^{t'})$ if and only if $t \sim t'$ as Čech cocycles. The latter equivalence means that there are $t_j \in \mathrm{U}(1)$ with

$$t'_{jk} = t_{jk}\frac{t_j}{t_k}\,.$$

In fact, for $t \sim t'$ the t_j induce an isomorphism $L^t \to L^{t'}$ since (I) is satisfied. Conversely, if L^t and $L^{t'}$ are equivalent as line bundles with the same connection determined by the local connection forms α_j, there are $h_j : U_j \to \mathrm{U}(1)$ with $g^{t'}_{jk} = g^t_{jk} h_j h_k^{-1}$, which describe the isomorphism and preserve the connection forms α_j. Consequently p

$$0 = \alpha_j - \alpha_j = \frac{1}{2\pi i}\frac{dh_j}{h_j}\,.$$

Hence, the h_j are constant and yield the equivalence of t and t' as Čech cocycles. □

As a first application:

Proposition 8.9. *Let M be simply connected and let $\omega \in \mathcal{A}^2(M)$ satisfy* (E)*. Then there exists exactly one line bundle L with connection, such that $Curv(L, \nabla) = \omega$ up to equivalence.*

Here, we use

$$\pi_1(M) = 0 \implies \check{H}^1(M, \mathrm{U}(1)) = 0\,.$$

In fact, the abelianization of the fundamental group $\pi_1(M)$ is isomorphic to the homology group $H_1(M)$: $\pi_1(M)/[\pi(M), \pi_1(M)] \cong H^1(M) = H^1(M, \mathbb{Z})$. $\check{H}^1(M, \mathbb{Z}) \cong H^1(M, \mathbb{Z})$ is naturally isomorphic to the dual of $H_1(M, \mathbb{Z})$. This implies $\check{H}^1(M, \mathrm{U}(1)) = 0$ in case of $\pi_1(M) = 0$. See also Observation 8.23 at the end of the next section.

Corollary 8.10. *A simply connected quantizable symplectic manifold (M, ω) has exactly one prequantum line bundle up to equivalence.*

Example 8.11. We continue the Example 7.6 of the sphere $\mathbb{S}^2$ which is simply connected. We have seen in 7.6 that with the form $\omega := \frac{1}{4\pi}\mathrm{vol}$ the symplectic manifold $(\mathbb{S}^2, C\omega)$ is quantizable if and only if $C \in \mathbb{Z}$, $N = C \neq 0$. With the preceding result we conclude that for each $N \in \mathbb{Z}$, $N \neq 0$, there is exactly one prequantum line bundle (L^N, ∇^N) up to isomorphism on $(\mathbb{S}^2, N\omega)$. We determine the line bundles with connection (L^N, ∇^N) at the end of the next section.

Remark 8.12 (Chern Class)**.** Let M be a manifold and let $\mathfrak{U} = (U_j)_{j\in I}$ be an open cover such that all $U_{j_0\cdots j_n}$ are contractible. The transition functions (g_{jk}) of any complex line bundle L over M with respect to the cover $\mathfrak{U}$ always exist and define a cohomology class in

$$\check{H}^1(\mathfrak{U}, \mathcal{E}^\times) = \check{H}^1(M, \mathcal{E}^\times)\,,$$

and an entire cohomology class in $\check{H}^2(\mathfrak{U}, \mathbb{Z})$. In fact, let

$$z_{ijk} = \frac{1}{2\pi i}\left(\log g_{ij} + \log g_{jk} + \log g_{ki}\right) \ \text{ on } \ U_{ijk}.$$

Locally, z_{ijk} is well-defined and integer-valued, since $e^{2\pi i z_{ijk}} = g_{ij}g_{jk}g_{ki} = 1$. Therefore, (z_{ijk}) defines an element in $\check{Z}^2(\mathfrak{U}, \mathbb{Z})$. There is a problem in getting a global definition of z_{ijk}, due to the fact that the logarithm is ambiguous in $C^\times$, but the corresponding Čech cohomology class $[(z_{ijk})]$ in $\check{H}^2(\mathfrak{U}, \mathbb{Z}) = H^2(M, \mathbb{Z})$ is well-defined: Any other choices of branches of the logarithms lead to another cocycle $z'_{ijk} \in \check{Z}^2(\mathfrak{U}, \mathbb{Z})$ such that $z'_{ijk} = z_{ijk} + m_{ij} + m_{jk} + m_{ki}$ with entire $m_{jk} \in \mathbb{Z}$. Hence, $z'_{ijk} \sim z_{ijk}$.

The class $c(L) := [(z_{ijk})] \in \check{H}^2(M, \mathbb{Z})$ is called the CHERN CLASS of the line bundle L. $c(L)$ is an important invariant of the equivalence class of the line bundle. In case of a prequantum bundle $(L, \nabla), H)$ on a symplectic manifold (M, ω) the Chern class of L is given by the symplectic form ω: $c(L) = [\omega]$.

As an immediate consequence, we obtain:

Assertion 8.13. *A symplectic manifold is quantizable if and only if the symplectic form satisfies the integrality condition* (E)*.*

We know this result already by using Weil's Theorem (i.e. (G) $\iff$ (E)), but the assertion can now be deduced from the above results in the following way without referring to Weil's theorem:

1. "(E) $\implies$ prequantum bundle exists" according to Proposition 8.1,
2. The converse "prequantum bundle exists $\implies$ (E)" follows from the last remark.

8.3 Flat Line Bundles

Flat bundles on a manifold are used to present an additional approach to understand the variety of non equivalent line bundles with connection whose curvature form is prescribed.

The basic idea comes from the fact that the set of equivalence classes of line bundles on M is an abelian group where the group multiplication is induced by the multiplication of the transition functions or, equivalently, is induced by the tensor product $L \otimes L'$ as we have seen in Section 3.4 of Chapter 3. For two line bundles L, L' with transition functions g_{jk}, g'_{jk} the tensor product $L \otimes L'$ is a line bundle with transition functions $g_{jk}g'_{jk}$.

For connections ∇, ∇' on the line bundles L, L' the corresponding tensor product connection $\nabla^{L\otimes L'}$ is given by

$$\nabla^{L\otimes L'}(s \otimes s') = \nabla s \otimes s' + s \otimes \nabla' s' ,$$

for local sections s of L and s' of L'. It is denoted by $\nabla + \nabla'$ for reasons which becomes clear regarding the result of the next Lemma. For a common open cover (U_j) of M which allows local trivializations ψ_j resp. ψ'_j of L resp. L', let α_j, α'_j be the local connection forms of the respective connections.

Lemma 8.14. *The local connection forms of $\nabla + \nabla'$ on $L \otimes L'$ are $\alpha_j + \alpha'_j$.*

Proof. Let $s_j(a) = (\psi_j)^{-1}(a,1)$, $s'_j(a) = (\psi'_j)^{-1}(a,1)$ the standard non vanishing sections defined on U_j and let fs_j , $f's'_j$ local sections. Then

$$\begin{aligned}
&(\nabla + \nabla')(fs_j \otimes f's'_j) = (df + 2\pi i\alpha_j f)s_j \otimes f's'_j + fs_j \otimes (df' + 2\pi i\alpha'_j f')s'_j \\
&= \big((df + 2\pi i\alpha_j f)f' + f(df' + 2\pi i\alpha'_j f')\big)\,(s_j \otimes s'_j) \\
&= \big(d(ff') + 2\pi i(\alpha_j + \alpha'_j)ff'\big)\, s''_j ,
\end{aligned}$$

where s''_j is the corresponding standard section of $L \otimes L'$ over U_j. □

For a line bundle with connection (L, ∇) with transition functions g_{jk} let $\nabla^\vee$ be the "dual" connection on the dual line bundle $L^\vee$ associated to the cocycle (g_{jk}^{-1}). Similar to the proof of the preceding lemma one can show for the local connection forms α_j of ∇:

Lemma 8.15. *The local connection forms of $\nabla^\vee$ on $L^\vee$ are $-\alpha_j$.*

As a result, the local connection forms of $\nabla + \nabla^\vee$ on $L \otimes L'$ vanish. In particular, the curvature $\mathrm{Curv}(L \otimes L^\vee, \nabla + \nabla^\vee)$ is zero.

Observation. The set of all line bundles with connection on a given manifold form a group with respect to the composition

$$((L, \nabla), (L')\nabla') \mapsto (L \otimes L', \nabla + \nabla')) .$$

Definition 8.16. A line bundle with connection (L, ∇) is called FLAT if the curvature vanishes. The connection ∇ is called flat as well.

Lemma 8.17. *The curvatures satisfy*

$$\mathrm{Curv}\,(L \otimes L', \nabla + \nabla')) = \mathrm{Curv}\,(L, \nabla) + \mathrm{Curv}\,(L', \nabla')) .$$

As a consequence, the flat connections form a subgroup of the group of all line bundles with connection.

Proposition 8.18. *Any flat line bundle with connection (L, ∇) is equivalent to a suitable line bundle with connection (L', ∇_0), where L' has constant transition functions and ∇_0 is the trivial connection with local connection forms $\alpha_j = 0$.*

Proof. Let α_j be the local connection forms of the flat connection ∇. Because of flatness, $d\alpha_j = 0$ and there exist functions $g_j \in \mathcal{E}(U_j)$ with $dg_j = \alpha_j$. These g_j induce an equivalent line bundle L' with transition functions $g'_{jk} := g_{jk} h_j h_k^{-1}$ where $h_j := \exp(2\pi i g_j)$. We have

$$\begin{aligned}\frac{1}{2\pi i}\frac{dg'_{jk}}{g_{jk}} &= \frac{1}{2\pi i}\frac{dg_{jk}}{g_{jk}} + \frac{1}{2\pi i}\frac{dh_j}{h_j} - \frac{1}{2\pi i}\frac{dh_k}{h_k} \\ &= \alpha_k - \alpha_j + dg_j - dg_k \\ &= \alpha_k - \alpha_j + \alpha_j - \alpha_k = 0\,. \end{aligned} \tag{38}$$

Hence g'_{jk} is constant and $\alpha'_j = 0$ defines a flat connection ∇' on L'. The isomorphism $L \to L'$ is given by $h_j := \exp(2\pi i g_j)$:

$$g'_{jk} = g_{jk}\frac{h_j}{h_k}\,,\ \alpha_j - 0 = dg_j\,.$$

□

As a consequence of the above lemmata, the tensor product $(L \otimes L^\vee, \nabla + \nabla^\vee)$ is equivalent to the trivial flat bundle with trivial connection: $L_0 = M \times \mathbb{C}$ is isomorphic to $L \otimes L^\vee$ and (L_0, d) is the trivial line bundle with trivial connection d with connection form $\alpha = 0$. Evidently, (L_0, d) is flat.

Given a closed two form $\omega \in \mathcal{A}^2(M, \mathbb{C})$ satisfying (E) we want to determine the set E_ω of equivalence classes $[(L, \nabla)]$ of line bundles with connections (L, ∇) with $\mathrm{Curv}(L, \nabla) = \omega$. E_ω is an abelian group induced by the tensor product of line bundles with connection. Given (L, ∇), for an equivalent line bundle with connection (L', ∇') the tensor product $(L \otimes L'^\vee, \nabla + \nabla'^\vee)$ is equivalent to the trivial flat bundle with connection $\nabla + \nabla'^\vee$ determined by the connection forms $\alpha_j - \alpha_j = 0$. The connection is flat.

Since for (L, ∇) with $\mathrm{Curv}(L, \nabla) = \omega$ and a flat line bundle with connection (L', ∇') the tensor product $(L \otimes L', \nabla + \nabla')$ has the same curvature ω we conclude:

Lemma 8.19. *E_ω is in one-to-one correspondence with E_0.*

Furthermore, E_0 is a group with respect to the tensor product as composition. In fact, E_0 is the quotient group of the group of all flat connections (cf. Lemma 8.17).

Therefore, in order to determine E_ω it is enough to determine E_0. This is a major point of investigating flat bundles and their equivalence classes.

Proposition 8.20. *The equivalence classes of flat line bundles are in one-to-one correspondence with the Čech classes of* $\check{H}^1(\mathfrak{U}, \mathrm{U}(1))$*:*

$$E_0 \cong \check{H}^1(\mathfrak{U}, \mathrm{U}(1)) \cong \check{H}^1(M, \mathrm{U}(1))\,.$$

Proof. This result is known already from the preceding section (Theorem 8.8). Here is a proof using flat line bundles:

Each equivalence class of E_0 contains a line bundle L with connection having constant transition functions $g_{jk} \in \mathrm{U}(1)$ (cf. Lemma 8.16). Two such bundles $L^g, L^{g'}$, given by $g = (g_{jk}), g' = (g'_{jk})$, are equivalent as flat line bundles, if and only if there are $h_j \in \mathrm{U}(1)$ with $g'_{jk} = g_{jk} h_j h_k^{-1}$ (according to condition (I)). But this means exactly that (g_{jk}) defines a class $[(g_{jk})]$ in $\check{H}^1(\mathfrak{U}, \mathrm{U}(1))$. Furthermore, it follows that the assignment $E_0 \to \check{H}^1(\mathfrak{U}, \mathrm{U}(1))\,,\ L^g \mapsto [g]$, is well-defined and bijective.

□

Example 8.21 (2-Sphere)**.** We continue the Example 8.11 of the sphere $\mathbb{S}^2$ and want to describe the line bundles (L^N, ∇^N) on $(\mathbb{S}^2, N\omega)$ explicitly. For L^N one can take the line bundle $H(N)$ which we have introduced in Section 3.3 in the context of line bundles over the Riemann sphere $\mathbb{P}^1$ which is diffeomorphic to $\mathbb{S}^2$. The $[H(N)]$ describe the group of equivalence classes of line bundles on $\mathbb{S}^2$ which is isomorphic to $\check{H}^1(\mathbb{S}^2, \mathcal{E}^\times) \cong \mathbb{Z}$. In the section 3.4 the bundles $H(N)$ have been described as N-fold tensor products of the hyperplane bundle $H(1)$ which in turn is the dual of the tautological bundle $H(-1)$ (which is also the tangent bundle of the sphere $\mathbb{S}^2$).

In the Example 6.16 we have equipped $H(-1)$ with a connection $\nabla^{(-1)}$ compatible with the Hermitian structure. And from the above we know the following: If this connection $\nabla^{(-1)}$ is given by the local one forms (α_j) with respect to an open cover then the bundle $H(N)$ obtains the connection ∇^N determined by the one forms $(-N\alpha_j)$. Hence, $(L^N, \nabla^N) \sim (H(N), \nabla^N)$.

With the notation of Example 6.16 the connection ∇^N is

$$\nabla^N s = -N \frac{\bar{z}_j dz^j}{\sum |z^j|^2} s\,.$$

Another elementary example is the cylinder:

Example 8.22 (Cylinder)**.** Let $M = T^*\mathbb{S}^1$ be the symplectic phase space with the circle $\mathbb{S} = \mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ as the configuration space. This is a special case of the cotangent bundle T^*Q with configuration space Q. The cotangent bundle $T^*\mathbb{S}$ is trivial (as a real 1-dimensional line bundle): There exists a nowhere vanishing vector field Z on $\mathbb{S}$ generated by the curve γ

$$\gamma : \mathbb{R} \to \mathbb{S}\,,\ \gamma(t) = \exp 2\pi i t\,,\ t \in \mathbb{R}\,,$$

$Z(a) := \dot{\gamma}(t_0) = [\gamma(t_0 + t)]_a$, when $\gamma(t_0) = a$. The dual one form β on $\mathbb{S}$ is defined by $\beta(Z) = 1$ and also nowhere vanishing. Hence, the cotangent bundle is trivial[37] and looks like a cylinder $\mathbb{S} \times \mathbb{R}$. The form β is locally given as dq where q is a local coordinate of $\mathbb{S}$ given by the angle

$$q = \gamma^{-1}(t) = \frac{1}{2\pi i} \log \exp(2\pi i t) = t \,,$$

t in an open interval of length ≤ 1.

On any cotangent bundle there is the natural (Liouville) one form λ (cf. Construction 1.17) given in local canonical coordinates as $\lambda = pdq$. $-\lambda$ serves for the cotangent bundle as symplectic potential with symplectic form $\omega = d(-\lambda) = dq \wedge dp$. Globally: $\omega = \beta \wedge dp$. In particular, ω is exact whereas β is not exact.

As for general cotangent bundles, a natural first choice for a prequantum line bundle is the trivial bundle $L_0 = L$ with the connection

$$\nabla f s_1 = (df - 2\pi i \lambda f)\, s_1 \,,$$

and the Hermitian structure H induced from $L = M \times \mathbb{C}$.

The inequivalent flat connections we expect according to Proposition 8.20 (because of $E_0 \cong \check{H}^1(M, \mathrm{U}(1)) \cong \mathrm{U}(1))$ are given by $\nabla s_1 = +2\pi i(\kappa) dq s_1$, where $\kappa \in \mathbb{R}$. Although the one form κdq on $\mathbb{S}^1$ looks like an exact one form, it is not so, because q is not globally defined, as we mentioned above. As a consequence, the $\nabla^{\kappa}, \nabla^{\kappa'}$ (resp. the one forms $\kappa dq, \kappa' dq$) are pairwise inequivalent for $\kappa, \kappa' \in\,]0,1]$, and they are equivalent for general $\kappa, \kappa' \in \mathbb{R}$ if and only if $\kappa - \kappa' \in \mathbb{Z}$. This describes the result of Proposition 8.20 explicitly:

$$E_0 \to \check{H}^1(T^*\mathbb{S}^1, \mathrm{U}(1)) = \mathrm{U}(1)\,, \;\; [(L_0, \nabla^{\kappa})] \mapsto \exp(2\pi i \kappa) \in \mathrm{U}(1)\,.$$

Therefore we have the connection

$$\nabla^{\kappa}(f s_1) = (df - 2\pi i \lambda f + 2\pi i \kappa dq f)\, s_1$$

with local connection form $-\lambda + \kappa dq$, $\kappa \in \mathbb{R}$. And we obtain a family of inequivalent prequantum line bundles (L, ∇^{κ}, H).

The corresponding prequantum operators P_{κ} assigned to the observable $F = p$ are

$$P_{\kappa} := q(p) = -\frac{i}{2\pi} \frac{\partial}{\partial q} + \kappa \,.$$

This leads to different quantizations which are not equivalent and describe different physics as one can read off the respective spectra of the P_{κ}. As Hilbert space we

[37] In general, the tangent bundle of a Lie group is trivial, see Proposition C.21, and hence the cotangent bundle is trivial as well.

choose $L^2(\mathbb{S}^1)$, thereby reducing the number of variables (choosing the vertical real polarization in the sense of the notions of the next chapter.)

Then the spectrum of P^κ is $\sigma(P^\kappa) = \{n + \kappa \mid n \in \mathbb{Z}\}$. As a consequence, there exists no unitary operator $U : \mathbb{H} \to \mathbb{H}$ with $U \circ P^\kappa = P^{\kappa'} \circ U$ unless $\kappa - \kappa' \in \mathbb{Z}$,

Observation 8.23 (Holonomy)**.** The group E_0 of equivalence classes of flat line bundles has a physical interpretation as the Moduli Space of Flat Connections modulo Gauge Transformations. The gauge transformations are the isomorphisms of flat line bundles respecting the Hermitian structure, they are given by the multiplication by functions $h \in \mathcal{E}(M, \mathrm{U}(1))$ as before: $L \to L\,,\ p \mapsto h(a)p$ when $p \in L_a\,,\ a \in M$.

This approach leads to the Holonomy Representation of a flat bundle (L, ∇). We can introduce a Hermitian metric which is compatible with ∇ since the local connection forms are real. Therefore we can introduce the corresponding $\mathrm{U}(1)$-bundle $L^1 = S = \{p \in L \mid |p| = 1\} \to M$ (also being the frame bundle) and the induced connection there. The associated parallel transport defines, for closed curves γ starting and ending in a fixed point $a \in M$ the parallel transport map $Q(\gamma) : L^1_a \to L^1_a$ which can be called the Holonomy (see Section 5.4 for the definition of $Q(\gamma)$). $Q(\gamma)$ is given as a multiplication with a complex number of norm 1, which we again denote by $Q(\gamma)$. Since the curvature is zero, the parallel transport is locally independent of the respective curves. As a consequence, for homotopic closed curves $\gamma \sim \gamma'$ starting and ending in $a \in M$ the holonomies agree: $Q(\gamma) = Q(\gamma')$. This defines the map

$$\mathrm{Hol}_\nabla : \pi_1(M) \to \mathrm{U}(1)\,, \quad \mathrm{Hol}_\nabla([\gamma]) := Q(\gamma)\,,$$

which is a group homomorphism. In our Example 8.22 above the local connection form of the flat connection ∇ determined by the real parameter κ is κdq. And $Q(\gamma) = \exp(-2\pi i \int_\gamma A)$ with $A = \kappa dq$. For $\gamma(t) = \exp(2\pi i t)$ we obtain $Q(\gamma) = \exp(-2\pi i \kappa)$ Now, $\pi_1(T^*\mathbb{S}) = \pi_1(\mathbb{S}) = \{m[\gamma] \mid m \in \mathbb{Z}\} \cong \mathbb{Z}$ and the connection ∇ given by κ yields $\mathrm{Hol}_\nabla(m[\gamma]) = \exp(-2\pi i \kappa m)$.

As a consequence, there is a natural mapping

$$\mathrm{Hol} : E_0 \to \mathrm{Hom}\,(\pi_1(M), \mathrm{U}(1))\,,\ \nabla \mapsto \mathrm{Hol}_\nabla\,,$$

which turns out to be an injective group homomorphism. Now $\mathrm{Hom}\,(\pi_1(M), \mathrm{U}(1)) \cong \mathrm{Hom}\,(H_1(M), \mathrm{U}(1))$ since $\mathrm{U}(1)$ is abelian. And $\mathrm{Hom}\,(H_1(M), \mathrm{U}(1)) \cong \check{H}^1(M, \mathrm{U}(1))$.

We are back in the situation where we have worked before: Each class $(t_{jk}) \in \check{H}^1(\mathfrak{U}, \mathrm{U}(1))$ defines a line bundle with the transition functions t_{jk}. And the line bundle with t_{jk} is isomorphic with the line bundle with t'_{jk} if and only if the cocycles t_{jk} and t'_{jk} are equivalent. This shows that $\mathrm{Hol} : E_0 \to \mathrm{Hom}\,(\pi_1(M), \mathrm{U}(1))$ is surjective, hence an isomorphism and all the groups are isomorphic:

$$E_0 \cong \mathrm{Hom}\,(\pi_1(M), \mathrm{U}(1)) \cong \mathrm{Hom}\,(H_1(M), \mathrm{U}(1)) \cong \check{H}^1(M, \mathrm{U}(1))\,.$$

Summary:

9 Polarization

In Quantum Mechanics one can represent the Hilbert space of states as the space of square integrable complex functions on the spectrum of a given complete set of commuting observables. In the classical situation a natural choice of a "complete set of commuting observables" on a given symplectic manifold (M, ω) of dimension $2n$ is a set of $n = \frac{1}{2} \dim M$ functions $F_1, F_2, \ldots, F_n \in \mathcal{E}(M, \mathbb{R})$, which are independent at each point of M, commuting in the sense of

$$\{F_i, F_j\} = 0, \ \forall i, j \in \{1, \ldots, n\},$$

and such that the corresponding Hamiltonian vector fields $X_{F_1}, X_{F_2}, \ldots X_{F_n}$ are all complete. We thus arrive at the notion of a completely integrable system.

In general, however, such classical observables do not exist. As a consequence, one has to relax the condition of globally defined F_i and one considers instead distributions which locally describe the above situation with a complete set of commuting variables. Such distributions have to be integrable, i.e. they are foliations, and they have to be adapted to the symplectic structure. These requirements lead to the notion of a real polarization. However, some symplectic manifolds do not admit real polarizations, for instance the 2-sphere $\mathbb{S}^2$ with its natural volume form. Therefore, one generalizes this concept to complex polarizations, i.e. complex Lagrangian subbundles of the complexification $TM^{\mathbb{C}}$ of the tangent bundles which are involutive.

In this chapter we cannot present much more than the formal definitions of the different types of real and complex polarizations leaving aside the rich geometric theory of foliations and polarizations. The goal is to prepare the way to the correct Hilbert representation space by "cutting down the number of variables" in the direction of polarizations (which is discussed in Chapter 7), and, moreover, to show that the notion of a polarization is well adapted to the symplectic structure. In the special case of a Kähler polarization the interplay of the symplectic structure with the polarization leads to a complex (holomorphic) structure on the manifold.

9.1 Distribution

Definition 9.1. A DISTRIBUTION D on a manifold M is a subbundle D of the tangent bundle $TM : D \subset TM$.

A distribution is therefore a real vector bundle (cf. Appendix Section D.1) D such that D_a is a real vector subspace of the tangent space T_aM at each point. Moreover, at each point $a \in M$ there exist an open neighbourhood $U \subset M$ and k smooth vector fields $X_1, \ldots, X_k \in \mathfrak{V}(U)$ with

$$D_b = \operatorname{span}_{\mathbb{R}}\{X_j(b) \mid 1 \leq j \leq k\}, \quad b \in U. \tag{39}$$

In this description, the natural number k can be chosen to be the rank of the vector bundle D.

If the distribution D is only given by a collection $\mathcal{X}$ of vector fields at each point by $D_b := \operatorname{span}_{\mathbb{R}}\{X(b) \mid X \in \mathcal{X} \text{ defined on } U\}, \quad b \in U$, then in addition one has to require that $\dim_{\mathbb{R}} D_a$ is constant. Or, equivalently, that the generating vector fields (X_j) in (39) are linear independent.

Definition 9.2. A distribution D is called INTEGRABLE if for each $a \in M$ there exists a k-dimensional submanifold N in an open neighbourhood U of a so that for each $b \in N$: $T_b N = D_b$.

Any such submanifold $N \subset U$ is called an INTEGRAL MANIFOLD of the distribution.

Distributions $D \subset TM$ with $\dim_{\mathbb{R}} D_a = 1 = \operatorname{rank} D$ are always integrable, since locally $D_a = \mathbb{R}X_a, a \in U$, for a (local) nowhere vanishing vector field X and the N_a are given by the integral curves of X.

Example 9.3. Let $M = \mathbb{S}^2$ be the 2-sphere with the the volume form as symplectic form ω. There is no 1-dimensional distributions D on $\mathbb{S}^2$. This property can be deduced from the fact that real line bundles on any manifold M are classified by $H^1(M, \mathbb{Z}/2\mathbb{Z})$, and $H^1(\mathbb{S}^2, \mathbb{Z}/2\mathbb{Z}) = 0$ since $\mathbb{S}^2$ is simply connected. Hence, every real line bundle is trivial. As a consequence, a distribution $D \subset T\mathbb{S}^2$ would be a trivial subbundle of $T\mathbb{S}^2$ which would have a nowhere vanishing section. Therefore $T\mathbb{S}^2$ would have a nowhere vanishing section in contradiction to the "Hairy Ball Theorem".

Without proof, we state the fundamental theorem of Frobenius.

Theorem 9.4. *A necessary and sufficient condition for a distribution to be integrable is that the global sections of D form a Lie subalgebra of $\mathfrak{V}(M)$:*

$$X, Y \in \Gamma(M, D) \implies [X, Y] \in \Gamma(M, D).$$

An integrable distribution is also called a FOLIATION. The maximal connected integral manifolds, i.e. $N \subset M$ with $T_b N = D_b$ for all $b \in N$, are called the LEAVES of the foliation.

Let M/D be the space of leaves.

Definition 9.5. A foliation is called REDUCIBLE or ADMISSIBLE if M/D exists as a quotient manifold (see Section A.4) and the canonical map $\pi : M \to M/D$ is a submersion.

Remark 9.6. Recall, that the last condition is equivalent to π having maximal rank at all points of M. Using a result concerning quotient manifolds (cf. Proposition A.29) we know for concrete instances the following criterium in the case that the quotient topology on M/D is Hausdorff: If M/D has a differentiable structure with respect to the quotient topology and if there exists a surjective submersion $p : M \to M/D$ with respect to this differentiable structure, then M/D with this differentiable structure is the quotient manifold and p is the quotient map (up to isomorphism).

Examples 9.7.

1. The vertical distribution: Let $Q \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ with the standard coordinates $q^1, q^2, \dots, q^n$. The cotangent bundle $T^*Q \cong Q \times \mathbb{R}^n$ with standard (and canonical with respect to $\omega = dq^k \wedge dp_k = -d\lambda$) coordinates $q^1, \dots q^n, p_1, \dots, p_n$ has the VERTICAL DISTRIBUTION D spanned by
$$\left\{ \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n} \right\}, \quad D \subset T(T^*Q).$$
D is integrable and the leaves are given by
$$\{(q,p) \in Q \times \mathbb{R}^n : q = c\} = \{c\} \times \mathbb{R}^n = \tau^{-1}(c)\,.$$
Here, $c \in Q$ is constant and $\tau : T^*Q \to Q$ is the canonical projection. Moreover, $T^*Q/D \cong Q$. Thus, D is integrable and reducible.
In this example, the distribution D is generated by the hamiltonian vector fields of the globally defined functions $q^1, \dots, q^n \in \mathcal{E}(T^*Q, \mathbb{R})$ in involution ($\{q^i, q^k\} = 0$): $X_{q^k} := \frac{\partial}{\partial p^k}$ and
$$D = \operatorname{span}_{\mathcal{E}(T^*Q,\mathbb{R})}\{X_{q^k} : k = 1, 2, \dots, n\}$$
This relation shows that this special distribution and the symplectic structure fit together; in the terms of the following section, D is a polarization. Also, we have the global definiton of D as described in the beginning of this chapter.

2. The cotangent bundle $M = T^*Q$ with respect to a general manifold Q of dimension n has again the vertical distribution D given locally by $\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}$ in suitable canonical bundle charts (q,p) and leaves $\tau^{-1}(c) \cong \mathbb{R}^n$, $c \in Q$. D is integrable and reducible. However, in general, we do not have global $F_1, \dots, F_n$ which Poisson-commute and whose Hamiltonian vector fields $X_{F_1}, \dots, X_{F_n}$ generate D.

3. The horizontal distribution: Let $Q \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and let $M = T^*Q \cong Q \times \mathbb{R}^n$ as before in the first example. The horizontal distribution D is given by $\left\{ \frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^n} \right\}$. The leaves are
$$\{(q,p) \in Q \times \mathbb{R}^n \mid p = c\} = Q \times \{c\} = pr_2^{-1}(c)\,,\ c \in \mathbb{R}^n$$
and $M/D \cong \mathbb{R}^n$ [38]. So, D is integrable and reducible.

4. The radial distribution: $M = \mathbb{R}^2 \setminus \{0\}$ with coordinates (q,p). Let D be the distribution generated by
$$q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}.$$
D is a distribution whose leaves are the circles $\{p^2 + q^2 = r^2\}$, $r \in \mathbb{R}, r > 0$. The quotient exists as a manifold and satisfies $M/D \cong \mathbb{R}_+$. So, D is integrable and reducible.

[38] As a reminder, we assume our manifolds to be connected

Note, that the vertical distribution D on $M = \mathbb{R}^2 \setminus \{0\} \subset \mathbb{R}^2$ induced by the projection $\mathrm{pr}_1 : M \to \mathbb{R}$ is not reducible. The leaves are the vertical lines through $(0,p), p \neq 0$ and the two rays

$$\{(q,0) : q > 0\} \text{ and } \{(q,0) : q < 1\}.$$

The quotient M/D is not Hausdorff in the quotient topology.

9.2 Real polarization

Definition 9.8. Let (M,ω) be a symplectic manifold. A REAL POLARIZATION on M is a foliation (i.e. an integrable distribution) $D \subset TM$ on M, which is MAXIMALLY ISOTROPIC,i.e. for all $a \in M$:

$$\omega_a(X,Y) = 0 \text{ for } X, Y \in D_a,$$

and no strictly larger subspace of T_aM which contains D_a has this property. A real polarization is called REDUCIBLE or ADMISSIBLE if it is reducible as a distribution.

This definition is modeled after the notion of a completely integrable system:

Example 9.9 (Completely Integrable System)**.** Let $F_1, F_2, \ldots, F_n \in \mathcal{E}(M,\mathbb{R})$ be n independent functions on a symplectic manifold (M,ω) of dimension $2n$ which commute, i.e. $\{F_i, F_j\} = 0$, and whose corresponding Hamiltonian vector fields $X_{F_1}, \ldots X_{F_n}$ are all complete. In particular, $D := \mathrm{span}_{\mathcal{R}(M,\mathbb{R})}\{X_{F_1}, \ldots X_{F_n}\} \subset TM$ is a real polarization.

Conversely, any real polarization looks locally like this example:

Proposition 9.10. *Let (M,ω) be a symplectic manifold of dimension $2n$. Then a given smooth distribution $D \subset TM$ is a real polarization if and only if for each $a \in M$ there exists an open neighbourhood U of a and n independent smooth functions $F_1, \ldots, F_n \in \mathcal{E}(U,\mathbb{R})$ such that*

1. *For each $a \in U$, $D_a = \mathrm{span}_{\mathbb{R}}\{X_{F_1}(a), \ldots, X_{F_n}(a)\}$;*
2. *$\{F_j, F_k\} = 0$, i.e. $\omega|_U(X_{F_j}, X_{F_k}) = 0$ for all $j, k \in \{1, 2, \ldots, n\}$.*

Proof. Let D be a real polarization. Since D is in particular an integrable distribution, there exist locally n independent smooth functions $F_1, \ldots, F_n \in \mathcal{E}(U,\mathbb{R})$ such that the integral manifolds are of D are locally of the form

$$\{a \in U : F_1(a) = c_1, \ldots, F_n(a) = c_n\}$$

with suitable constants $c_1, \ldots, c_n \in \mathbb{R}$. For each vector field $X \in \Gamma(U,D)$ we have

$$L_X F_i = X(F_i) = 0\,,$$

for $i = 1, 2, \ldots, n$. Hence

$$\omega(X_{F_i}, X) = dF_i(X) = L_X(F_i) = 0.$$

It follows that $X_{F_i} \in \Gamma(U, D)$, since D is maximally isotropic. As a consequence, the hamiltonian vector fields $X_{F_1}, \ldots, X_{F_n}$ span D locally. Note, that the $F_1, \ldots F_n$ are independent, i.e. $dF_1(a), \ldots, dF_n(a)$ are linearly independent for each $a \in U$. By isotropy, we have $\omega(X_{F_i}, X_{F_j}) = 0$, hence

$$\{F_i, F_j\} = 0\,,$$

thus we have shown that the 2 properties are satisfied.

Conversely, by conditions 1. a distribution D is well-defined. The local conditions "$\{F_i, F_j\} = 0 \iff \omega(X_i, X_j) = 0$ on U" imply that D is isotropic. Since the F_i are independent, $\dim_{\mathbb{R}} D_a = n$. Finally, again by the independence, the distribution D is integrable whose integral manifolds on U are the following

$$\{a \in U | F_1(a) = c_1, \ldots, F_n(a) = c_n\}\,.$$

□

Concrete examples are the first three out of the four examples in 9.7: They are reducible real polarizations. Moreover:

Example 9.11. On a two-dimensional symplectic manifold (M, ω) any distribution of rank 1 is automatically integrable and Lagrangian, hence a real polarization. In particular, the fourth example in 9.7 of a radial distribution on $M = \mathbb{R}^2 \setminus \{0\}$ is a reducible real polarization. It is the distribution spanned by the hamiltonian vector field X_H defined by the energy $H(q,p) = \frac{1}{2}(p^2 + q^2)$ of the harmonic oscillator. Note, that M is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}_+$ by the diffeomorphism

$$(q,p) \mapsto \left(\frac{(q,p)}{r}, r\right)\,,\; r := \sqrt{q^2 + p^2}\,.$$

Closely related is the cylinder $T^*\mathbb{S}^1 \cong \mathbb{S}^1 \times \mathbb{R}$ with the symplectic form $\omega = dq \wedge dp$ as in Example 8.22 and with the horizontal distribution D given by the non vanishing vector field $\frac{\partial}{\partial q}$. The leaves of D are the circles $\{(\exp 2\pi i t, p) \mid t \in \mathbb{R}\}\,,\, p \in \mathbb{R}$. Thus, $T^*\mathbb{S}^1$ can be seen as an extension of $M : \mathbb{S}^1 \times \mathbb{R}_+ \subset T^*\mathbb{S}^1$.

Horizontal distributions on T^*Q for general Q are quite special from the point of view of polarizations. In fact, the (connected components of the) leaves have to be of the form of open subsets of $(\mathbb{S}^1)^k \times \mathbb{R}^{n-k}$, due to general results for completely integrable systems, which does not give much freedom.

In general, real polarizations need not exist on a given symplectic manifold, in particular, when there does not exist any distribution of rank $\frac{1}{2}\dim_{\mathbb{R}} M$, as we have seen for the 2-sphere $M = \mathbb{S}^2$ (see Example 9.3).

For the purpose of geometric quantization we thus need a generalization of the notion of a real polarization: the complex polarization!

9.3 The Complex Linear Case

Before introducing the definition of a complex polarization on a symplectic manifold, we first study the linear case, i.e. we consider a $2n$-dimensional symplectic vector space (V,ω) as the prototype of the tangent space T_aM , $a \in M$, of a symplectic manifold (M,ω). Here, $\omega : V \times V \to \mathbb{R}$ is bilinear over $\mathbb{R}$, antisymmetric and non-degenerate. We have studied this structure in the beginning (see Section 1.1).

Let $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} \cong V \oplus iV$ be the complexification of V with the obvious $\mathbb{C}$-bilinear extension of $\omega : V \times V \to \mathbb{R}$ to $V^{\mathbb{C}}$

$$\omega = \omega^{\mathbb{C}} : V^{\mathbb{C}} \times V^{\mathbb{C}} \to \mathbb{C},$$

$$\omega(v + iw, v' + iw') := \omega(v, v') - \omega(w, w') + i(\omega(v, w') + \omega(w, v')),$$

and with conjugation

$$\overline{v + iw} := v - iw \, , \, v, w \in V \, .$$

Definition 9.12. A complex linear subspace $P \subset V^{\mathbb{C}}$ is called a Complex Lagrangian Subspace, if P is

- isotropic (i.e. for all $z, w \in P : \omega(z, w) = 0$) and
- maximally isotropic, i.e. whenever $Q \subset V^{\mathbb{C}}$ is a complex isotropic subspace containing P, $P \subset Q$, it follows that $P = Q$.

Our symplectic form ω defines a sesquilinear and skew-symmetric form on $V^{\mathbb{C}}$ by:

$$\langle z, z' \rangle_P := -\frac{1}{2} i\omega(\bar{z}, z') \, , \tag{40}$$

for $z, z' \in V^{\mathbb{C}}$. In particular, for $v, w \in V$ we have

$$\langle v + iw, v + iw \rangle_P := \omega(v, w) \, .$$

Let P be a complex Lagrangian subspace. Then, in general, the form $\langle \, , \, \rangle_P$ fails to be non-degenerate on P. The null space of $\langle \, , \, \rangle_P |_{P \times P}$ will be denoted by N:

$$N = \{z \in P \mid \langle z, w \rangle_P = 0 \ \ \forall \ \ w \in P\} = P^{\perp}.$$

It is easy to see that $N = P \cap \overline{P}$ where $\overline{P} := \{\bar{z} \mid z \in P\}$. Therefore, the form is non-degenerate if and only if $N = P \cap \overline{P} = \{0\}$ which is the case if $P + \overline{P} = V^{\mathbb{C}}$.

In case of $N \neq \{0\}$ the form $\langle \, , \, \rangle_P$ projects to a non-degenerate sesquilinear form on P/N. This form is, in general, not positive definite. Let (r, s) be the signature of this form. Then its matrix with respect to a suitable basis of P/N is

$$\operatorname{diag}\left(\underbrace{1, 1, \dots, 1}_{r}, \underbrace{-1, \dots, -1}_{s}\right), 0 \leq r + s = n - \dim_{\mathbb{C}} N = \dim_{\mathbb{C}} P/N \, . \tag{41}$$

The complex Lagrangian subspace P is said to be

- of TYPE (r, s) if the above signature is (r, s).
- POSITIVE in the case of $s = 0$[39].
- REAL in case of $r = s = 0$. Then $P = D^{\mathbb{C}}$ for a real Lagrangian subspace $D \subset V$. Then $\langle\ ,\ \rangle_P = 0$ and $N = P$.
- KÄHLER[40] in case of $r + s = n$, i.e. $N = P \cap \overline{P} = \{0\}$. In that case, the form $\langle\ ,\ \rangle_P$ is non-degenrate and $N = P$. In some places the term Kähler is reserved for complex Lagrangian subspaces which are positive and satisfy $N = \{0\}$. Then the complex Lagrangian subspaces P with merely $N = \{0\}$ are called pseudo Kähler or of Kähler type.

Remark 9.13. The choice of sign in 40 is arbitrary, in many places in the literature $\frac{1}{2}i\omega(\bar{z}, z)$ is used as the corresponding form induced by P. As a consequence, with the alternative choice of sign "positive" and "negative" interchanges for the complex Lagrangian subspaces (and later Lagrangian subbundles). For the results this is not much of a difference. In particular, what can be proved for positive Lagrangians mostly can be proved for negative Lagrangian, as well.

The subspaces

$$D := P \cap \overline{P} \cap V \quad \text{and} \quad E := (P + \overline{P}) \cap V$$

are of special interest in the following. Note, that $D^{\mathbb{C}} = P \cap \overline{P} = N$ and $E^{\mathbb{C}} = P + \overline{P}$.

The number $k := n - (r+s) = \dim_{\mathbb{C}} P \cap \overline{P} = \dim_{\mathbb{R}} D$ is sometimes called the number of "real directions" in P.

In the following we illustrate the definitions and the different spaces $D, E, P, \overline{P}, N$ by an example which uses a special basis of V. A basis $(u_1, u_2, \ldots, u_n; v_1, v_2 \ldots, v_n) = (u; v)$ is called a symplectic frame of the symplectic vector space (V, ω) if the matrix of the 2-form ω with respect to this basis is the block matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where 1 is the unit $n \times n$-matrix.

Example 9.14. Let (V, ω) be a symplectic vector space with a symplectic frame $(u; v)$. We define a complex Lagrangian subspace P of $V^{\mathbb{C}}$ of type (r, s) with $\dim_{\mathbb{C}} N = k = n - (r + s)$ by

$$\begin{aligned}
&D = \operatorname{span}_{\mathbb{R}} \{u_1, \ldots, u_k\}, D^{\mathbb{C}} = P \cap \overline{P} = N\,, \\
&P = \operatorname{span}_{\mathbb{C}} (\{u_1, \ldots, u_k\} \cup \{u_j + iv_j \mid k < j + r \le n\} \cup \{u_j - iv_j \mid k + r < j \le n\})\,, \\
&\overline{P} = \operatorname{span}_{\mathbb{C}} (\{u_1, \ldots, u_k\} \cup \{u_j - iv_j \mid k < j + r \le n\} \cup \{u_j + iv_j \mid k + r < j \le n\})\,, \\
&E = \operatorname{span}_{\mathbb{R}} (\{u_1, \ldots, u_n\} \cup \{v_{k+1}, \ldots v_n \mid k < j \le n\}), E^{\mathbb{C}} = P + \overline{P}\,, \\
&C := \operatorname{span}_{\mathbb{R}} \{v_1, \ldots, v_k\}\,, \text{ with } E + C = V^{\mathbb{C}}\,.
\end{aligned}$$

[39] see the Remark 9.13 below
[40] also called PURELY COMPLEX

Note, that $\langle u_j + iv_j, u_j + iv_j\rangle_P = \omega(u_j, v_j)$ and $\langle u_j - iv_j, u_j - iv_j\rangle_P = -\omega(u_j, v_j)$. As a consequence, the form induced by $\langle\ ,\ \rangle_P$ on P/N has the following diagonal matrix as its matrix with respect to the basis of P/N induced by $(u_j + iv_j \mid k+1 \le j \le n)$

$$\operatorname{diag}\left(\underbrace{1,\dots,1}_{r},\underbrace{-1,\dots,-1}_{s}\right),\quad r+s=n-k.$$

Analogously, on $\overline{P}/N$ the corresponding matrix is

$$\operatorname{diag}\left(\underbrace{-1,-1,\dots,-1}_{r},\underbrace{1,1,\dots,1}_{s}\right).$$

The Kähler case we study in a later section in some detail.

9.4 Complex Polarization

We now come to the definition of a general complex polarization on a symplectic manifold.

Definition 9.15. Let (M,ω) be a symplectic manifold of dimension $2n$. A COMPLEX POLARIZATION P of (M,ω) is a complex vector subbundle $P \subset TM^{\mathbb{C}}$ of complex dimension n such that

1. For all $X, Y \in \Gamma(M,P)$ we have $[X,Y] \in \Gamma(M,P)$ (P is INVOLUTIVE),
2. $P_a \subset T_aM^{\mathbb{C}}$ is maximally isotropic for all $a \in M$ (P is LAGRANGIAN), i.e. a complex Lagrangian subspace ,
3. $D_a := P_a \cap \overline{P_a} \cap T_aM$ has constant rank $k \in \{0,1,\dots,n\}$: $\dim_{\mathbb{R}} D_a = k$, $a \in M$.

By 3. $D = P \cap \overline{P} \cap TM$ is a vector bundle and hence a distribution. When P is a complex polarization then $\overline{P}$ is a complex polarization as well. It follows that $D \subset TM$ is an integrable distribution.

Since we have the constant rank condition in 3. the assertion concerning the non-degenerate form on P_a/N_a (see (41)) leads to a type (r,s) for all $a \in M$. (Here we need M to be connected which has been assumed in general.)

The complex polarization P is said to be

- of TYPE (r,s) if P_a/N_a is of type (r,s) for all $a \in M$.
- POSITIVE if $s = 0$ (but see Remark 9.13).

- REAL if $P_a = D_a^{\mathbb{C}}$ for at least one $a \in M$ and then for all $a \in M$. This is equivalent to $P = \overline{P}$.

- KÄHLER if $D_a = \{0\}$ for all $a \in M$ (sometimes Kähler polarization means $D = 0$ and positive and the condition $D = 0$ is called pseudo Kähler).

- STRONGLY INTEGRABLE if the distribution $E \subset TM$ ($E_a := (P_a + \overline{P}_a) \cap T_aM$) is integrable.

- REDUCIBLE if the orbit space M/D exists as a differentiable manifold and the projection $M \to M/D$ is a submersion.

Note, that the vertical distribution on $T^*\mathbb{R}^n$ as well as the horizontal distribution induce the complex polarizations $P := D^{\mathbb{C}}$, which are real polarizations in this terminology.

A special and well-known example of a complex polarization is the following.

Example 9.16 (Simple Phase Space)**.** Let $M = (T^*\mathbb{R}^n, \omega)$ be the symplectic manifold with the standard symplectic form $\omega = dq^j \wedge dp_j$, where q^j, p_j, $1 \le j \le n$ are the standard canonical coordinates of M. The vector fields

$$\frac{\partial}{\partial p_j}, \frac{\partial}{\partial q^j}, 1 \le j \le n,$$

form a basis for each tangent space $T_aM \cong \mathbb{R}^{2n}$. They form a basis of the complexified $T_aM^{\mathbb{C}}$ (see below) as well, now over the complex numbers $\mathbb{C}$.

Let $z_j := p_j + iq^j$ complex coordinates, hence we understand $M \cong \mathbb{R}^n \times \mathbb{R}^n$ as a complex vector space and as a complex manifold $M = \mathbb{C}^n$. As a reminder (cf. Proposition B.23), the following vector fields form a basis of $T_aM^{\mathbb{C}}$ as well

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial p_j} - i\frac{\partial}{\partial q^j}\right),$$

$$\frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial p_j} + i\frac{\partial}{\partial q^j}\right).$$

The polarization we want to introduce in this example is defined as

$$P := \operatorname{span}_{\mathcal{E}(M)}\left\{\frac{\partial}{\partial \bar{z}_j} \;\middle|\; 1 \le j \le n\right\}.$$

(In Chapter B on Complex Analysis P is denoted by $T^{(0,1)}$, cf. B.24.) Using the identities

$$X_{z_j} = -2i\frac{\partial}{\partial \bar{z}_j},\ X_{\bar{z}_j} = 2i\frac{\partial}{\partial z_j},\ 1 \le j \le n,$$

we see that P can be defined as the span of the corresponding hamiltonian vector fields

$$P := \operatorname{span}_{\mathcal{E}(M)} \left\{ X_{z_j} \mid 1 \leq j \leq n \right\} . \tag{42}$$

It is easy to show that $P \subset TM^{\mathbb{C}}$ is involutive: For $V, W \in \Gamma(M, P)$ there are $w^j, v^j \in \mathcal{E}(M)$ such that

$$V = v^j \frac{\partial}{\partial \bar{z}_j}, W = w^j \frac{\partial}{\partial \bar{z}_j} .$$

Now for $f \in \mathcal{E}(M)$

$$\begin{aligned}
[V, W] f &= v^j \frac{\partial}{\partial \bar{z}_j} \left(w^k \frac{\partial}{\partial \bar{z}_k} f \right) - w^k \frac{\partial}{\partial \bar{z}_k} \left(v^j \frac{\partial}{\partial \bar{z}_j} f \right) \\
&= v^j \frac{\partial w^k}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{z}_k} f - w^k \frac{\partial v^j}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{z}_j} f \quad \left(\text{since } \frac{\partial}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{z}_k} f = \frac{\partial}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{z}_j} f \right) \\
&= \left(v^j \frac{\partial w^k}{\partial \bar{z}_j} - w^j \frac{\partial v^k}{\partial \bar{z}_j^j} \right) \frac{\partial}{\partial \bar{z}_k} f .
\end{aligned}$$

And

$$\left(v^j \frac{\partial w^k}{\partial \bar{z}_j} - w^j \frac{\partial v^k}{\partial \bar{z}_j^j} \right) \frac{\partial}{\partial \bar{z}_k} \in P .$$

Moreover, $dz_j \wedge d\bar{z}_j = d(p_j + iq^j) \wedge d(p_j - iq^j) = idq^j \wedge dp_j - idp_j \wedge dq^j = 2idq^j \wedge dp_j$. Therefore, our standard symplectic form $\omega = dq^j \wedge dp_j$ can be written as:

$$\omega = \frac{1}{2i} \sum_{j=1}^n dz_j \wedge d\bar{z}_j = -\frac{1}{2i} \sum_{j=1}^n d\bar{z}_j \wedge dz_j .$$

Using this form of ω we can immediately conclude that

$$\omega \left(\frac{\partial}{\partial \bar{z}_k}, \frac{\partial}{\partial \bar{z}_l} \right) = 0 ,$$

i.e. P is isotropic (which can be deduced from (42) as well).

P is maximally isotropic (Lagrangian) because of $\dim_{\mathbb{C}} P = n$.

Since, furthermore

$$\overline{P}_a := \operatorname{span}_{\mathbb{C}} \left\{ \left. \frac{\partial}{\partial z_j} \right|_a : 1 \leq j \leq n \right\} ,$$

it follows that

$$P_a \cap \overline{P}_a = \{0\}, \quad P_a \oplus \overline{P}_a = T_a M^{\mathbb{C}}.$$

As a result, P is a complex polarization. Because of the last identity $P \cap \overline{P}_a = D = \{0\}$, P is a Kähler polarization. Moreover P is positive.

Note, that $\overline{P}$ defines a Kähler polarization, too. But this Kähler polarization is negative (see Remark9.13). P is called the HOLOMORPHIC POLARIZATION and $\overline{P}$ the antiholomorphic polarization[41].

Examples 9.17. We present three further elementary examples:

1. The two-sphere $\mathbb{S}^2$ has no real polarization as we know by Example 9.3. But it has, similar to the cotangent bundle $T^*\mathbb{R}^n$ in the preceding example, the holomorphic polarization P. It is defined locally by $\frac{d}{d\bar{z}}$ for local complex coordinates z. P is Kähler and positive.

2. The same holds true for the cylinder $M = T^*\mathbb{S}^1$.

3. Let $M = T^*\mathbb{R}^2$ with the usual symplectic form, the standard canonical coordinates p_1, p_2, q^1, q^2 and the complex coordinates $z_j = p_j + iq^j$ as before. Define

$$P := \operatorname{span}_{\mathcal{E}(M)} \left\{ \frac{\partial}{\partial p_1} + i\frac{\partial}{\partial q^1}, \frac{\partial}{\partial p_2} \right\} .$$

Then P is Lagrangian with

$$D = P \cap \overline{P} \cap TM = \operatorname{span}_{\mathcal{E}(M,\mathbb{R})} \left\{ \frac{\partial}{\partial p_2} \right\} ,$$

with $\dim_{\mathbb{R}} D_a = 1$, $\dim_{\mathbb{R}} E_a = 3$. Moreover, P and $P + \overline{P}$ are involutive. Hence, P is a complex polarization. P is reducible and positive.

The example can be generalized to $n > 1$ by looking at Example 9.14. For instance, a positive polarization $P \subset TM^{\mathbb{C}}$, $M = T^*\mathbb{R}^n$, is given by

$$\begin{aligned}
P &= \operatorname{span}_{\mathcal{E}(M,\mathbb{C})} \left(\left\{ \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_k} \right\} \cup \left\{ \frac{\partial}{\partial p_j} + i\frac{\partial}{\partial q^j} \mid k < j \leq n \right\} \right) , \\
\overline{P} &= \operatorname{span}_{\mathcal{E}(M,\mathbb{C})} \left(\left\{ \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_k} \right\} \cup \left\{ \frac{\partial}{\partial p_j} - i\frac{\partial}{\partial q^j} \mid k < j \leq n \right\} \right) , \\
D &= \operatorname{span}_{\mathcal{E}(M,\mathbb{R})} \left\{ \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_k} \right\} , \\
E &= \operatorname{span}_{\mathcal{E}(M,\mathbb{R})} \left(\left\{ \frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n} \right\} \cup \left\{ \frac{\partial}{\partial q^{k+1}} \dots, \frac{\partial}{\partial q^n} \right\} \right) .
\end{aligned}$$

Remark 9.18. The product of complex polarizations P of (M, ω) and P' of (M', ω') is defined as

$$P \oplus P' \subset TM^{\mathbb{C}} \oplus TM'^{\mathbb{C}} \cong T(M \times M')^{\mathbb{C}} .$$

where the symplectic form on $M \times M'$ is $\omega \oplus \omega'$: $P \oplus P'$ is a complex polarization of the symplectic manifold $(M \times M', \omega \oplus \omega')$. For instance, the generalized example P of $T^*\mathbb{R}^n$ just described is essentially the product of the vertical polarization P_k of $M = T^*\mathbb{R}^k$ and the holomorphic polarization P_{n-k} of $T^*\mathbb{R}^{n-k}$.

[41]In some texts, $\overline{P}$ is called the holomorphic, and P the antiholomorphic polarization.

The contribution of complex polarizations to the reduction of the prequantum Hilbert space to obtain a correct Hilbert space as the representation space will be treated in the next chapters.

9.5 Kähler Polarization

Kähler polarizations P are those complex polarizations on a symplectic manifold (M, ω) which satisfy $P \cap \overline{P} = \{0\}$. They obtained this name since every Kähler manifold, i.e. a complex manifold M with a its symplectic (Kähler) form ω, has the holomorphic polarization as a natural polarization which is Kähler (and positive). This will be explained in this section in some detail, together with the converse, namely the fact, that the existence of a Kähler polarization on a symplectic manifold (M, ω) induces a complex structure on M such that M is Kähler with Kähler form ω.

We first study the structure of a complex vector space on a real vector space V of dimension $2n$.

Definition 9.19. Any $\mathbb{R}$-linear map $J : V \to V$ with $J^2 = -1$ is called an Almost Complex Structure [42].

Note, that this definition appears also as Definition B.21 in the Appendix about Complex Analysis.

It is easy to see that if J is an almost complex structure, then by

$$(\alpha + i\beta)(v) := \alpha v + \beta J(v), \quad v \in V, \alpha, \beta \in \mathbb{R}$$

a scalar multiplication $\mathbb{C} \times V \to V$ is defined, introducing on V the structure of a complex vector space. On the other hand, if V is the underlying real vector space of a complex vector space, then the multiplication with i (or $-i$): $I : V \to V$, $v \mapsto iv$ is an almost complex structure: I is $\mathbb{R}$-linear and $I(I(v)) = I(iv) = i^2 v = -v$, $v \in V$.

Let ω be a symplectic form on V (i.e. $\mathbb{R}$-bilinear, non-degenerate and alternating). ω is called to be Compatible with J if

$$\omega(v, w) = \omega(Jv, Jw)\,,$$

for all $v, w \in V$, i.e. if J is a (linear) canonical transformation.

Remark 9.20. If J_ω is the matrix representing ω with respect to a basis of V and if the matrix representing J will be denoted be the same symbol J, then the compatibilty condition is equivalent to: $J \circ J_\omega = J_\omega \circ J$. In fact: $\omega(v, w) = v^\top J_\omega w$. Thus, because of $\omega(Jv, Jw) = (Jv)^\top J_\omega Jw = v^\top J^\top J_\omega Jw$ and $J^{-1} = J^\top$ the compatibility is equivalent to $J^{-1} J_\omega J = J_\omega$ or $J \circ J_\omega = J_\omega \circ J$.

[42] For example, the symplectic involution σ introduced in Section 1.1 is an almost complex strucutere.

In particular, we can choose the basis of V such that J_ω is the block matrix

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where 1 is the unit $n \times n$-matrix. (Such a basis is called a symplectic frame.)

We conclude that the complex structures J such that ω is compatible with J can be parametrized by certain elements of the (real) symplectic group $\mathrm{Sp}(n)$ ($\mathrm{Sp}(n) = \{S \in \mathrm{GL}(2n, \mathbb{R}) \mid S^\top \circ \sigma \circ S = \sigma\}$, see (85)), namely those $S \in \mathrm{Sp}(n)$ with $S^2 = -1$.

For example, if we choose a real $n \times n$-matrix Y which is invertible and symmetric, the block matrix

$$J = \begin{pmatrix} 0 & Y^{-1} \\ -Y & 0 \end{pmatrix}$$

satisfies $J^2 = -1$ and $J^\perp \sigma J = \sigma$ and thus defines an almost complex structure compatible with the symplectic structure. In particular we see that there exist many compatible almost complex structures.

Another set of compatible almost complex structures is given by the block matrices

$$J = \begin{pmatrix} -X & -1 \\ 1 + X^2 & X \end{pmatrix},$$

where X is an arbitrary real symmetric $n \times n$-matrix. Moreover, any compatible J has as its matrix the block matrix

$$J = \begin{pmatrix} -Y^{-1}X & -Y^{-1} \\ Y + XY^{-1}X & XY^{-1} \end{pmatrix},$$

X, Y as above.

A compatible almost complex structure on V induces a symmetric bilinear form $g : V \times V \to \mathbb{R}$ and a sesquilinear form $h : V \times V \to \mathbb{C}$ on V defined by

$$g(v, w) := \omega(v, Jw)$$
$$h(v, w) := g(v, w) + i\omega(v, w)$$

g is non-degenerate, but in general not positive definite. One could have defined $g'(v, w) = \omega(Jv, w)$ to obtain $g = -g'$ (see Remark 9.13).

Let us now consider these structures on the complexification $V^{\mathbb{C}}$ of V as in Section 9.3. The $\mathbb{R}$-linear map J can be linearly extended to $V^{\mathbb{C}}$ as a $\mathbb{C}$-linear map $J^{\mathbb{C}}$ by $J^{\mathbb{C}}(v + iw) := Jv + iJw$ with $(J^{\mathbb{C}})^2 = -1$. The $4n$-dimensional space $V^{\mathbb{C}}$ carries now two almost complex structures, namely the extension $J^{\mathbb{C}}$ of J and the almost complex structure coming from the complexification: $v + iw \mapsto i(v + iw) = -w + iv$.

The extension $J^{\mathbb{C}}$ can be diagonalized with respect to the two eigenvalues $i, -i$: In fact, for $u \in V$:

$$J^{\mathbb{C}}(u - iJu) = J(u) + iu = i(u - iJu)$$

and

$$J^{\mathbb{C}}(u + iJu) = J(u) - iu = -i(u + iJu)$$

with the eigenspaces

$$\begin{aligned} V^{(1,0)} &:= \{z - iJ^{\mathbb{C}}z \mid z \in V^{\mathbb{C}}\} = \{z \in V^{\mathbb{C}} \mid J^{\mathbb{C}}z = +iz\} = \operatorname{Ker}(J^{\mathbb{C}} - i) \\ V^{(0,1)} &:= \{z + iJ^{\mathbb{C}}z \mid z \in V^{\mathbb{C}}\} = \{z \in V^{\mathbb{C}} \mid J^{\mathbb{C}}z = -iz\} = \operatorname{Ker}(J^{\mathbb{C}} + i). \end{aligned} \tag{43}$$

Note, that $V^{\mathbb{C}} = V^{(1,0)} \oplus V^{(0,1)}$: For each $v \in V^{\mathbb{C}}$

$$v = \frac{1}{2}(v - iJ^{\mathbb{C}}v) + \frac{1}{2}(v + iJ^{\mathbb{C}}v)\,.$$

By definition, the two almost complex structures agree on $V^{(1,0)}$ ($J^{\mathbb{C}}v = iv\,,\ v \in V^{(1,0)}$) and differ on $V^{(0,1)}$ by a minus sign ($J^{\mathbb{C}}v = -iv\,,\ v \in V^{(0,1)}$).

The notation $V^{(1,0)}$, $V^{(0,1)}$ is due to the notation of the direct sum representation of the complexified tangent bundle $TM^{\mathbb{C}} = T^{(1,0)}M \oplus T^{(0,1)}M$ in order to define differential forms of degree (r, s), among others (see Definition B.26).

Proposition 9.21. *Let (V, ω) be a symplectic vector space.*

1. *Assume, in addition, that J is an almost complex structure on V which is compatible with ω. Then the eigenspace $P := V^{(0,1)}$ of $J^{\mathbb{C}}$ with eigenvalue $-i$ is a complex Lagrangian subspace with $P \cap \overline{P} = \{0\}$, hence Kähler. Analogously for $\overline{P}$.*

2. *Conversely, a complex Lagrangian subspace P of $V^{\mathbb{C}}$, which is Kähler, defines a compatible almost complex structure J on V such that P is the eigenspace of $J^{\mathbb{C}}$ with respect to i or $-i$.*

Proof. To show 1., we see that P is isotropic since for $v, w \in P$

$$\omega(v, w) = \omega(J^{\mathbb{C}}v, J^{\mathbb{C}}w) = \omega(-iv, -iw) = (-i)^2\omega(v, w) = -\omega(v, w)\,,$$

i.e. $\omega(v, w) = 0$. P is maximally isotropic since $\dim_{\mathbb{C}} P = n$. Finally, for $v \in P \cap \overline{P} = \{0\}$ we have $iz = J^{\mathbb{C}}z = -iz$, hence $z = 0$.

To show the converse, i.e. 2., we use the decomposition $V^{\mathbb{C}} = P \oplus \overline{P}$ and the corresponding projections $\pi\,,\ \bar{\pi} : V^{\mathbb{C}} \to V^{\mathbb{C}}$ onto the subspaces $P = \operatorname{Im}\pi\,,\ \overline{P} = \operatorname{Im}\bar{\pi}$. Any $v \in V^{\mathbb{C}}$ has the representation $v = \pi v \oplus \bar{\pi} v$. When $v \in V$ this decomposition implies

$$v = \bar{v} = \overline{\pi v \oplus \bar{\pi} v} = \overline{\pi v} \oplus \overline{\bar{\pi} v}\,,$$

hence $\pi v = \overline{\bar{\pi} v}$, $\bar{\pi} v = \overline{\pi v}$. As a consequence:

$$\overline{i\pi(v) - i\bar{\pi}(v)} = i\pi(v) - i\bar{\pi}(v)\,,$$

i.e. $i\pi(v) - i\bar{\pi}(v) \in V$ Therefore, the map $J : V \to V$

$$Jv := i\bar{\pi}(v) - i\pi(v)\,,\ v \in V,$$

(which is roughly the "imaginary part" of πv) is well-defined and $\mathbb{R}$-linear. Moreover, J satisfies $J^2 = -1 = -\mathrm{id}_V$:

$$i\bar{\pi}(i\bar{\pi}v - i\pi v) - i\pi(i\bar{\pi}v - i\pi v) = -(\bar{\pi})^2 v - (\pi)^2 v = -\bar{\pi}v - \pi v = -v\,.$$

P is the eigenspace $\mathrm{Ker}\,(J^{\mathbb{C}} + i)$ since for $z \in V^{\mathbb{C}}$

$$J^{\mathbb{C}} z = -iz \Longleftrightarrow i\bar{\pi}(z) - i\pi(z) = -i(\pi(z) + \bar{\pi}(z)) \Longleftrightarrow \bar{\pi}(z) = 0 \Longleftrightarrow z \in P\,.$$

Finally, J is compatible with ω by the isotropy of P and $\overline{P}$. □

As a result, for a given symplectic vector space (V,ω) there is a natural bijective correspondence between the set of complex Lagrangians on $V^{\mathbb{C}}$ (with respect to ω), which are Kähler, and the set of compatible almost complex structures on (V,ω). And there is a natural bijective correspondence between the set of positive Kähler polarization and the set of positive almost complex structures J (i.e. where g, $g(v,w) = \omega(v, Jw)$, is positive definite). Moreover, by Remark 9.20 the set of compatible almost complex structures can be identified with a subset of $\{J \mid J \circ \sigma = \sigma \circ J\} \cong \mathrm{Sp}(2n)$, the symplectic group (see (85)).

Before we discuss the relation between Kähler polarizations and complex manifold structure we present a generalization of Example 9.16 which shows the effect of changing the almost complex structure on a given symplectic vector space. In particular, this example indicates in which way non-positive Kähler polarizations occur.

Example 9.22. We start with the simple phase space $M = T^*\mathbb{R}^n$ with its standard symplectic form $\omega) = dq^k \wedge dp_k$ with respect to the usual canonical coordinates (q,p) of M. The corresponding basis $\left\{\frac{\partial}{\partial q^j}, \frac{\partial}{\partial p_k}\right\}$ is a symplectic frame, i.e. the matrix representing ω is the block matrix

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let J be the almost complex structure on $M = \mathbb{R}^{2n}$ given by a real symmetric and invertible $n \times n$-matrix Y through the block matrix

$$J = \begin{pmatrix} 0 & Y^{-1} \\ -Y & 0 \end{pmatrix}.$$

We introduce complex coordinates $z^k := g^{jk}p_j + iq^k$, where $Y = (g_{jk})\,, Y^{-1} = (g^{jk})$. In this way multiplication by i reflects the almost complex structure J. Note, that the complex coordinates in Example 9.16 are given by $Y = 1$, i.e. $z^k = p_k + iq^k$. In particular, the new coordinates determine the holomorphic structure in the sense that the holomorphic function are the functions f such that for all $k = 1, ...n$

$$\frac{\partial}{\partial \bar{z}^k} f = 0\,.$$

Here,

$$\frac{\partial}{\partial \bar{z}^k} := \frac{1}{2}\left(g_{jk}\frac{\partial}{\partial p_j} + i\frac{\partial}{\partial q^k}\right)\,.$$

The symplecic form can be written as

$$\omega = dq^j \wedge dp_j = \frac{1}{2} i g_{jk} d\bar{z}^j \wedge dz^k\,.$$

The symplectic potential is

$$\alpha = \frac{1}{2} i g_{jk} \bar{z}^j dz^k\,.$$

The holomorpic polarization $P = P_J$ determined by the compatible almost complex structure J is generated by the hamiltonian vector fields $\frac{\partial}{\partial \bar{z}^k}$. The potential α is adapted in the sense that $\alpha(X) = 0$ for $X \in \Gamma(M, P)$.

Finally, let (r, s) be the signature of the matrix $Y = (g_{jk})$ then this signature is the signature of the polarization. In particular, P_J is positive if and only if $s = 0$ which is the same as Y being positive definite.

Kähler Manifolds

After this digression about the linear case we come back to manifolds. We have seen, that on a real vector space V an almost complex structure is essentially the same as a complex structure, in the sense that J determines on V the structure of a complex vector space. An analogous property for manifolds is no longer true. A complex structure on a manifold is given by an atlas of holomorphically compatible charts. These charts induce an almost complex structures in the tangent spaces, as is explained in Section B.21, but the converse need not hold.

Definition 9.23. An Almost Complex Structure on a manifold M is a section (a tensor field) $J \in \Gamma(M, \mathrm{End}(TM))$ with $J^2 = -\mathrm{id}_{TM}$. In particular, for each $a \in M$ the map $J_a : T_aM \to T_aM$ is an almost complex structure in the linear sense (cf. Definition 9.19).

Any complex manifold M has a natural almost complex structure, namely the multiplication by i in the tangent spaces T_aM , $a \in M$, induced by the holomorphic charts (see Example 3. in B.22 for the description in local coordinates). However, there exist examples of differentiable manifolds with an almost complex structure which cannot be induced by a complex manifold structure.

Definition 9.24. When an almost complex structure J on a manifold comes from a complex structure, i.e. from a holomorphic atlas as required above, J it is called INTEGRABLE.

The following result of Newlander and Nirenberg can be found, e.g., in [Huy05].

Theorem 9.25 (Newlander-Nirenberg). *An almost complex structure on a differentiable manifold M is integrable if the induced $T^{(0,1)}M = TM^{(0,1)}$ is involutive, i.e. $\left[T^{(0,1)}M, T^{(0,1)}M\right] \subset T^{(0,1)}M$.*

Note, that an almost complex structure J induces a direct sum decomposition $T^{(1,0)}M \oplus T^{(0,1)}M = TM^{\mathbb{C}}$ of the complixified tangent bundle $TM^{\mathbb{C}}$ into the eigenspaces $T^{(1,0)}M = \mathrm{Ker}\,(J^{\mathbb{C}} - i)$ and $T^{(0,1)}M = \mathrm{Ker}\,(J^{\mathbb{C}} + i)$ of the complexification $J^{\mathbb{C}}$ of J.

Proposition 9.26. *A symplectic manifold (M, ω) with a Kähler polarization P induces a compatible almost complex structure, which is integrable. In particular, M is a complex manifold in a natural way and $P = T^{(0,1)}M$ is the holomorphic polarization.*

Proof. We have to transfer the above results for symplectic vector spaces to the manifold case. For each $a \in M$, there is a natural and compatible almost complex structure $J_a : T_aM \to T_aM$ whose complexification satisfies $T_a^{(0,1)}M = \mathrm{Ker}\,(J_a^{\mathbb{C}} + i) = P_a \subset TM^{\mathbb{C}}$ according to Proposition 9.21. The induced section $J \in \Gamma(M, \mathrm{End}\,(TM))$ is an almots complex structure. Since $P = T^{(0,1)}$ is involutive as a complex polarization, by the theorem of Newlander-Nirenberg J is integrable. □

Definition 9.27. A symplectic manifold (M, ω) with a positive Kähler polarization P is called a Kähler manifold.

This is a definiton in the spirit of symplectic manifolds and polarizations.

By the last result a Kähler manifold is, in particular, a complex manifold such that the complex structure is compatible with ω and such that P is the holomorphic polarization. Moreover, since the Kähler polarization is positive, the symmetric form $g(X, Y) = \omega(X, JY)$, $X, Y \in \mathfrak{V}(M)$, is positive and therefore a Riemannian metric g on M.

The usual definition of a Kähler manifold is the following.

Definition 9.28. A Kähler manifold is a complex manifold M with a Riemannian metric g such that the almost complex structure J is compatible with g:

$$g(X, Y) = g(JX, JY) \ \ \forall X, Y \in T_aM, \ \ a \in M.$$

Moreover, the induced form $\omega(X, Y) := g(JX, Y)$ is closed.

The two definitions are equivalent. Eventually, a Kähler manifold (M, J, g) carries the structure of a symplectic manifold $(M.\omega)$ and a positive polarization P such that all the structures J, g, ω, P are compatible with each other.

10 Representation Space

With all the ingredients:

- a symplectic manifold (M,ω);
- a prequantum line bundle (L,∇,H)
- a complex polarization $P \subset TM^{\mathbb{C}}$

developed so far, one now can construct – as an essential part of the programme of Geometric Quantization – the Representation Space, i.e. the Hilbert space of the quantum model, on which the quantum observables, which correspond to classical observables of a given subset $\mathfrak{o} \subset \mathcal{E}(M)$, act as self-adjoint operators[43] in an irreducible way. Of course, this will be done on the basis of the prequantum operator

$$q(F) : \mathbb{H} \to \mathbb{H}\,,\ F \in \mathcal{E}(M)\,,$$

constructed for (L,∇,H) in Chapter 7, where $\mathbb{H} = \mathbb{H}(M,L)$ is the prequantum Hilbert space of the square integrable global sections of L.

To construct the reduced representation space with respect to a complex Polarization P one considers polarized sections s of L (i.e. $\nabla_X s = 0$ for all vector fields $X \in \Gamma(M,P)$) as the starting point. Intuitively, polarized sections are the sections of L which are constant along the leaves of the polarization P, more precisely, along the leaves of the induced distribution D.

The construction of the representation space can be quite complicated. Among other problems[44], one needs to know how to integrate polarized sections. In general, the natural volume induced by ω^n on M does not work any more. In the case of a Kähler polarization, however, – which we investigate in Section 2 – it is clear how to integrate and a general programme of Geometric Quantization can be carried through directly. Moreover, for the vertical polarization on the simple phase space $M = T^*U$, U open in $\mathbb{R}^n$, the integration is no problem, as well as for the horizontal polarization.

In the general case, we assume for this chapter, that there is a natural volume form $d\,\mathrm{vol}$ on the quotient space M/D[45] where D is the integrable real distribution with $D^{\mathbb{C}} = P \cap \overline{P}$. Now, the representation space $\mathbb{H}_P$ can be defined as the completion of the prehilbert space

$$\{s \in \Gamma(M,L) \mid s \ \text{ polarized } \ \text{ and } \ \int_{M/D} H(s,s)d\,\mathrm{vol} < \infty\}\,.$$

[43]or at least as symmetric operators

[44]In some cases there are no non-zero polarized sections at all, cf. the next chapter.

[45]In general one should use a density on the quotient M/D, see Chapter 12.

Finally, for a first full version of Geometric Quantization one needs the concept of a directly quantizable observable, that is an real function $F \in \mathcal{E}(M)$ such that for all polarized sections s the derivative $\nabla_{X_F} s$ is also polarized. For such an observable F the prequantum operator $q(F)$ has a natural restriction to $\mathbb{H}_P$ yielding a quantum operator, which is denoted again by $q(F) \in \mathcal{S}(\mathbb{H}_P)$, in such a way that the Dirac conditions (see Chapter 2) are satisfied.

10.1 Polarized Sections

The purpose of introducing polarizations is to reduce the set of wave functions (i.e.the elements ψ) in the prequantum Hilbert space $\mathbb{H}$ of prequantization (cf. Chapter 7) in order to make the representation irreducible. This is done by restricting to those functions, sections, vector fields, which are parallel to the given polarization, or, in other words, which are polarized.

Definition 10.1. For a complex polarization P on a symplectic manifold (M, ω) the Polarized Functions are the functions $f \in \mathcal{E}(M, \mathbb{C})$ with

$$L_X f = 0 \quad \text{for all} \quad X \in \Gamma(M, P)\,.$$

The Polarized sections in a line bundle L over M with connection ∇ are the sections $s \in \Gamma(M, L)$ with

$$\nabla_X s = 0 \quad \text{for all} \quad X \in \Gamma(M, P).$$

The basic idea is to consider only those sections of a prequantum bundle (L, ∇, H) on (M, ω) as possible "wave functions" that are "constant along the directions of P" with respect to a polarization $P \subset TM^{\mathbb{C}}$. This idea is made precise by using the definition of a polarized section and by constructing the corresponding Hilbert space of "wave functions" based on the space

$$\Gamma_{\nabla,P}(M, L) := \{s \in \Gamma(M, L) \mid \nabla_X s = 0 \;\text{ for all }\; X \in \Gamma(M, P)\}$$

of polarized sections. $\Gamma_{\nabla,P}(M, L)$ is a vector space over $\mathbb{C}$. However, $\Gamma_{\nabla,P}(M, L)$ is not a module over $\mathcal{E}(M)$. For a section $s \in \Gamma_{\nabla,P}(M, L)$ and a function $f \in \mathcal{E}(M)$ the covariant derivative ∇fs need not be polarized, in general. It is polarized, whenever $\nabla_X fs = (L_X f)s + f\nabla_X s = 0$ for all $X \in \Gamma(M, P)$. This holds true if f is a polarized function. Therefore, $\Gamma_{\nabla,P}(M, L)$ is a module over the ring $\mathcal{E}_P(M)$ of polarized functions.

Note, that $\nabla_X s$ is in general not polarized for polarized s and $X \in \mathfrak{V}(M)$. This is a serious problem when it comes to determine the quantum operator $q(F)$ appropriately, which we will address in the third section of this chapter.

In order to construct the Hilbert space using the space of polarized sections, we have to describe what kind of scalar product on $\Gamma_{\nabla,P}(M, L)$ is reasonable. One can not simply integrate along the volume form ω^n on M which is the right choice for the prequantum Hilbert space. For instance, with respect to the vertical polarization $P = D^{\mathbb{C}}$ on $M = T^*\mathbb{R}^n$, a non-zero polarized section s in the trivial bundle is constant along the fibres of the canonical projection $\tau : T^*\mathbb{R}^n \to \mathbb{R}^n$, the leaves of the vertical distribution D. These fibres are $\{(p,q) \mid p \in \mathbb{R}^n\}$, $q \in \mathbb{R}^n$ fixed. As a consequence, $\int_M \langle s, s\rangle d\,\mathrm{vol} = \infty$ for $s \neq 0$. In this case, one can integrate along $\mathbb{R}^n$ which is essentially M/D with respect to the natural measure on $M/D \cong \mathbb{R}^n$, the Lebesgue measure.

In the following elementary example we show that the original purpose to reduce the number of variables by using the vertical polarization on $T^*\mathbb{R}^n$ and using the natural measure is successful insofar that it leads to the expected results and gives the right representation spaces. The same holds for the horizontal polarization.

Examples 10.2 (Simple Phase Space)**.** Let $M = T^*U$, $U \subset \mathbb{R}^n$ open, with the standard symplectic form $dq^j \wedge dp_j$ and let $L = M \times \mathbb{C}$ be the trivial complex line bundle with the global section $s_1 \in \Gamma(M,L)$, $s_1(a) = (a,1)$, for $a \in M$, and with the induced Hermitian structure H on L.

1. Consider the vertical polarization $P := D^{\mathbb{C}}$ associated to the vertical distribution D (see Example 9.7) which is

$$P := \mathrm{span}_{\mathcal{E}(M)} \left\{ \frac{\partial}{\partial p_j} \,\middle|\, j = 1, \dots, n \right\} ,$$

and consider the connection ∇ on L given by the negative of the Liouville form: $-\lambda = -p_k dq^k$. (L, ∇, H) is a prequantum line bundle. Recall

$$\nabla_X f s_1 = \big(L_X f - 2\pi i p_k dq^k(X) f\big)\, s_1 \,,\ X \in \mathfrak{V}(M).$$

The polarized sections are

$$\Gamma_{\nabla,P}(M,L) := \{ f s_1 \in \Gamma(M,L) \mid f \in \mathcal{E}(M) \ \text{ with } \ \frac{\partial}{\partial p_j} f = 0\,,\ j = 1, \dots, n\} ,$$

since $-p_k dq^k(X) = 0$ for $X \in \Gamma(M,P)$.

This space of polarized sections can be identified with the space $\mathcal{E}(U)$ of functions $f = f(q,p)$ on M which only depend on the variable q. We integrate over U with respect to the Lebesgue measure and get the Hilbert space $\mathbb{H}_P = L^2(U, d\lambda(q))$ of square integrable functions on U as the reduced representation space.

The quantum operators $q(F)$ for $F = p_j, q^k$, have the form

$$q(q^j) = -\frac{i}{2\pi}\left(-\frac{\partial}{\partial p_j} - 0\right) + q^j = q^j =: Q^j\,,\ \text{ on } \mathcal{E}(U)\,,$$

$$q(p_j) = -\frac{i}{2\pi}\left(\frac{\partial}{\partial q^j} - 2\pi i p_j\right) + p_j = -\frac{i}{2\pi}\frac{\partial}{\partial q^j} =: P_j\,,\ \text{ on } \mathcal{E}(U)\,,$$

as we expect it from elementary quantum mechanics in the Schrödinger representation.

2. Now let $P' := D^{\mathbb{C}}$ be the horizontal polarization associated to the horizontal distribution D (see Example 9.7). Then

$$P' := \operatorname{span}_{\mathcal{E}(M)} \left\{ \frac{\partial}{\partial q^j} \,\middle|\, j = 1, \dots, n \right\} .$$

Consider the connection ∇' given by the 1-form: $\alpha = q_k dp^k$. Recall

$$\nabla'_X f s_1 = \big(L_X f + 2\pi i q_k dp^k(X) f \big)\, s_1 .$$

The polarized sections are

$$\Gamma_{\nabla', P'}(M, L) := \{ f s_1 \in \Gamma(M, L) \mid f \in \mathcal{E}(M) \text{ with } \frac{\partial}{\partial q_j} f = 0\,,\, j = 1, \dots, n \}$$

since $\alpha(X) = 0$ for $X \in \Gamma(M, Q)$.

This space of polarized sections can be identified with the space $\mathcal{E}(\mathbb{R}^n)$ of functions $f = f(q, p)$ on M which only depend on the variable p. The corresponding representation space is $\mathbb{H}_{P'} = L^2(\mathbb{R}^n)$.

The quantum operators $q(F)$ for $F = p_j, q^k$ have the form

$$q(q^j) = -\frac{i}{2\pi} \left(-\frac{\partial}{\partial p_j} + 2\pi i q_j \right) + q^j = \frac{i}{2\pi} \frac{\partial}{\partial p_j} =: Q^j\,,$$

$$q(p_j) = -\frac{i}{2\pi} \left(\frac{\partial}{\partial q^j} + 0 \right) + p_j = p_j := P_j\,,$$

as we expect it from elementary quantum mechanics in the Heisenberg representation.

Remark 10.3. The prequantum operators satisfy the canonical commutation relations

$$[Q^j, P_k] = \frac{i}{2\pi} \delta^j_k$$

in both examples. See Section F.3 in the Appendix for the relevance of the canonical commutation relations.

Remark 10.4. The two connections used in the preceding example are not the same, but they induce representations which are unitarily equivalent. This can be seen directly by using the Fourier transform $T : \mathbb{H}_P \to \mathbb{H}_Q$ as intertwining operator satisfying

$$T \circ Q^j = Q^j \circ T\,,\, T \circ P_j = P_j \circ T\,.$$

Unitary equivalence follows also by applying the Theorem of Stone-von Neumann F.47.

What happens when we use the connection ∇ defined by $-\lambda$ also for the horizontal polarization P'? We obtain another representation, which is unitarily equivalent in a natural way to the second representation: $s = fs_1$ is polarized if and only if

$$\frac{\partial}{\partial q_j} f - 2\pi i p_j f = 0\,,\ j = 1\ldots, n.$$

A general solution of this system of partial differential equations is $f = g(p)\exp(2\pi i qp)$, where $g = g(p) \in \mathcal{E}(\mathbb{R}^n)$ and $qp := q^j p_j$. Hence, the space $\Gamma_{\nabla,P'}(M,L)$ of polarized sections can be identified with the space $\{g(p)\exp(2\pi i qp) \mid g = g(p) \in \mathcal{E}(\mathbb{R}^n)\}$ and the corresponding representation space is

$$\mathbb{H} = \{g\exp(2\pi i qp) \mid g = g(p) \in L^2(\mathbb{R}^n)\} \cong \mathbb{H}_{P'}\,.$$

The quantum operators $q(F)$ for $F = p_j, q^k$ are

$$q(q^j)\,(g\exp(2\pi i qp)) = \left(\frac{i}{2\pi}\frac{\partial}{\partial p_j} g\right)\exp(2\pi i qp)\,,$$

$$q(p_j) = p_j\,.$$

$\mathbb{H}$ together with q is unitary equivalent to $\mathbb{H}_P$ and $\mathbb{H}_{P'}$ together with its quantum operators q.

The preceding considerations show, that in the case of a simple phase space T^*U, $U \subset \mathbb{R}^n$ open, the reduction by polarization leads to the right representation space known from elementary quantum mechanic. Before we confirm this result also in the case of a Kähler polarization, let us consider the general case.

For a reducible complex polarization P we have, as before, the induced integrable distribution D with $P \cap \overline{P} = D^{\mathbb{C}}$ and the quotient M/D, the space of leaves $M/P = M/D$. We assume that M/D is endowed with some natural volume form vol.[46].

Definition 10.5 (Representation Space)**.** The representation space $\mathbb{H} = \mathbb{H}_P = \mathbb{H}_{\nabla,P}$ is the completion of

$$\mathbb{H}_{pre} := \left\{ s \in \Gamma_{\nabla,P}(M,L) \,\middle|\, \langle s,s\rangle := \int_{M/D} H(s,s)d\,\mathrm{vol} < \infty \right\}.$$

Here, for a polarized section $s \in \Gamma(M,L)$ the term $H(s,s)$ in the integral is the following function on M/D. $H(s,s)(x) = H(s(a),s(a))$ for $a \in x$. This is well-defined since the polarized s is constant on the leaves x of D, i.e. $s(a) = s(a')$ for $a, a' \in x$.

[46]According to Chapter 12 we can relax this condition and integrate instead along a density on M/D which is always possible, and which leads to representation spaces unitarily equivalent to each other in a natural way.

Remark 10.6. The representation space may be trivial in the sense that $\mathbb{H}_P = 0$ because for no polarized section s the integral $\int_{M/D} H(s,s)d\,\mathrm{vol}$ is finite, see for insance in the coming Example 10.13.

Before we describe Geometric Quantization using the new representation space $\mathbb{H}_P$ we study the construction of the reduced representation space in the relatively easy case of a Kähler polarization, where $M/D = M$ and the symplectic form ω induces a natural volume form.

10.2 Kähler Quantization

Let P be a Kähler polarization $P \subset TM^{\mathbb{C}}$ on our symplectic manifold (M,ω), that is $P \cap \overline{P} = \{0\}$. In this situation, there exists a unique complex structure on the manifold M (i.e. a structure of a complex manifold, or in other words an almost complex structure which is integrable) such that P is the holomorphic polarization (cf. Proposition 9.26), $P = T^{(0,1)}M$. Thus, for all local holomorphic charts

$$\varphi = z = (z_1, \dots z_n) : U \to V \subset \mathbb{C}^n, \quad U, V \text{ open},$$

of the complex structure we have

$$P_a = \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_1}, \dots, \frac{\partial}{\partial \bar{z}_n} \right\} \subset T_a M^{\mathbb{C}} = T_a M \oplus i T_a M, \quad a \in U.$$

In addition, let (L, ∇, H) be a prequantum bundle on the symplectic manifold (M,ω).

Observation 10.7. The polarization P induces on the complex line bundle L a natural structure of a holomorphic line bundle.[47] Hence, in the following, we regard L is a holomorphic line bundle. The polarized sections $s : M \to L$ are nothing else than the holomorphic sections.

Proof. For each $a \in M$ there exists an open neighbourhood $U \subset M$ and a nowhere zero polarized section $s \in \Gamma(U, L)$. Let (U_j) be an open cover of M together with nowhere zero polarized sections $s_j \in \Gamma(U_j, L)$. On $U_{jk} = U_j \cap U_k \neq \emptyset$ there exists $g_{jk} \in \mathcal{E}(U_{jk})$ with $s_j = g_{kj} s_k$ since s_j, s_k are nowhere zero. The functions g_{jk} are holomorphic because of the fact that s_j, s_k are polarized: For $X \in \Gamma(M, P)$:

$$0 = \nabla_X s_j = (L_X g_{kj}) s_k + g_{kj} \nabla_X s_k = (L_X g_{kj}) s_k\,,$$

hence $L_X g_{kj} = 0$.

Now, by

$$\psi_j : L|_{U_j} \to U_j \times \mathbb{C}\,, \; z s_j(a) \mapsto (a, z)\,, \; (a,z) \in U_j \times \mathbb{C}\,,$$

[47] The holomorphic line bundle structure is compatible with ∇.

a system of local trivializations of L is defined whose transition functions are holomorphic. In fact,

$$\psi_k \circ \psi_j^{-1}(a,z) = \psi_k(zs_j(a)) = \psi_k(zg_{kj}s_k(a)) = (a, zg_{kj}(a)) = (a, g_{kj}.z)\,,\ (a,z) \in \mathrm{U}_{jk}\,.$$

and it follows that $\psi_k \circ \psi_j^{-1} : U_{jk} \to U_{jk}$ is biholomorphic. As a consequence, the local trivializations (ψ_j) determine the structure of holomorphic line bundle on L. It is clear, that this structure is inedependent of the choice of the cover (U_j) and the choice of the sections (s_j). □

The symplectic form induces a natural volume form $\mathrm{vol} = C\omega^n$ on M ($C > 0$ some constant) and the prehilbert space we are looking for is:

$$\mathbb{H}_{pre} := \{s \in \Gamma_{hol}(M,L) \mid \int_M H(s,s)d\,\mathrm{vol} < \infty\}.$$

$\mathbb{H}_{pre}$ can be completed in order to yield a separable quantum Hilbert space $\mathbb{H}_P$, the reduced representation space. $\mathbb{H}_P$ is a closed subspace of the full prequantum Hilbert space $\mathbb{H}(M,L)$ (see Section 7.3) of square integrable smooth sections of L considered previously. One can even prove, that $\mathbb{H}_{pre}$ is already closed in $\mathbb{H}$ ([Woo80] and Proposition B.14) and there is no need of completing $\mathbb{H}_{pre}$.

How can the prequantum operators act on the reduced representation space $\mathbb{H}_P$? In general, it might happen that for a polarized section s the covariant derivative $\nabla_{X_F}s$ is no longer polarized. In this situation the operator $q(F)$ might not be well-defined on a suitable dense subspace of $\mathbb{H}_P$. The natural approach to overcome this difficulty is to focus on a subset $\mathfrak{o}$ of the Poisson algebra $\mathcal{E}(M)$ of classical observables such that for all $F \in \mathfrak{o}$ there is a dense subspace $D_F \subset \mathbb{H}_P$ such that $q(F)(D_F) \subset \mathbb{H}_P$. These F are called directly quantizable. We will not discuss these matters further at the moment (see, however, the next section), but rather continue to present elementary examples after the following remark.

Remark 10.8 (Kähler Quantization)**.** Summarizing the above considerations, the subspace $\mathbb{H}_P$ of holomorphic sections in the prequantum representation space $\mathbb{H}$ is the new representation space of the Kähler polarization P and the restrictions of the prequantum operators $q(F)$ to $\mathbb{H}_P$ for the directly quantizable observables F yield a geometric quantization satisfying (D1) aund (D2). This construction is sometimes called Kähler quantization.

Example 10.9 (Simple Phase Space. Holomorphic Polarization)**.** We come back to the example of $M = T^*\mathbb{R}^n$ with the standard symplectic form $\omega = dq^j \wedge dp_j$ and introduce, as before, the holomorphic coordinates

$$z_j := p_j + iq^j,$$

(which is essentially introducing a complex vector space structure on $M \cong \mathbb{R}^n \times \mathbb{R}^n$ such that $M \cong \mathbb{C}^n$). Let

$$P = \operatorname{span}_{\mathcal{E}(M)} \left\{ \frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n} \right\} \subset TM^{\mathbb{C}}.$$

be the holomorphic polarization. The line bundle $L \to M$ is trivial (there are only trivial line bundles on M) and the connection is unique up to isomorphism as well (since $H^1_{dR}(M, \mathbb{C}) = 0$). A potential α of

$$\omega = \frac{i}{2} \sum d\bar{z}_j \wedge dz_j \,,$$

which is adapted to P is given by

$$\alpha = \frac{i}{2} \sum_j \bar{z}_j dz_j \,.$$

α is adapted in the sense that $\alpha(X) = 0$ for all vector fields $X \in \Gamma(M, P)$.

Instead of defining the connection of our prequantum bundle (L, ∇, H) with respect to the non-vanishing section s_1, where $s_1(a) = (a, 1)$, $a \in M$, as before, we choose the section

$$s_e(a) = \left(a, \exp \left(- \sum_{j=1}^{n} \frac{\pi}{2} \bar{z}_j z_j \right) \right) = \exp(-\frac{\pi}{2} \bar{z} z) s_1(a) \,,$$

(with $\bar{z}z := \|z\|^2$) and define the connection ∇ by

$$\nabla_X s_e := 2\pi i \alpha(X) s_e \,,$$

$X \in \mathfrak{V}(M)$.

Note, that the polarized sections depend on the choice of the connection resp. on the choice of the potential α of ω and the non-vanishing section s with $\nabla s = 2\pi i \alpha s$. Even equivalent connections have different polarized sections, see the subsequent observation and the remark above. The spaces of polarized sections, however, are isomorphic, and the resulting representation spaces with its prequantum operators will be unitarily equivalent.

Each section $s \in \Gamma(M, L)$ is of the form $s = f s_e$ with $f \in \mathcal{E}(M)$. Because of

$$\nabla_X f s_e = (L_X f + 2\pi i \alpha(X) f) s_e \,,$$

the section $f s_e$ will be polarized if and only if $L_X f = 0$ for all $X \in \Gamma(M, P)$ (recall $\alpha(X) = 0$) and this in turn is equivalent to f being a holomorphic function[48]. As a result

$$\Gamma_{\nabla, P}(M, L) = \{ f s_e \mid f \in \mathcal{O}(\mathbb{C}^n) \} \cong \mathcal{O}(\mathbb{C}^n) \,.$$

[48] f is partially holomorphic because of $L_X f = 0$ for the generators $X = \partial/\partial \bar{z}_j \in P$.

The natural scalar product is

$$\langle f, g \rangle = \int_{\mathbb{C}^n} \bar{f} g \exp(-\pi \|z\|^2) dz$$

for those $f, g \in \mathcal{O}(\mathbb{C}^n)$ for which the integral is defined (dz is Lebesgue integration or integration with respect to ω^n, which is the same up to a constant). The space of polarized sections with the scalar product is essentially the Bargmann-Fock space

$$\mathbb{F} := \left\{ f \in \mathcal{O}(\mathbb{C}^n) \;\middle|\; \int_{\mathbb{C}^n} |f|^2 \exp(-\pi \|z\|^2) dz < \infty \right\} .$$

Note, that this space of holomorphic functions is already complete in the norm given by the scalar product (cf. Proposition B.14). Thus, $\mathbb{F}$ is already a Hilbert space. It is a proper subspace of $\mathcal{O}(\mathbb{C}^n)$ as a vector space and a proper and closed subspace of the Hilbert space $L^2(\mathbb{C}^n, \exp(-\pi z\bar{z})dz)$ of functions which are square integrable with respect to $\exp(-\pi z\bar{z})dz$.

We have constructed the reduced representation space $\mathbb{H}_P = \mathbb{F}$ for the simple phase space $T^*\mathbb{R}^n \cong \mathbb{C}^n$ and the holomorphic polarization P. This space will be denoted by $\mathbb{H}_P := \mathbb{F}$ in the following. In comparison, the unreduced Hilbert space in the simple case is the prequantum Hilbert space $\mathbb{H} = L^2(\mathbb{R}^{2n}, d\lambda)$ of square integrable functions on $\mathbb{R}^{2n} \cong M \cong \mathbb{C}^n$ with respect to Lebesgue integration $d\lambda$.

Let us finish the example by determining the prequantum operators

$$q(F) = -\frac{i}{2\pi} \nabla_{X_F} + F$$

for $F = z$ and $F = \bar{z}$[49]:

The Hamiltonian vector field X_F of an observable $F \in \mathcal{E}(M)$ has the following expression in the complex coordinates

$$X_F = 2i \sum_j \left(\frac{\partial F}{\partial \bar{z}_j} \frac{\partial}{\partial z_j} - \frac{\partial F}{\partial z_j} \frac{\partial}{\partial \bar{z}_j} \right) . \tag{44}$$

Hence,

$$X_{z_j} = -2i \frac{\partial}{\partial \bar{z}_j} , \quad X_{\bar{z}_j} = 2i \frac{\partial}{\partial z_j} ,$$

For holomorphic f we have $\nabla_{X_{z_j}} f s_e = 0$ since $X_{z_j} \in \Gamma(\mathbb{C}^n, P)$. Therefore,

$$q(z_j) = z_j$$

is the multiplication operator $f \mapsto z_j f$ on $\mathbb{H}_P$.

[49] We allow complex classical observables in this example.

Furthermore,

$$\nabla_{X_{\bar{z}_j}} f s_1 = \left(2i\frac{\partial}{\partial z_j} f + 2\pi i\alpha(2i\frac{\partial}{\partial z_j})f\right) s_1$$
$$= \left(2i\frac{\partial}{\partial z_j} f + 2\pi i\frac{i}{2}(2i\bar{z}_j)f\right) s_1$$
$$= \left(2i\frac{\partial}{\partial z_j} f - 2\pi i\bar{z}_j f\right) s_1 ,$$

hence

$$q(\bar{z}_j) = \frac{1}{\pi}\frac{\partial}{\partial z_j} - \bar{z}_j + \bar{z}_j = \frac{1}{\pi}\frac{\partial}{\partial z_j} .$$

With the notations

$$Z_k := q(z_k) = z_k \, , \, \overline{Z}^j := q(\bar{z}_j) = \frac{1}{\pi}\frac{\partial}{\partial z_j}, \ 1 \leq j, k \leq n \, ,$$

we obtain

$$\overline{Z}^j Z_k \phi = \frac{1}{\pi}\delta^j_k \phi + \frac{1}{\pi} z_k \frac{\partial}{\partial z_j}\phi \, ,$$

and finally

$$[\overline{Z}^j, Z_k] = \frac{1}{\pi}\delta^j_k \, .$$

These are essentially the canonical commutation relations (CCR).

Further classical observables F will be discussed later, for instance, the energy $H = \frac{1}{2}\sum_{j=1}^n z_j \bar{z}_j$ of the harmonic oscillator in the example below.

Remark 10.10. With $a_j^* := Z_j$, $a_j := \overline{Z}^j$ we see that the $Z_j, \overline{Z}_j$ act as the familiar raising and lowering operators. According to Remark B.18 the operator a_j^* is a closed operator with the space $\mathcal{P}$ of complex polynomials as its domain. Moreover, as the notation suggests, a_j^* is the adjoint of a_j. The "position" and "momentum" operators in this context are the self-adjoint operators[50]

$$A_j := z_j + \frac{1}{\pi}\frac{\partial}{\partial z_j} = a^* + a_j \, ,$$

$$B_j := i\left(z_j - \frac{1}{\pi}\frac{\partial}{\partial z_j}\right) = i(a^* - a_j) \, .$$

The CCR are satisfied in the following form

$$[A_j, B_k] = \frac{i}{2\pi}\delta_{jk} \, .$$

[50] In Section F.2 closed, adjoint and self-adjoint operators are treated.

Observation. The covariant derivative ∇s_1 of the section $s_1 = \exp(\frac{\pi}{2}\left\|z\right\|^2)s_e = gs_e$ is

$$\begin{aligned}\nabla s_1 = \nabla g s_e &= \left(dg + 2\pi i \frac{i}{2}\sum \bar{z}_j dz_j g\right) s_e \\ &= \left(\frac{\pi}{2}\sum(\bar{z}_j dz_j + z_j d\bar{z}_j) - \pi \bar{z}_j dz_j\right) s_1 \\ &= 2\pi i \left(\frac{i}{4}\sum \bar{z}_j dz_j - z_j \bar{d}z_j\right) s_1\end{aligned}$$

In other words, when the connection ∇ shall be defined with respect to the section s_1 this has to be done with the potential $\beta = {}^i\!/\!{}_4(\sum \bar{z}_j dz_j - z_j \bar{d}z_j)$ of ω instead of α in order to obtain the same connection. Let us denote by ∇' the connection defined using α and s_1:

$$\nabla' s_1 = 2\pi i \alpha s_1 \,.$$

The polarized sections are

$$\begin{aligned}\Gamma_{\nabla',P}(M,L) &= \{f s_1 \mid f \in \mathcal{O}(\mathbb{C}^n)\}\,,\\ \Gamma_{\nabla,P}(M,L) &= \{f s_e \mid f \in \mathcal{O}(\mathbb{C}^n)\}\,.\end{aligned}$$

Remark 10.11. The representation space in the preceding example is called the BARGMANN-FOCK REPRESENTATION[51]. Similar to this example we have presented the examples with the vertical resp. horizontal distribution (and real polarization) in $M = T^*\mathbb{R}^n$ in 1. and 2. of Examples 10.2: Here the reduced Hilbert space is $L^2(\mathbb{R}^n)$.

Examples 10.12 (Simple Phase Space With Different Polarizations)**.** We collect our results of geometric quantization of position and momentum in the classical case of a simple phase space $M = T^*\mathbb{R}^n$ and the three main natural polarizations

1. The vertical distribution D on $T^*\mathbb{R}^n$ induces the vertical polarization $P = D^{\mathbb{C}}$. It will be used to reduce the prequantum representation space $\mathbb{H} = \mathbb{H}(M,L)$ to obtain as the reduced Hilbert space the SCHRÖDINGER REPRESENTATION $\mathbb{H}_P = L^2(\mathbb{R}^n, d\lambda)$ of square integrable functions $\phi = \phi(q)$ with respect to Lebesgue integration $d\lambda = d\lambda(q)$ on the configuration space $\mathbb{R}^n$. The prequantum operators corresponding to position and momentum are (cf. 1. in Examples 10.2)

   $$Q^j := q^j \ \text{ and } \ P_j := -\frac{i}{2\pi}\frac{\partial}{\partial q^j}\,.$$

   They satisfy the following canonical commutation relations

   $$\left[Q^j, P_k\right] = \frac{i}{2\pi}\delta^j_k\,.$$

[51] also called Segal-Bargmann representation or simply Bargmann representation

2. The horizontal distribution D on $T^*\mathbb{R}^n$ leads to the horizontal polarization $Q = D^{\mathbb{C}}$). The reduced Hilbert space is sometimes called the HEISENBERG REPRESENTATION $\mathbb{H}_P = L^2(\mathbb{R}^n, d\lambda)$ with $d\lambda = d\lambda(p)$ the Lebesgue measure but now consisting of wave functions depending on the momentum variables p_j only. One obtains the quantized operators (cf. 2. in Examples 10.2)

$$Q^j := \frac{i}{2\pi}\frac{\partial}{\partial p_j} \text{ and } P_k := p_k\,,$$

and consequently the canonical commutation relations

$$\left[Q^j, P_k\right] = \frac{i}{2\pi}\delta^j_k\,.$$

3. Using the holomorphic polarization P on $T^*\mathbb{R}^n$ in order to reduce the prequantum representation space one obtains as the reduced Hilbert space the BARGMANN SPACE $\mathbb{H}_P = \mathbb{F}$ of the preceding example. The prequantum operators corresponding to the "classical" observables $z_j, \bar{z}_j$ are

$$Z_k = z_k\,,\ \overline{Z}^j = \frac{1}{\pi}\frac{\partial}{\partial z_j},\ 1 \leq j,k \leq n\,,$$

They satisfy the following canonical commutation relations

$$[\overline{Z}^j, Z_k] = \frac{1}{\pi}\delta^j_k\,.$$

All three representations are unitarily equivalent to each other as is explained in Section F.3.

The Bargmann-Fock representation is useful for quantization of the harmonic oscillator in n dimensions (see below) and of a simplified model of a Bose-Einstein field.

Further Kähler quantizations are given by a family of almost complex structures J on our simple phase space $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ generalizing Example 10.9:

Example 10.13 (Simple Phase Space. Variants of Holomorphic Polarization)**.** We come back to the example of $M = T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n = V$ with the standard symplectic form $\omega = dq^j \wedge dp_j$. But in contrast to Example 10.9 we do not introduce the holomorphic polarization of the standard almost complex structure $q^k \mapsto p_k\,,\ p_k \mapsto -q^k$ but the holomorphic polarization $P = P_J$ determined by the almost complex structure J (see Proposition 9.21) given by the block matrix

$$J = \begin{pmatrix} 0 & Y^{-1} \\ -Y & 0 \end{pmatrix}\,,$$

where $Y = (g_{jk})$ is a symmetric and real $n \times n$-matrix with inverse $Y^{-1} = (g^{jk})$. In this way Example 9.22 will be continued.

The corresponding holomorphic coordinates with respect to J are

$$z^k := g^{jk}p_j + iq^k \,,$$

The symplectic form is

$$\omega = dq^j \wedge dp_j = \frac{1}{2} i g_{jk} d\bar{z}^j \wedge dz^k \,,$$

with symplectic potential

$$\alpha = \frac{1}{2} i g_{jk} \bar{z}^j dz^k \,.$$

The holomorphic polarization is $P = P_J = \ker(J^{\mathbb{C}} + i) \subset TV^{\mathbb{C}}$ is generated by the vector fields $\frac{\partial}{\partial \bar{z}^k}$.

The prequantum line bundle $L \to V$ is trivial. The connection ∇ on L will be defined with respect to the non-vanishing section s_e

$$s_e(a) = (a, \exp(-\frac{\pi}{2} g_{jk} \bar{z}_j z_k)) = (a, \exp(-\frac{\pi}{2} g(\bar{z}, z))) \,,$$

$(g(\bar{z}, z) := g_{jk} \bar{z}^j z^k)$ by

$$\nabla_X s_e := 2\pi i \alpha(X) s_e \,,$$

$X \in \mathfrak{V}(M)$, and the hermitian structure is the one induced from $L = V \times \mathbb{C}$.

Each section $s \in \Gamma(V, L)$ is of the form $s = f s_e$ with $f \in \mathcal{E}(V)$. Because of

$$\nabla_X f s_e = (L_X f + 2\pi i \alpha(X) f) s_e \,,$$

the section $f s_e$ will be polarized if and only if $L_X f = 0$ for all $X \in \Gamma(V, P)$ (recall $\alpha(X) = 0$) and this in turn is equivalent to f being a holomorphic function with respect to J. Let $\mathcal{O}(V)$ denote the space of these holomorphic functions, a more precise notation is $\mathcal{O}(V_J)$ when V_J denotes the complex vector space with underlying V and almost complex structure J. As a result

$$\Gamma_{\nabla, P}(M, L) = \{ f s_e \mid f \in \mathcal{O}(V) \} \cong \{ f \exp(-\frac{\pi}{2} g(\bar{z}, z)) \mid f \in \mathcal{O}(V) \} \cong \mathcal{O}(V) \,.$$

The natural scalar product on $\{ f \exp(-\frac{\pi}{2} g(\bar{z}, z)) \mid f \in \mathcal{O}(V) \}$ is

$$\langle f, f' \rangle = \int_V \bar{f} f' \exp(-\frac{\pi}{2} g(\bar{z}, z)) dz$$

for those $f, f' \in \mathcal{O}(V)$ for which the integral is defined (dz is Lebesgue integration or integration with respect to ω^n, which is the same up to a constant). The space of polarized sections with the scalar product is essentially the Bargmann space

$$\mathbb{F}_J = \mathbb{F} := \left\{ f \in \mathcal{O}(V) \;\middle|\; \int_V |f|^2 \exp(-\frac{\pi}{2} g(\bar{z}, z))) dz < \infty \right\} \,.$$

This space of holomorphic functions is already complete in the norm given by the scalar product (cf. Proposition B.14), i.e. $\mathbb{F}$ is already a Hilbert space. It is a proper subspace of $\mathcal{O}(V)$ as a vector space and a proper and closed subspace of the Hilbert space $L^2(\mathbb{C}^n, \exp(-\frac{\pi}{2}g(\bar{z},z))dz)$ of functions which are square integrable with respect to $\exp(-\frac{\pi}{2}g(\bar{z},z))dz$.

Note, that the representation space $\mathbb{F}_J$ is trivial whenever Y is not positive definite since then only the holomorphic f, $f = 0$, has a finite integral

$$\int_V |f|^2 \exp(-\frac{\pi}{2}g(\bar{z},z))dz\,.$$

Therefore, we mostly restrict to the positive case.

We have constructed the reduced representation space $\mathbb{H}_P = \mathbb{F}_J$ for the simple phase space $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ and the holomorphic polarization P_J.

Example 10.14 (2-sphere)**.** We continue the example of the 2-sphere $M = \mathbb{S}^2$. Let ω be the symplectic form such that $\omega = \frac{1}{4\pi}\mathrm{vol}$ for the natural volume form. We know from Example 7.6 that $(\mathbb{S}^2, C\omega)$ is quantizable if and only if $C \in \mathbb{Z}$, $N = C \neq 0$, and from Example 8.11 that up to equivalence each $(\mathbb{S}^2, N\omega)$ has only the prequantum bundles $(H(N), \nabla^N)$, $N \in \mathbb{Z}$, with ∇^N as in Example **??**. Here, $H(1)$ is the hyperplane line bundle and $H(-1)$ the tautological line bundle. $H(N)$ for $N \in \mathbb{N}$ is the N-fold tensor product of $H(1)$

As a differentiable manifold $\mathbb{S}^2$ is naturally diffeomorphic to the projective line $\mathbb{P}^1$. With respect to the holomorphic resp. Kähler polarization on $\mathbb{P}^1$, the polarized sections of $H(N)$ are the holomorphic sections. In Proposition 3.22 it is shown, that the space of holomorphic sections $\mathbb{V}_N = \Gamma_{hol}(\mathbb{P}^1, H(N))$ is finite dimensional of dimension $N+1$ for $N \geq 0$ and 0 for $N < 0$. For the Kähler quantization this implies, that the space of polarized sections of the line bundle $H(N)$ is $\mathbb{V}_N \cong \mathbb{C}^{N+1}$. In particular, no special integration is necessary, all Hermitian structures on $\mathbb{V}_N$ are equivalent. And, once $q(F)$ for $F \in \mathcal{E}(\mathbb{S}^2)$ can be implemented at all on $\mathbb{V}_N$ (see next section) then $q(F)$ is self-adjoint. The reduced representation space is $\mathbb{V}_N$. This result agrees with the spin-$\frac{1}{2}$-representations, including the dimension of the eigenspaces.

It can be shown, that on each $\mathbb{H}^N$ an irreducible representation of SU(2) is induced, thereby generating all irreducible representations of SU(2).

Examples 10.15. Now the door is open to investigate Kähler manifolds in general and use the complex structure as well as the Kähler geometry as powerful tools. We list some interesting cases:

1. $\mathbb{C}^n$ and all its open complex submanifolds.
2. $M = T^*\mathbb{S}^1 \cong \mathbb{C}^*$
3. $M = \mathbb{S}^1 \times \mathbb{S}^1$ and all other 2-dimensional compact oriented manifolds without boundary.

4. $\mathbb{P}^1 \times \mathbb{P}^1$ (Kepler problem) and other compact complex surfaces.

5. The projective spaces $\mathbb{P}^n$ introduced earlier and all its closed complex submanifolds $M \subset \mathbb{P}^n$. These are the so-called projective manifolds.

10.3 Directly Quantizable Observables

We come back to the case of a general complex polarization: As before, let (L, ∇, H) be a prequantum bundle on a symplectic manifold (M, ω). Let P be a reducible complex polarization P on M with its integrable distribution $D = P \cap \overline{P} \cap TM$. Moreover, let μ be a measure on M/D.

The question arises, for which $F \in \mathcal{E}(M)$ the operator $q(F)$ leads to an operator in the representation space $\mathbb{H}_P$. The following example illustrates the possible problems for $q(F)$ becoming an operator in $\mathbb{H}_P$.

Example 10.16 (Harmonic Oscillator)**.** It is rather a counterexample showing that even with a natural measure on M/D a naive implementation of geometric quantization is not always possible. Let $M = T^*\mathbb{R}^n$ be as before with the form $\omega = dq^j \wedge dp_j$. The energy of the harmonic oscillator is $H = \frac{1}{2}(p^2 + q^2), (q, p) \in M$ (we restrict to the case $n = 1$ in the following).

We try to use the vertical distribution D with $P = D^{\mathbb{C}}$ in order to quantize H. In this case the quotient manifold $M/D \cong \mathbb{R}$ is the configuration space and has the Lebesgue measure as a familiar measure.

The Hamiltonian vector field is

$$X_H = q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\,.$$

The covariant derivative with respect to the potential $-pdq = -\lambda$ reads

$$\nabla_{X_H} f s_1 = \left(q\frac{\partial}{\partial p} - p\frac{\partial}{\partial q}\right) f s_1 - 2\pi i p(-p) f s_1\,.$$

and the operator $q(H)$ for $\phi \in \mathbb{H}_P$ is

$$q(H)\phi = \frac{1}{2\pi i}\left(-p\frac{\partial}{\partial q}\phi\right) + p^2\phi + H\phi\,.$$

This will not be a function in $\mathbb{H}_P$, in general. A change in the potential (for instance $\alpha = qdp$) will not help.

As a result, although the vertical distribution looks good from the point of the space of leaves $M/D \cong \mathbb{R}$, the energy H of the harmonic oscillator cannot be quantized directly when the representation space is reduced to $\mathbb{H}_P$. To come around this problem one can consider another polarization, e.g. the holomorphic polarization or the radial polarization, see below, or one has to introduce other methods to incorporate classical observables like H in the geometric quantization program, as will be done in later chapters.

In order to verify whether the prequantum operator $q(F)$ leads to an operator in $\mathbb{H}_P$ the following fundamental question has to be answered: When s is a polarized section of L, will $q(F)s$ be polarized as well?

Definition 10.17. Let P be a complex polarization on a symplectic manifold. An observable $F \in \mathcal{E}(M)$ will be called DIRECTLY QUANTIZABLE (with respect to P) if for all $X \in \Gamma(M,P)$ the Lie bracket $[X_F, X]$ remains in $\Gamma(M,P)$. In another description: $\mathrm{ad}_{X_F}(X) := [X_F, X]$ fulfills $\mathrm{ad}_{X_F}(\Gamma(M,P)) \subset \Gamma(M,P)$.

$\mathfrak{R}_P = \mathfrak{R}_P(M)$ denotes the set of all directly quantizable classical observables. Note, that we consider complex-valued observables F although in physics only real-valued F are relevant.

Lemma 10.18. *$\mathfrak{R}_P$ is a Lie algebra with respect to the Poisson bracket on $\mathcal{E}(M)$.*

Proof. Let $F, G \in \mathfrak{R}_P$ and $X \in \Gamma(M,P)$. Directly from

$$[X_{\{F,G\}}], X] = [-[X_F, X_G], X] = [[X_G, X], X_F] + [[X, X_F], X_G]$$

one can read off that $[X_{\{F,G\}}, X] \in \Gamma(M,P)$, hence $\{F,G\} \in \mathfrak{R}_P$. □

Remark 10.19. The condition $\mathrm{ad}_{X_F}(\Gamma(M,P)) \subset \Gamma(M,P)$ means that ad_{X_F} preserves the polarization. This invariance property has a nice interpretation using the Lie derivative L_X of a vector field Y. It is defined as

$$L_X Y := \frac{d}{dt}((\Phi_{-t})_* Y)|_{t=0}\,,$$

where $\Phi_t = \Phi_t^X$ is the local flow of the vector field X on M. In Proposition A.24 it is shown that $L_X Y = [X, Y]$, hence $\mathrm{ad}_X = L_X$.

Proposition 10.20. *For a prequantum bundle (L, ∇, H) on a symplectic manifold (M, ω) and a complex polarization P we have: When $F \in \mathfrak{R}_P$ and $s \in \Gamma_{\nabla,P}(M,L)$, then $q(F)s$ is polarized.*

Proof. Let $X \in \Gamma(M,P)$. We have to show $\nabla_X(q(F)s) = 0$ for $s \in \Gamma_{\nabla,P}(M,L)$. Now

$$\begin{aligned}
\nabla_X(q(F)s) &= \nabla_X\left(-\frac{i}{2\pi}\nabla_{X_F}s + Fs\right)\\
&= -\frac{i}{2\pi}\left(\nabla_X\nabla_{X_F}s\right) + \nabla_X(Fs)\\
&= -\frac{i}{2\pi}\left(\nabla_X\nabla_{X_F}s\right) + L_X Fs + \nabla_X s\\
&= -\frac{i}{2\pi}\left(\nabla_X\nabla_{X_F}s\right) + L_X Fs
\end{aligned}$$

since s is polarized. By definition of the curvature

$$\omega(X,Y) = \frac{1}{2\pi i}\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right)$$

we obtain

$$-\frac{i}{2\pi}\nabla_X\nabla_{X_F}s = -\frac{i}{2\pi}\nabla_{X_F}\nabla_X s + \omega(X,X_F)s - \frac{i}{2\pi}\nabla_{[X,X_F]}s = \omega(X,X_F)s$$

since $\nabla_{X_F}\nabla_X s = 0$ and $\nabla_{[X,X_F]}s = 0$. We complete the above list of equations and obtain

$$\begin{aligned}\nabla_X(q(F)s) &= -\frac{i}{2\pi}\left(\nabla_X\nabla_{X_F}s\right) + L_X F s \\ &= +\omega(X,X_F)s + L_X F s \\ &= -L_X f s + L_X f s = 0\end{aligned}$$

□

10.4 Main Result

As the main result so far we obtain the first complete geometric quantization: The set of classical observables $\mathfrak{R}_P$ – defined in the preceding section – together with the other ingredients yields a full geometric quantization scheme in the following sense:

Theorem 10.21. *Let (L,∇,H) be a prequantum bundle on a symplectic manifold (M,ω) together with a reducible complex polarization P on M. Assume, that there exists a measure μ on the quotient $Q = M/D$.*[52]

Each $F \in \mathfrak{R}_P$ induces a quantum operator

$$q(F) = -\frac{i}{2\pi}\nabla_{X_F} + F$$

in the representation space $\mathbb{H}_P$ (defined in 10.5) such that the following results hold true

1. *$\mathfrak{R}_P$ is a Lie subalgebra of the Poisson algebra $\mathcal{E}(M)$.*
2. *$q(F) \in \mathcal{S}(\mathbb{H}_P)$*[53] *for $F \in \mathfrak{R}_P$, and the quantization map $q : \mathfrak{R}_P \to \mathcal{S}(\mathbb{H}_P)$ is $\mathbb{R}$-linear and satisfies* (D1) *and* (D2) *for $\mathfrak{o} = \mathfrak{R}_P$.*
3. *If X_F is complete, $F \in \mathfrak{R}_P$, then $q(F)$ is self-adjoint.*

[52] The existence of such a measure is not needed, integrals can be taken with respect to a density on M/D, see Chapter 12.

[53] $\mathcal{S}(\mathbb{H})$ denotes the symmetric and densely defined operators on a Hilbert space $\mathbb{H}$

Proof. By Proposition 10.20 the map $q(F) : \Gamma(M, L) \to \Gamma(M, L)$ maps polarized sections to polarized sections. Hence, it is well-defined on $D_F := \{s \in \Gamma_{\nabla,P}(M, L) \cap \mathbb{H}_P \mid q(F)s \in \mathbb{H}_P\}$. D_F is dense n $\mathbb{H}_P$ since it contains the sections with compact support in M/D.

Now 1. has been shown in Lemma 10.18. 2. Symmetry of $q(F)$ can be proved as in the proof of Proposition 2.4. The Dirac conditions and 3. follow in the same way as for the prequantization (cf. Chapter 7). □

Proposition 10.22. *Determination of $\mathfrak{R}_P$ in special cases of the simple phase space:*

1. In case of the vertical distribution / polarization P on $M = T^*Q$, $Q \subset \mathbb{R}^n$ open:

$$\mathfrak{R}_P = \{A(q) + B^j(q)p_j \mid A, B^j \in \mathcal{E}(Q)\} .$$

2. Analogously, for the horizontal polarization P on $M = T^*\mathbb{R}^n$, swapping the roles of q and p.

3. In case of the holomorphic polarization P on $M = T^*\mathbb{R}^n = \mathbb{C}^n$:

$$\mathfrak{R}_P = \{A(z) + B^j(z)\bar{z}_j \mid A, B^j \in \mathcal{O}(\mathbb{C}^n)\} .$$

Proof. 1. It is enough to restrict to local coordinates q^j of Q and to investigate

$$[\frac{\partial}{\partial p_j}, X_F].$$

Recall that

$$X_F = \frac{\partial F}{\partial q^j}\frac{\partial}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial}{\partial q^j} .$$

The two equations

$$\left[\frac{\partial}{\partial p_j}, \frac{\partial F}{\partial q^k}\frac{\partial}{\partial p_k}\right] = \frac{\partial^2 F}{\partial p_j \partial q_k}\frac{\partial}{\partial p_k} + \frac{\partial F}{\partial q^k}\frac{\partial}{\partial p_j}\frac{\partial}{\partial p_k} - \frac{\partial F}{\partial q^k}\frac{\partial}{\partial p_k}\frac{\partial}{\partial p_j}$$

and

$$\left[\frac{\partial}{\partial p_j}, -\frac{\partial F}{\partial p_k}\frac{\partial}{\partial q^k}\right] = -\frac{\partial^2 F}{\partial p_j \partial p_k}\frac{\partial}{\partial q_k} - \frac{\partial F}{\partial p_k}\frac{\partial}{\partial p_j}\frac{\partial}{\partial q^k} + \frac{\partial F}{\partial p_k}\frac{\partial}{\partial q^k}\frac{\partial}{\partial p_j}$$

show that

$$[\frac{\partial}{\partial p_j}, X_F] \in P$$

if and only if

$$\frac{\partial^2 F}{\partial p_j \partial p_k} = 0 .$$

Hence, $F \in \mathfrak{R}_P$ if and only if

$$\frac{\partial^2 F}{\partial p_j \partial p_k} = 0$$

for all $1 \leq j, k \leq n$. As a consequence, F is of the form $A(q) + B^j(q)p_j$ with suitable $A, B^j \in \mathcal{E}(Q)$.

2. and 3. are shown in the same way. □

The last result in the preceding proposition asserts that the energy $H = \frac{1}{2}\sum z_j\bar{z}_j \in \mathcal{E}(\mathbb{C}^n)$ of the harmonic oscillator is a directly quantizable observable with respect to the holomorphic polarization P:

$$H \in \mathfrak{R}_P.$$

We determine the quantum operator $q(H)$ in the next example.

Example 10.23 (Harmonic Oscillator)**.** This example can be viewed as a continuation of Example 10.9. Consider the Hamiltonian

$$H = \frac{1}{2}\sum p_j^2 + (q^j)^2 = \frac{1}{2}\sum_{k=1}^{n} z_k\bar{z}_k$$

of the harmonic oscillator with phase space $M = T^*\mathbb{R}^n \cong \mathbb{C}^n$. The Hamiltonian vector field

$$X_H = q^j\frac{\partial}{\partial p_j} - p_j\frac{\partial}{\partial q^j}$$

does not fit to the vertical distribution as we have seen before. Hence, it is reasonable to use the holomorphic polarization P with respect to the holomorphic coordinates $z_j = p_j + iq^j$ (No summation!). P is the polarization generated by the vector fields Xz_k:

$$P = \operatorname{span}\left\{\frac{\partial}{\partial\bar{z}_j}\right\}.$$

Using (44) or the above form of X_H we obtain

$$X_H = i\left(\sum z_j\frac{\partial}{\partial z_j} - \bar{z}_j\frac{\partial}{\partial\bar{z}_j}\right).$$

With $\alpha = \frac{i}{2}\sum_j \bar{z}_j dz_j$ and $\omega = \frac{i}{2}\sum \bar{z}_j \wedge dz_j$ as before one has for holomorphic functions $f : \mathbb{C}^n \to \mathbb{C}$:

$$\begin{aligned}
-\frac{i}{2\pi}\nabla_{X_H} f s_1 &= -\frac{1}{2\pi i}\left(i\sum_j z_j\frac{\partial}{\partial z_j}f + 2\pi i\frac{i}{2}\sum_j \bar{z}_j i z_j f\right)s_1 \\
&= \frac{1}{2\pi}\sum_j z_j\frac{\partial}{\partial z_j}f s_1 - \frac{1}{2}\sum_j z_j\bar{z}_j f s_1.
\end{aligned}$$

Thus, the geometric quantization for the holomorphic polarization P yields

$$q(H) = \frac{1}{2\pi}\sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}$$

– essentially the EULER OPERATOR – as the quantized Hamiltonian H on the Bargmann space $\mathbb{H}_P$.

The eigenvalues E will be determined by the equation

$$\frac{1}{2\pi}\sum z_k \frac{\partial}{\partial z_k}\phi = E\phi\,.$$

Claim: For an entire holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ this equality can be satisfied if and only if f is a N-homogeneous complex polynomial and $E = E_N = \frac{1}{2\pi}N\,,\ N \in \mathbb{N}$. To show this claim let us recall, that f has the power series expansion $f = \sum P_N f$ with suitable N-homogeneous polynomials P_N converging uniformly on all compact subsets of $\mathbb{C}^n$ to f (cf. Proposition B.5). $P_N f$ has the form

$$P^N f(z) = \sum_{j_1+\ldots+j_n=N} \frac{1}{j_1!\cdots j_n!}\frac{\partial^{j_1+\cdots+j_n} f(0)}{\partial {z_1}^{j_1}\cdots\partial {z_n}^{j_n}} z_1^{j_1}\cdots z_n^{j_n}\,.$$

For a monomial $c_{\mu_1\mu_2\ldots\mu_n} z_1^{\mu_1} z_2^{\mu_2}\ldots z_n^{\mu_n}$ (no summation!) of degree $N = \mu_1+\mu_2+\ldots+\mu_n$ it is clear that

$$\sum_k z_k \frac{\partial}{\partial z_k} c_{\mu_1\ldots\mu_n} z_1^{\mu_1}\ldots z_n^{\mu_n} = (\mu_1+\ldots+\mu_n)\, c_{\mu_1\ldots\mu_n} z_1^{\mu_1}\ldots z_n^{\mu_n} = N c_{\mu_1\ldots\mu_n} z_1^{\mu_1}\ldots z_n^{\mu_n}\,,$$

hence,

$$\sum_k z_k \frac{\partial}{\partial z_k}(P_N f) = N P_N f$$

and eventually

$$\sum_k z_k \frac{\partial}{\partial z_k} f = \sum_N \sum_k z_k \frac{\partial}{\partial z_k} P_N f = \sum_N N P_N f\,.$$

The claim follows from the last equality. So the eigenvalues of our operator $q(H)$ on $\mathbb{H}_P$ are $E_N = \frac{1}{2\pi}N$ and the eigenspaces $\mathbb{V}_N$ are the spaces of N-homogenous polynomials (note, that the homogeneous complex polynomials in N variables are in the domain of $q(H)$). And we obtain a complete decomposition $\mathbb{H}_P = \bigoplus \mathbb{V}_N$ into eigenspaces. By the way, this shows that $q(H)$ is self-adjoint.

From the physical side this result is not correct. The observed eigenvalues are $\frac{1}{2\pi}(N + n/2)$ instead. So the term $\frac{n}{2}$ is missing, which is related to the zero point energy.

By comparison, we see that as a correct quantized operator $q(H)$ one should take

$$q(H) := \frac{1}{2\pi}\left(\sum_j z_j \frac{\partial}{\partial z_j}\right) + \frac{n}{2}\,. \tag{45}$$

This can be achieved by replacing L with the line bundle $L \otimes S$ where $S \to M$ is a geometrically induced complex line bundle over M respecting the symplectic geometry of (M, ω) and the polarization (see Chapter 15 and the correction scheme in the next section).

Let us recall how in conventional canonical quantization the quantum operator (45) can be obtained. The energy is written in the form $H = \frac{1}{4}\sum_j (z_j \bar{z}_j + \bar{z}_j z_j)$ and the quantum operators

$$q(z_j) = z_j \ , \ q(\bar{z}_j) = \frac{1}{\pi}\frac{\partial}{\partial z_j}$$

(cf. Example 10.9) are used by replacing the classical coordinates $z_j, \bar{z}_j$ directly. As a result we get for $f \in \mathcal{O}(\mathbb{C}^n)$:

$$q(H)f = \frac{1}{4\pi}\sum_j \left(z_j \frac{\partial}{\partial z_j} + \frac{\partial}{\partial z_j} z_j \right) f = \frac{1}{4\pi}\sum_j \left(2z_j \frac{\partial}{\partial z_j} f + f \right) = \frac{1}{2\pi}\left(\sum_j z_j \frac{\partial}{\partial z_j} + \frac{n}{2} \right) f.$$

Evidently, this "conventional correction" has to do with operator ordering.

Remark 10.24. Let us consider the simple phase space $T * \mathbb{R}^n \cong \mathbb{C}^n$ with the holomorphic polarization P. The directly quantizable observable are the functions $A(z) + B^j(z)\bar{z}_j$ with A, B^j holomorphic. Hence, the real-valued directly quantizable observables are the functions

$$F(z, \bar{z}) = A + \bar{D}^j z_j + D^j \bar{z}_j + sumC^j z_j \bar{z}_j \ ,$$

where $A, , C^j$ are real constants and D^j are complex constants. We conclude from the calculatios of the preceding examples that we know the quantum operator $q(F)$ for all real $F \in \mathfrak{R}_P$:

$$q(A) = A\, \mathrm{id}_{\mathbb{H}} \ , \ q(\bar{D}^k z_k + D^k \bar{z}_k) = \bar{D}^k z_k + D^k \frac{1}{\pi}\frac{\partial}{\partial z_k}$$

and

$$q(\sum C^j z_j \bar{z}_j) = \frac{1}{\pi}\sum C^j z_j \frac{\partial}{\partial z_j} \ .$$

10.5 Sketch of Correction Scheme

Let (L, ∇, H) a prequantum bundle on a symplectic manifold (M, ω) with a complex Kähler polarization P. The "polar" $P^0 := \{\mu \in T^*M^{\mathbb{C}} \mid \mu|_P = 0\}$ is a complex vector bundle of rank n over M, subbundle of the complexified cotangent bundle $T^*M^{\mathbb{C}}$ of M.

The n-fold wedge product $\Lambda^n P^0$ is a complex line bundle which is called the CANONICAL LINE BUNDLE[54] of P and denoted by K_P.

[54]For general complex vector bundles V an alternative definition is used: The canonical bundle is $K(V) := \Lambda^r(V^\vee)$,where r is the rank of the bundle V (see 15.1) and 15.7.

Let P' be another complex polarization on M with the condition $P \cap P' = \{0\}$ (for instance $P' = \overline{P}$ in case of a Kähler polarization). The line bundle $K_P \otimes K_{P'}$ is naturally isomorphic to the complex line bundle $\Lambda^{2n}T^*M^{\mathbb{C}}$ over M. This line bundle is trivial having ω^n as a nowhere vanishing section. For sections $\alpha \in \Gamma(M, K_P)$, $\beta \in \Gamma(M, K_{P'})$ the wedge $\bar{\alpha}\wedge\beta$ is a scalar multiple of ω^n: $\langle\alpha,\beta\rangle\omega^n = \bar{\alpha}\wedge\beta$, which provides a sesquilinear pairing $(\alpha,\beta) \mapsto \langle\alpha,\beta\rangle$.

K_P has the natural (partial) connection $\nabla_X\alpha = i_X\alpha$, $X \in \Gamma(M, P)$, which is flat.[55]

Now, let us assume that K_P has a square root S in the sense that for a line bundle S over M there exists an isomorphism $K_P \cong S \otimes S$. (The existence of such a line bundle S depends on a topological property of M.) We set $K_P^{1/2} := S$ and denote the corresponding induced (partial) connection by $\nabla^{1/2}$.

Our prequantum bundle will be replaced by

$$L^{\text{corr}} = L \otimes K_P^{1/2}\,,\ \nabla^{\text{corr}} := \nabla + \nabla^{1/2} = \nabla \otimes 1 + 1 \otimes \nabla^{1/2}.$$

The polarized sections are the sections $s \otimes \alpha \in \Gamma(M, L \otimes K_P^{1/2})$ satisfying

$$\left(\nabla + \nabla^{1/2}\right)(s \otimes \alpha) = 0$$

which is the same as $\nabla s = 0$ and $\nabla^{1/2}\alpha = 0$.

The representation space $\mathbb{H}_P$ is the completion of the space of polarized sections $s \otimes \alpha \in \Gamma(M, L \otimes K_P^{1/2})$ with respect to the scalar product by

$$\int H(s,s)\langle\alpha,\alpha\rangle\omega^n = \int H(s,s)\bar{\alpha}\wedge\alpha < \infty.$$

The quantum operator is defined by

$$q^{\text{corr}}(F)(s\otimes\alpha) := -\frac{i}{2\pi}\left(\nabla_X s \otimes \alpha + s \otimes L^{1/2}_{X_F}\alpha\right) + F = q(F)s\otimes\alpha - \frac{i}{2\pi}s \otimes L^{1/2}_{X_F}\alpha$$

for directly quantizable observables $F \in \mathcal{E}(M)$, where $s \otimes \alpha \in \Gamma(M, L \otimes K_P^{1/2})$ are polarized sections. In this way $\mathbb{H}_P$ and $q^{corr}(F)$ yields a full geometric quantization.

This modification is called half-form correction and can be extended to more general cases with $P \cap \overline{P} \neq \{0\}$. This will be the subject of Chapter 15.

Example 10.25. Applied to the harmonic oscillator on $M = T^*\mathbb{R} = \mathbb{C}$ with Hamiltonian $H(z) = \frac{1}{2}|z|^2$ and holomorphic polarization P we get the following: The polarized sections of the (trivial) line bundle L are essentially the holomorphic functions $f : \mathbb{C} \to \mathbb{C}$. K_P is trivial and generated by dz. $K_P^{1/2}$ is generated by a section denoted by

[55] See Chapter 15

$dz^{1/2}$. From $X_H = i(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}})$ we deduce $L_{X_H} dz = idz$ and from this $L^{1/2}_{X_H} dz^{1/2} = \frac{i}{2} dz^{1/2}$ using

$$L_{X_H} dz = 2dz^{1/2} L^{1/2}_{X_H} dz^{1/2} \,.$$

With the result $q(H) = \frac{1}{2\pi} z \frac{d}{dz}$ (from Example 10.23) and inserting the last formula in $q^{\mathrm{corr}}(H)$ the quantum operator $q^{\mathrm{corr}}(H)$ we obtain

$$q^{\mathrm{corr}}(H)(f \otimes dz^{1/2}) = \frac{1}{2\pi} z \frac{d}{dz} f \otimes dz^{1/2} - \frac{i}{2\pi} f \otimes \frac{i}{2} dz^{1/2} = \frac{1}{2\pi}(z\frac{d}{dz} + \frac{1}{2})(f \otimes dz^{1/2}) \,.$$

Hence $q^{\mathrm{corr}}(H)$ acts on the space $\mathcal{O}(\mathbb{C})$ of holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ in the following way

$$\boxed{q^{\mathrm{corr}}(H) := \frac{1}{2\pi}\left(z\frac{d}{dz} + \frac{1}{2}\right) .} \tag{46}$$

Therefore, the corrected quantum operator yields the right eigenvalues with the right multiplicities.

Note, that the very sketchy description of the representation space $\mathbb{H}_P$ is not needed for the calculation of the quantum operator $q^{\mathrm{corr}}(H)$ acting on the space $\Gamma_P(T^*\mathbb{R}, L^{\mathrm{corr}}) \cong \mathcal{O}(\mathbb{C})$. We have mentioned the definition of the Hilbert space $\mathbb{H}_P$, since in half-density and half-form quatization it is important to obtain general and corrected representation spaces and since it opens the way to pairing the spaces $\mathbb{H}_P$ and $\mathbb{H}_{P'}$ for different polarizations.

Summary: By implementing a complex polarization P as an additional geometric data complementing the prequantum bundle on the symplectic manifold the reduced representation space $\mathbb{H}_P$ can now be constructed as the completion of the space of polarized and square integrable sections. To complete the first basic version of Geometric Quantization the concept of a directly quantizable observable F has to be introduced in order to confirm that for a polarized section s the derivative $\nabla_{X_F} s$ is polarized as well. For such F it is easy to define the quantum operator $q(F)$ in the new representation space $\mathbb{H}_P$ to achieve a full geometric quantization.

Several elementary examples are presented in detail in this chapter in order to illustrate the action of Geometric Quantization. In particular, for the simple case of $M = T^*\mathbb{R}^n$ three different polarizations yield the well-known Schrödinger, Heisenberg and Bargmann representations. Also, the quantization of the harmonic oscillator is described, leading to a slightly incorrect result. A sketch of a correction is added which is presented in greater generality in Chapter 15.

In many situations the first version of Geometric Quantization developed so far is not satisfying. Nevertheless, the previous 10 chapters cover a rather complete picture of the basic principles of Geometric Quantization and they describe essentially the content of the course given in winter 2021/22.

Among the weaknesses are, for example, that the representation space can be zero since there do not exist non-trivial polarized sections, a phenomenon which we treat in the next chapter. Moreover, eigenvalues are not correct as in the case of the harmonic oscillator. In addition, there is a need the quantize many more classical observables than merely the directly quantizable ones. As a consequence one has to modify or complement the basic Geometric Quantization presented in the first 10 chapters. This is the subject of the next chapters.

11 Existence of Polarized Sections and Holonomy

So far we have seen polarizations in action only for the following two cases: For the Kähler polarizations and for the vertical and the horizontal polarizations of the simple phase space $M = T^*\mathbb{R}^n$. In these two cases it was no question of whether or not there exist polarized sections at all (except possibly for the compact Kähler manifolds as phase spaces).

11.1 Bohr-Sommerfeld Variety

The following example exhibits a general obstacle to the existence of global polarized sections different from zero[56].

Example 11.1. Let $M = T^*\mathbb{S}$ the phase space with the circle $\mathbb{S}$ as configuration space (cf. Example 8.22) and with the HORIZONTAL POLARIZATION. The cotangent bundle $T^*\mathbb{S} = M$ is trivial and we write $M = \mathbb{S} \times \mathbb{R}$. As before, as prequantum line bundle we take the trivial bundle $L = M \times \mathbb{C}$ with connection ∇ given by the Liouville form $\lambda = pdq$, i.e.

$$\nabla f s_1 = (df + 2\pi i(-\lambda) f)\, s_1\,,$$

where $s_1(a) = (a, 1)\,,\ a \in M$ and $f \in \mathcal{E}(M)$, and H induced from $L = M \times \mathbb{C}$.

Differently from the previous discussion of this example in 8.22 we now study the horizontal polarization instead of the vertical polarization. The horizontal polarization P is generated by the vector field

$$\frac{\partial}{\partial q}\,,$$

where q is essentially the angle variable. Consequently a general section $s = f s_1$ is polarized if $L_X f - 2\pi i \lambda(X) f = 0$ for all $X \in \Gamma(M, P)$, i.e. if

$$\frac{\partial}{\partial q} f = 2\pi i p f$$

in local coordinates. This differential equation has the general solution

$$f(q,p) = g(p) \exp 2\pi i p q\,, \tag{47}$$

with an arbitrary function $g \in \mathcal{E}(\mathbb{R})$. Moreover, f has to be periodic in the variable q with period 1. Therefore, only for $p \in \mathbb{Z}$ there is a non-zero solution of this equation, depending on $(q,p) \in \mathbb{S} \times \{p\}$ for fixed $p \in \mathbb{Z}$. It follows, by the continuity of f, that $f = 0$. Altogether there do not exist nontrivial global polarized sections.

[56] If one asks for local sections which are polarized: In the situation of a general complex polarization, there exist non-zero local sections which are polarized. We do not need this result, we are interested in this section only in global polarized sections.

There is another way to arrive at this result which exhibits a general pattern behind this result and which generalizes to arbitrary reducible polarizations: The submanifolds $\mathbb{S} \times \{p\} =: S_p$, $p \in \mathbb{R}$, are the leaves (integral manifolds) of the horizontal distribution. The restriction $\nabla|_{S_p}$ of ∇ to a leave S_p is a flat connection on the line bundle $L|_{S_p} \to S_p$, the restriction of L to S_p. Let $a = (1,p) \in S_p$. Parallel transport $Q(\gamma) : L_a \to L_a$, $v \mapsto Q(\gamma)v$, along the curve $\gamma(t) = (\exp 2\pi i t, p)$, $t \in [0,1]$, is given by the integral (cf. Proposition 5.17)

$$Q(\gamma) = \exp\left(2\pi i \int_\gamma p dq\right) = \exp\left(2\pi i p \int_0^1 dt\right) = \exp(2\pi i p)\,. \tag{48}$$

Assume that $s \in \Gamma(M, L)$ is a global polarized section. Then the restriction of s to S_p is a horizontal section $S_p \to L|_{S_p}$ and therefore determines the parallel transport given by $\nabla|_{S_p}$. In particular, if $s(a) \neq 0$, the parallel transport is $s(\gamma(0)) \mapsto s(\gamma(1))$ with $\gamma(0) = \gamma(1) = a$ and therefore it is trivial in the sense that it is the identity $L_a \to L_a$. This implies $Q(\gamma) = 1$. But then it follows from (48) that $p \in \mathbb{Z}$. As a result, s has to be 0 outside the so called Bohr-Sommerfeld Variety

$$S := \bigcup \{S_p \mid p \in \mathbb{Z}\} = \mathbb{S} \times \mathbb{Z}\,,$$

which implies that $s = 0$ by continuity.

With respect to a different connection ∇, given by the one form $(\kappa - p)dq$, $\kappa \in \,]0,1[$ (see Example 8.22), one obtains essentially the same result. The corresponding parallel transport is given by the integral (cf. Proposition 5.17)

$$Q^\kappa(\gamma) = \exp\left(2\pi i \int_\gamma (p-\kappa) dq\right) = \exp(2\pi i (p-\kappa)\,. \tag{49}$$

A global polarized section s on $M = \mathbb{S} \times \mathbb{R}$ has to be zero outside of $S := \bigcup \{S_p \mid p - \kappa \in \mathbb{Z}\}$, and hence is the zero section.

The above considerations generalize directly to the case $M = T^*\mathbb{T}$ where $\mathbb{T} = (\mathbb{S})^n$ is the n-dimensional torus, $n > 0$.

It turns out that the non-existence of polarized sections is not a singular phenomenon of this example but rather is a general property whenever the leaves of the induced distribution are not simply connected. Before we present this result in the next proposition below, we want to give a different interpretation of the above example

Remark 11.2 (Energy Represention)**.** Consider the example $M := \mathbb{R}^2 \setminus \{(0,0)\} \subset T^*\mathbb{R} \cong \mathbb{R} \times \mathbb{R}$ with the usual symplectic form $\omega = dq \wedge dp$. Let the prequantization line bundle be $L = M \times \mathbb{C}$ with connection ∇ given by the connection form $-pdq = -\lambda$ as before.

Now, let D be the radial distribution given by the circles $p^2 + q^2 = 2E$, $E > 0$, i.e. the distribution with the circles around $(0,0)$ as its leaves. This distribution can be understood as the distribution generated by the energy function of the harmonic oscillator

$$H = \frac{1}{2}(p^2 + q^2)\,.$$

In fact, D is generated by the Hamiltonian vector field

$$X_H = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}\,.$$

Let $P = D^{\mathbb{C}}$ be the corresponding complex polarization.

There is a natural embedding

$$\Phi : M \to T^*\mathbb{S} = \mathbb{S} \times \mathbb{R}\,,\ (q,p) = p + iq = re^{2\pi it} \mapsto (e^{2\pi it}, \frac{1}{2}r^2)\,,$$

with $\Phi(M) = \{(q,p) \in T^*\mathbb{S} = \mathbb{S} \times \mathbb{R} \mid p > 0\} = \mathbb{S} \times \mathbb{R}_+$. The map Φ preserves the symplectic forms, the polarizations and the connections. Hence, the results of the preceding example show that there are no nonzero polarized sections on M.

As mentioned before we can transfer the arguments of the example to the case of general complex reducible distributions P: There exist severe restrictions to the existence of polarized sections, if the leaves of the induced distribution $D = P \cap \overline{P} \cap TM$ are not simply connected.

Holonomy Group

To explain this general result, we first recall some facts about parallel transport of a connection ∇ on a line bundle L over a general connected manifold X: Given a point $a \in X$ and a closed curve γ in X starting and ending in a the parallel transport along γ is an isomorphism $Q(\gamma) : L_a \to L_a$ given by a complex number which we denote with $Q(\gamma)$ (cf. Proposition 5.17). This number can be expressed as the integral

$$Q(\gamma) = \exp\left(-2\pi i \int_\gamma A\right),$$

where A is a local connection form of the connection. We have $Q(\gamma) \in \mathrm{U}(1)$.

The collection of all these $Q(\gamma)$ forms a group $G(a)$, a subgroup of $\mathrm{U}(1)$. Note, that $G(a)$ is a quotient of the loop group $\mathcal{L}(a)$, see Section 5.4. This group $G(a)$ is called the Holonomy Group of the connection at a. Since we assume X to be connected the holonomy groups are isomorphic to each other: $G(a) \cong G(b)$ for $a, b \in X$. In fact, they are even conjugate subgroups of $\mathrm{U}(1)$.

We have seen already in Observation 8.23, that in the case of a flat connection ∇ we obtain a natural group homomorphism

$$\mathrm{Hol}_\nabla : \pi_1(X) \to G(a)\,, \quad \mathrm{Hol}_\nabla([\gamma]) := Q(\gamma)\,,$$

since parallel transport is locally independent. Hol_∇ is surjective by definition of $G(a)$.

Observation 11.3. Let ∇ be a flat connection on the line bundle $L \to X$ and let X be simply connected. Then $G(a)$ is the trivial group $\{1\}$ for all $a \in X$, since the fundamental group $\pi_1(X)$ of X is trivial and Hol_∇ is surjective. This can also be seen directly by using the fact that each closed curve γ in X can be contracted to the constant curve $\gamma_0(t) = a$ and that the parallel transport $L_a \to L_a$ along γ and along γ_0 coincide since the parallel transport of a flat connection is locally independent.

We come now to the existence of global polarized sections in the general case: Let (M, ω) be a symplectic manifold with a prequantum line bundle (L, ∇, H). Let P a reducible complex polarization, i.e. P is a complex polarization such that with $D = P \cap \overline{P} \cap TM$ the quotient manifold M/D exists and the projection $\pi : M \to M/D$ is a submersion.

We fix a leaf (integral manifold) $\Lambda \subset M$ of the distribution D, i.e. $\Lambda = \pi^{-1}(x)$ for a suitable $x \in M/D$. The restriction of the connection ∇ to Λ induces a flat connection $\nabla|_\Lambda$ on the line bundle $L|_\Lambda \to \Lambda$.

Definition 11.4 (Bohr-Sommerfeld Variety)**.** Let P be a reducible complex polarization and D its induced real distribution. For each leaf Λ of D and $a \in \Lambda$ let $G_\Lambda(a)$ denote the holonomy group of the restricted connection $\nabla|_\Lambda$ on the restriction $L|_\Lambda \to \Lambda$ of the line bundle L. Then

$$S := \bigcup \{a \in M \mid G_\Lambda(a) = \{1\}\}$$

denotes the Bohr-Sommerfeld Variety.

In particular, let Λ be a leaf which is simply connected. Since the connection $\nabla|_\Lambda$ is flat, the holonomy group $G_\Lambda(a)$ has to be trivial: $G_\Lambda(a) = \{1\}$ which implies $\Lambda \subset S$. As a result we obtain the inclusion

$$\{a \in M \mid \Lambda(a) \text{ is simply connected}\} \subset S\,.$$

where $\Lambda(a)$ denotes the leaf through a: $a \in \Lambda(a)$. It follows $S = M$, when all the leaves Λ are simply connected. However, the condition $G_\Lambda(a) = \{1\}$ does not imply that Λ is simply connected as the Example 11.1 shows.

With the same arguments as in the above Example 11.1 we can deduce the following result.

Proposition 11.5. *Any polarized smooth section $s \in \Gamma(M, L)$ vanishes outside the Bohr-Sommerfeld variety.*

Proof. Let s be a polarized section and $a \in M$ with $s(a) \neq 0$. Let Λ be the leaf of the distribution D with $a \in \Lambda$. Then $s|_\Lambda \in \Gamma(\Lambda, L|_\Lambda)$ is a horizontal section of the restriction $\nabla|_\Lambda$ and thus determines the parallel transport $L_a \to L_b$ of $\nabla|_\Lambda$ over Λ. Recall that this means the following. If $\gamma : [0, 1] \to \Lambda$ is a curve in Λ starting in a and ending in $b \in \Lambda$, the parallel transport of $s(a) = s(\gamma(0)) \in L_a$ along γ is determined as $s(\gamma(1)) = s(b) \in L_b$. In case of $a = b$ which we consider here, the parallel transport is the identity, hence $Q(\gamma) = 1$. We conclude that the group $G_\Lambda(a)$ is trivial and therefore $a \in S$. □

As a consequence of this result, for a general reducible complex polarization there exist non-zero polarized global sections only if the Bohr-Sommerfeld variety has a non-empty interior. For instance, if all leaves are simply connected. Therefore, in many cases the space of global polarized sections is zero, and cannot be used to determine a meaningful representation space.

11.2 Distributional Sections

To overcome this difficulty one studies generalized sections, which could be defined in the same way as distributions[57].

Let us describe the introduction of distributional sections in the case of the example of the cylinder.

Example 11.6 (Continuation of Example 11.1)**.** The differential equation (47) allows the solutions

$$\phi_n(q) := \exp(2\pi i q n) = \left(e^{2\pi i q}\right)^n \ , \ q \in [0,1[\,,$$

defined on $S_n = \mathbb{S} \times \{n\}$. For fixed $n \in \mathbb{Z}$ these solutions are unique up to a multiplicative complex constant. We consider the Hilbert space $\ell^2(\mathbb{Z}) := \{(z_n) \in \mathbb{C}^{\mathbb{Z}} \mid \sum |z_n|^2 < \infty\}$ and understand an element $(z_n) \in \ell^2(\mathbb{Z})$ as a generalized function

$$\sum z_n \phi_n \,,$$

which is zero outside of the Bohr-Sommerfeld variety S and where $z_n\phi_n$ represents the corresponding polarized section in $\Gamma(S_n, L_{S_n})$ on the circle S_n. In this interpretation we denote the Hilbert space by $\mathbb{H}_P = \ell^2(\mathbb{Z})$ (with respect to the horizontal polarization P) and we regard $\mathbb{H}_P$ as the representation space of the model.

The height function $h := pr_2 : \mathbb{S} \times \mathbb{R} \to \mathbb{R}\,,\ (q,p) \to p$, is a classical observable whose Hamiltonian vector field $X_h = \frac{\partial}{\partial q}$ generates the horizontal distribution. As a consequence, the operator ∇_{X_h} vanishes on polarized sections and the prequantum operator $\hat{h} := q(h)$ is simply the multiplication $\phi \mapsto h\phi$ on the space of polarized sections. With respect to the representation space H_P this implies that $\hat{h}$ defines a self-adjoint operator in $\mathbb{H}_P$ with domain of definition

$$D(\hat{h}) := \{(z_n) \in \ell^2(\mathbb{Z}) \mid \sum n^2 |z_n|^2 < \infty\}\,,$$

[57] in the sense of generalized functions which are linear functionals, not to mix up with the notion of a distribution in the geometrical sense as a subbundle of the tangent bundle

by $\hat{h}((z_n)) = \sum h(q,n) z_n \phi_n = \sum n z_n \phi_n = (n z_n)$. Therefore, the representation space $\mathbb{H}_P$ decomposes into one dimensional eigenspaces $E_n := \mathbb{C}\phi_n = \operatorname{Ker}(\hat{h} - n)$ with eigenvalues n: $\hat{h}(\phi_n) = n\phi_n$.

Remark 11.7 (Continuation of Remark 11.2)**.** Applied to the Example in Remark 11.2 (energy representation) the introduction of distributional sections leads to the Hilbert space $\mathbb{H}_P = \ell^2(\mathbb{N}_+)$, with $\mathbb{N}_+ := \{n \in \mathbb{N} \mid n > 0\}$. In this situation, $\phi_n = e_n$, $n \in \mathbb{N}_+$ stands for a special distributional polarized section with support in the circle $C_{\sqrt{2n}} := H^{-1}(n) = \{(q,p) \mid q^2 + p^2 = (\sqrt{2n})^2\} \subset M = T^*\mathbb{R}^\times$ of radius $\sqrt{2n}$. And each $(z_n) = \sum z_n \phi_n \in \ell^2$ is a sum of such sections. Notice, that $H = h \circ \Phi$. Moreover, since X_H generates the horizontal distribution P in $M = T^*\mathbb{R}^\times$, the program of geometric quantization yields the prequantum operator $q(H)$ in $\mathbb{H}_P$ as the multiplication operator $\phi \mapsto H\phi$ as before. This multiplication operator, with $H(q,p) = n$ for $(q,p) \in C_{\sqrt{2n}}$, i.e. $q^2 + p^2 = 2n$, becomes a densely defined self-adjoint operator $q(H)$ in $\ell^2(\mathbb{N}_+) = \mathbb{H}_P$:

$$q(H) : (z_n) \mapsto \sum n z_n \phi_n = (n z_n).$$

The eigenspaces of $q(H)$ are $\mathbb{C}\{\phi_n\}, n \in \mathbb{N}_+$, with eigenvalues $n \in \mathbb{N}_+$. In particular, the eigenspaces of $q(H)$ are one dimensional which is in accordance with the known quantum mechanical model, but the eigenvalues are again not correct with the same defect as in the previous Example 10.23. The eigenvalues should be $n - \frac{1}{2}$ instead of n. This can be corrected with the use of half form quantization as is described in the preceding Section 10.5.

However, another correction is possible, which is closer in spirit to the geometry of the Bohr-Sommerfeld condition by choosing an alternative connection for the prequantum bundle. We have seen in Example 8.22 that the non-vanishing of $\check{H}^1(T^*\mathbb{S}, \mathrm{U}(1))$ allows non equivalent connection forms $-\lambda + \kappa dq$ depending on the real parameter $\kappa \in]0,1]$. With respect to the new connection ∇_κ over $T^*\mathbb{S}$ and the corresponding new prequantum bundle there appears a shift by $-\kappa$ in the formation of the Bohr-Sommerfeld variety. Indeed, by (49) it follows as in the case of $\kappa = 0$ that any global polarized section s on $M = \mathbb{S} \times \mathbb{R}$ has to be zero outside of the new Bohr-Sommerfeld variety $S := \bigcup \{S_p \mid p - \kappa \in \mathbb{Z}\} \subset T^*\mathbb{S}$.

The result of the correction in case of $\kappa = \frac{1}{2}$ now reads as follows: The representation space is essentially the same as above, it is $\ell^2(\mathbb{N}_+)$, but now with the interpretation that the polarized sections ϕ_n have their support in the circles $C_{\sqrt{2n-1}} := H^{-1}(n - \frac{1}{2})$. As a result, since $H = n - \frac{1}{2}$ on $C_{\sqrt{2n-1}}$, we obtain

$$q(H)(\phi_n) = \left(n - \frac{1}{2}\right)\phi_n\,, \quad \text{or} \quad q(H)(z_n) = \left(\left(n - \frac{1}{2}\right) z_n\right)$$

for $n \in \mathbb{N}_+$. This result leads again to a decomposition of the representation space $\mathbb{H}_P$ into one dimensional eigenspaces E_n, now with the correct spectrum

$$\sigma(q(H)) = \{n - \frac{1}{2} \mid n \in \mathbb{N}_+\}.$$

Summary:

12 Densities and Their Derivatives

The search for an appropriate representation space is not over. We have introduced complex polarizations $P \subset TM^{\mathbb{C}}$ on a symplectic manifold (M, ω) in Chapter 9 in order to reduce the prequantum Hilbert space but we have seen the effect of this reduction in the preceding chapter so far only in case of the vertical resp. horizontal polarization on the simple phase space $M = T^*\mathbb{R}^n$ (simple phase space) and in case of a Kähler polarization, i.e. a purely complex polarization.

In the following we use the concept of densities on a polarization P in order to obtain a representation space also for the non-Kähler case. In that case the newly constructed representation space is essentially a space of half-densities defined on the quotient manifold M/D, where $D = P \cap \overline{P} \cap TM$.

We have decided to include in these Lecture Notes a rather detailed exposition of densities and of the application of densities to geometric quantization, although this application, the so-called half-density quantization, leaves many problems of geometric quantization open. One reason for this decision is, that the new structure of r-densities on a vector bundle deserves an extra attention, and in the literature the treatment is mostly rather short. Another reason for presenting half-density quantization in detail is, that it can be understood as a preparation for the more involved half-form quantization which we develop later in Chapter 15.

The basic idea of half-density quantization for a quantizable symplectic manifold (M, ω) with prequantum bundle (L, ∇, H) and reducible polarization P is the following: In order to find a scalar product for polarized sections $s, s' \in \Gamma(M, L)$ which we use in order to construct the representation space $\mathbb{H}_P$ we observe that the expression $H(s, s')$ is a function on M which is constant on the leaves of the distribution D (recall: $D = P \cap \overline{P} \cap TM$). Hence, one could try to integrate $H(s, s')$ over the quotient manifold M/D. However, there is no natural measure or volume form on M/D. So 1-densities on M/D come into play since a 1-density on any manifold X can be naturally integrated over X, as we explain in Section 12.2. To use this fact, the expression $H(s, s')$, which can be viewed as to be a function on M/D, has to be transformed into a 1-density on M/D. This can be done by altering the original approach of geometric quantization in such a way that

- the line bundle L will be replaced by a line bundle $L \otimes \delta$ where $\delta \to M$ is a suitable line bundle of densities induced by P[58] which descend to half-densities on M/D and
- the connection ∇ will be replaced by $\nabla \otimes \nabla^{\delta}$, with a flat (partial) connection ∇^{δ} on δ (see Section 12.3).

If this is done properly, one considers the space H_P of sections $\psi = s \otimes \rho$ of $L \otimes \delta$ of compact support, which are polarized, i.e. which satisfy $(\nabla_X \otimes \nabla^{\delta}_X)(s \otimes \rho) = 0$ for

[58]more precisely, δ is the line bundle $\delta_{-1/2}(P)$ of $-1/2$-densities on P.

all $X \in \Gamma(M,P)$. For two such sections $\psi = s \otimes \rho, \psi' = s' \otimes \rho'$ the quantity $H(s,s')\bar{\rho}\rho'$ descends to a unique 1-density on M/D. Integrating this 1-density yields a scalar product on H_P, and the completion of H_P with respect to the induced norm is the representation space $\mathbb{H}_P^\delta = \mathbb{H}^\delta(M,L,P)$ one is looking for.

In this chapter a detailed exposition of the concept of a general r-density is presented. We treat the integration of a 1-density on a manifold and we introduce the partial connection as well as the partial Lie derivative of r-densities on a polarization P as a preparation for the half-density quantization in the following to chapters.

12.1 Densities

Let X be an m-dimensional manifold and let $\pi : V \to X$ be a complex vector bundle of rank k. (In our intended application the manifold X is, in one of the situations, the space M/D of leaves where $D = P \cap \overline{P} \cap TM$ is the real distribution induced by the complex polarization P and V is the complexified tangent bundle $TX^{\mathbb{C}}$, and in another situation $X = M$ and V is the polarization $P \subset TM^{\mathbb{C}}$, the distribution $D^{\mathbb{C}}$, or the quotient bundle $Z^D = TM^{\mathbb{C}}/D^{\mathbb{C}}$.)

The vector bundle V induces a frame bundle $R(V) \to X$ (see Construction D.4): $R(V)$ is a principal fibre bundle over X with structure group $G = \mathrm{GL}(k,\mathbb{C})$ whose fibre $R_x(V)$ over a point $x \in X$ is the set of all ordered bases (frames) of the complex vector space $V_x = \pi^{-1}(x)$, and whose transition functions are the g_{ij}, the transition functions determining the vector bundle V. The right action of $G = \mathrm{GL}(k,\mathbb{C})$ on $R(V)$

$$R(V) \times G \to R(V)$$

is given by

$$(bg)_\alpha := b_\beta g_\alpha^\beta, \quad (1 \le \alpha, \beta \le k)\,,$$

for a matrix $g = (g_\alpha^\beta) \in G$ and for $b = (b_1, b_2, \dots, b_k) \in R_x(V)$. Then $bg = ((bg)_1, \dots, (bg)_k) \in R_x(V)$ is another frame of V_x. Once a frame $b \in R_x(V)$ at $x \in X$ has been chosen, the map $G \to R_x(V), g \mapsto bg$, is bijective.

To motivate the notion of a density let us consider the top differential forms on a manifold X of dimension m. Any complex m-form $\eta \in \mathcal{A}^m(X)$ is a section of the complex line bundle $\Lambda^m(T^*X^{\mathbb{C}})$ and it can be evaluated at a complex frame $b \in R_x(TX^{\mathbb{C}})$, $x \in X$, to yield the value $\eta_x(b) \in \mathbb{C}$. In this sense η induces a (smooth) map $\eta^\sharp : R(TX^{\mathbb{C}}) \to \mathbb{C}$, $b \to \eta_x(b)$. This map $\eta^\sharp$ has the following transformation property:

$$\eta^\sharp(bg) = (\det g)\eta^\sharp(b)\,,\ b \in R(TX^{\mathbb{C}})\,,\ g \in \mathrm{GL}(m,\mathbb{C})\,. \tag{50}$$

Conversely, a map $w : R(TX^{\mathbb{C}}) \to \mathbb{C}$ with

$$w(bg) = (\det g)w(b)\,,\ b \in R(TX^{\mathbb{C}}), g \in \mathrm{GL}(m,\mathbb{C})\,,$$

defines an m-form η on X with $\eta^\sharp = w$.

Definition 12.1 (Density). Let $\pi : V \to X$ be a complex vector bundle of rank k and let r be a real number. A DENSITY[59] of weight r, also called r-density, on V is a function

$$u : R(V) \to \mathbb{C}\,,$$

$u \in \mathcal{E}(R(V),\mathbb{C})$[60], which transforms under $\mathrm{GL}(k,\mathbb{C})$ according to

$$u(bg) = |\det g|^r u(b)\,,\ g \in \mathrm{GL}(k,\mathbb{C})\,,\ b \in R(V)\,.$$

The $\mathcal{E}(M)$-module of r-densities $u : R(V) \to \mathbb{C}$ on V will be denoted by $\Delta_r(V)$[61].

An r-density on $TX^{\mathbb{C}}$ is also called an r-density on X and we write $\Delta_r(X) := \Delta_r(TX^{\mathbb{C}})$.

A slightly different but equivalent definition in case of $r = 1$ will be given at the beginning of the following Section 12.2 where the integration of 1-densities is introduced.

A straightforward example is the 1-density $|\eta|$ on V induced by a k-form $\eta \in \Gamma(M,\Lambda^k(V^\vee)) \cong \bigwedge^k(\Gamma(M,V),\mathcal{E}(M))$, where $V^\vee$ is the dual bundle of V and where $\Lambda^k(V^\vee)$ is the vector bundle of k-linear alternating forms on V. For an alternating and k-linear $\eta : \Gamma(M,V)^k \to \mathcal{E}(M)$ and a basis $(b_1, b_2, \dots b_k)$ of V_x , $x \in X$, one sets

$$|\eta|(b_1, b_2, \dots, b_k) := |\eta_x(b_1, b_2, \dots, b_k)|\,.$$

Then $|\eta|$ is a 1-density on V because of $\eta_x((bg)_1, \dots, (bg)_k) = (\det g)\,\eta_x(b_1, b_2, \dots, b_k)$, and thus

$$|\eta|(bg) = |\eta(bg)| = |\det g\eta(b)| = |\det g||\eta|(b)\,.$$

This example will be used, in particular, in the case of $V = TX^{\mathbb{C}}$ where $\Gamma(M,\Lambda^k(T^*X^{\mathbb{C}})) \cong \mathcal{A}^k(X)$ $(k = m = \dim M)$ is the $\mathcal{E}(X)$-module of differential k-forms.

Moreover, $|\eta|^r$ is an r-density on V for $r \in \mathbb{R}$.

Densities can be multiplied, conjugated, dualized, and more. These properties of densities can be best understood by transforming densities into sections of suitable line bundles.

Definition 12.2. Let $\pi : V \to X$ be a complex vector bundle and let r be a real number. The line bundle $R(V) \times_{\rho_r} \mathbb{C}$ over X associated to the frame bundle $R(V) \to X$ with respect to the representation $\rho_r : \mathrm{GL}(k,\mathbb{C}) \to \mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^\times$, $\rho_r(g) := |\det g|^{-r}$, is called the bundle of r-densities on V and denoted by $\delta_r(V)$.

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[59] in physics a density is sometimes called a pseudoscalar

[60] In the following, we mostly omit $\mathbb{C}$ in expressions like $\mathcal{E}(X,\mathbb{C})$ and write $\mathcal{E}(X)$, as before.

[61] This definition describes complex densities. In the same way one defines real r-densities on a real vector bundle. Moreover, by choosing X to be the manifold consisting of one point, we obtain the definition of an r-density on a vector space.

The concept of an associated vector bundle is explained in Section D.3. Recall, that the line bundle $R(V) \times_{\rho_r} \mathbb{C}$ is the quotient of $R(V) \times \mathbb{C}$ with respect to the equivalence relation

$$(b, z) \sim (bg, |\det g|^r z) = (bg, \rho_r(g^{-1})z)\,, \quad \text{for } g \in \mathrm{GL}(k, \mathbb{C})\,,$$

where $(b, z) \in R(V) \times \mathbb{C}$.

Remark 12.3. In particular, by Proposition D.7, we know that the sections $s \in \Gamma(U, \delta_r(V))$ on an open subset $U \subset X$ are equivalently described by the functions $u : R(V)_U \to \mathbb{C}$ transforming according to

$$u(bg) = |\det g|^r u(b)\,,$$

$b \in R(V)_U\,, g \in \mathrm{GL}(k, \mathbb{C})$. Hence, $\Delta_r(V)$, the $\mathcal{E}(M)$-module of r-densities on V, can be identified with the $\mathcal{E}(M)$-module of global sections $\sigma \in \Gamma(X, \delta_r(V))$ of the line bundle $\delta_r(V)$, and it is justified to call $\delta_r(V)$ the line bundle of r-densities. To simplify formulas, we often write $\Gamma_r(V)$ instead of $\Gamma(X, \delta_r(V))$ in the following.

We recall the isomorphism $\Delta_r(V) \cong \Gamma_r(V) = \Gamma(X, \delta_r(V))$ in detail by using the construction of the associated line bundle: Every $u \in \Delta_r(V)$ induces a section

$$s_u : X \to \delta_r(V) \text{ by } s_u(a) := [(b, u(b))]\,, b \in R_a(V)\,, a \in X.$$

The map s_u is well-defined: The elements of the fibre $\pi^{-1}(a)$ of the line bundle $\pi : R(V) \times_{\rho_r} \mathbb{C} \to X$ over $a \in X$ are the equivalence classes $[(b, z)]$ of the form $[(b, z)] = \{(bg, |\det g|^r z) \mid g \in \mathrm{GL}(k, \mathbb{C})\}$ with $b \in R_a(V)$. Hence,

$$(bg, u(bg)) = (bg, |\det g|^r u(b)) \sim (b, u(b))\,,$$

which shows that the assignment $s_u(a) = [(b, s(u))]$ is independent of the choice of $b \in R_a(V)$. Therefore, s_u is a well-defined global section of $\delta_r(V)$. Moreover, $u \mapsto s_u$, is $\mathcal{E}(X)$-linear and injective.

Conversely, a section $s \in \Gamma(X, \delta_r(V))$ defines an r-density $s^\sharp$: Let $b \in R(V)$ with $\pi(b) = a \in X$. Then $s(a) \in \delta_r(V)_a$ as the equivalence class $s(a) = [(b', z')]$ contains exactly one pair of the form $(b, z) \in s(a)$. Define $s^\sharp(b) := z$. Then $s^\sharp : R(V) \to \mathbb{C}$ is well-defined and satisfies $s^\sharp(bg) = |\det g|^r s^\sharp(b)$, since the unique $\zeta \in \mathbb{C}$ with $(bg, \zeta) \in s(a)$ is $\zeta = |\det g|^r z$: $(b, z) \sim (bg, |\det g|^r z)]$, hence, $s^\sharp(bg) = \zeta = |\det g|^r z$. Clearly, $(s_u)^\sharp = u$ and $(s^\sharp)_u = s$. Moreover, $s \mapsto s^\sharp$ is $\mathcal{E}(X)$-linear, and hence an isomorphism $\Gamma(X, \delta_r(V) \cong \Delta_r(V)$.

From Proposition D.9 we obtain

Fact 12.4. *Let g_{jk} be transition function for the vector bundle V with respect to an open cover (U_j) of X. Then $|\det g_{jk}|^{-r}$ are suitable transition functions for $\delta_r(V)$.*

Lemma 12.5. *Any line bundle $\delta_r(V)$ of r-densities on a complex vector bundle V over X is trivial. There exists a positive r-density which induces a a vector bundle isomorphism $\delta_r(V) \to X \times \mathbb{C}$.*

Proof. It is enough to determine a nowhere vanishing section $s = s_w \in \Gamma(X, \delta_r(V)$ given by a non-vanishing r-density $w : R(V) \to \mathbb{C}$: Let $(b_1, \ldots, b_k)$ be a fixed basis of $\mathbb{C}^k$. There is an open cover (U_j) of X with local trivializations $\psi_j : V_{U_j} \to U_j \times \mathbb{C}^k$ of the vector bundle V. On U_j the local sections $b_\kappa(x) := \psi_j^{-1}(x, b_\kappa)$, $x \in U_j$, $\kappa = 1, \ldots, k$, form a frame field $\hat{b}_j : U_j \to R(V)_{U_j}$ given by $\hat{b}_j(x) = (b_1(x), \ldots, b_k(x)) \in R_x V|_{U_j}$. Now, $w_j(\hat{b}_j g) := |\det g|^r$, $g \in \mathrm{GL}(k, \mathbb{C})$, defines a positive r-density w_j on the restriction V_{U_j} satisfying $w_j(\hat{b}_j) = 1$. Hence, with a smooth partition of unity (h_j) subordinated to the cover (U_j) we obtain a positive r-density

$$w := \sum h_j w_j : R(V) \to \mathbb{C}\,.$$

□

As a result, all density line bundles on a vector bundle V over X are the same from the viewpoint of isomorphism classes of line bundles.

Observation 12.6. Up to scaling, the line bundle $\delta_r(V)$ is determined completely by choosing one nowhere vanishing r-density $u \in \Delta_r(V)$: Then every $w \in \Delta_r(V) \cong \Gamma(X, \delta_r(V))$ is of the form $w = \lambda u$ with a unique $\lambda \in \mathcal{E}(X)$.

In particular, $\Delta_r(V)$ is an $\mathcal{E}(M)$-module of rank 1 and is isomorphic to $\mathcal{E}(M)$.

Using this result we obtain, by choosing a nowhere vanishing and positive 1-density u on V, an isomorphism of complex line bundles $\Delta_1(V) \to \Delta_r(V)$, $fu \to fu^r$, resp. $\delta_1(V) \cong \delta_r(V)$. This isomorphism is not of great interest, it depends on the choice of u.

Nevertheless, certain natural isomorphisms are of interest. The rest of this section is not directly needed in the sequel, it is only an elementary and detailed investigation around the new notion of r-density bundles and gives an impression how to work with them:

Proposition 12.7. *For complex vector bundles V, W, Z over the manifold X and $r, s \in \mathbb{R}$ there exist the following natural isomorphisms of density bundles:*

1. $\delta_r(V) \otimes \delta_s(V) \cong \delta_{r+s}(V)$
2. $(\delta_r(V))^\vee \cong \delta_r(V^\vee) \cong \delta_{-r}(V)$
3. $\delta_r(V \oplus Z) \cong \delta_r(V) \otimes \delta_r(Z)$
4. $\delta_r(W) \otimes \delta_{-r}(V) \cong \delta_r(Z)$ *and, equivalently,* $\delta_r(W) \cong \delta_r(V) \otimes \delta_r(Z)$ *if*

$$0 \longrightarrow V \longrightarrow W \longrightarrow Z \longrightarrow 0$$

is an exact sequence of vector bundles.

The isomorphisms, and hence their "naturality"[62] *are described in the proofs.*

Proof. Short proofs can be given by using the transition functions. For instance, in case 1., the transition functions of $\delta_t(V)$ for general t are $|\det g_{ij}|^{-t}$ when g_{ij} are the transition functions of the vector bundle V. Hence, the transition functions of $\delta_r(V) \otimes \delta_s(V)$ are $|\det g_{ij}|^{-r}|\det g_{ij}|^{-s} = |\det g_{ij}|^{-(r+s)}$. This equality implies that $\delta_r(V) \otimes \delta_s(V)$ and $\delta_{r+s}(V)$ are isomorphic as line bundles.

However, we present proofs which use the properties of densities directly in order to give a detailed impression about how one can work with r-densities on vector bundles which are interrelated.

Ad 1.: For a basis b of V_x, $x \in X$, i.e. $b \in R_xV$, and for $\sigma \in \Gamma_r(V)$, $\tau \in \Gamma_s(V)$ we define

$$(\sigma.\tau)^\sharp(b) := \sigma^\sharp(b)\tau^\sharp(b).$$

For $g \in \mathrm{GL}(k,\mathbb{C})$, $k = \mathrm{rk}\, V$, the following transformation property is satisfied:

$$(\sigma.\tau)^\sharp(bg) = \sigma^\sharp(bg)\tau^\sharp(bg) = |\det g|^r \sigma^\sharp(b)|\det g|^s \tau^\sharp(b) = |\det g|^{r+s}(\sigma.\tau)^\sharp(b)\,.$$

Hence, $\sigma.\tau$ is a well-defined $r+s$-density on V. And it is easy to check that the induced map $\sigma \otimes \tau \mapsto \sigma.\tau$ is an isomorphism of line bundles.

Ad 2.: For $b \in R_xV$ let $b^\vee$ be the dual basis $b^\vee \in R_xV^\vee$ with $b_j^\vee(b_k) = \delta_{jk}$. Then $\delta_r(V^\vee) \cong \delta_{-r}(V)$, $\tau \mapsto \sigma_\tau$, is given by

$$(\sigma_\tau)^\sharp(b) := \tau^\sharp(b^\vee)$$

for $\tau \in \delta_r(V^\vee)$. Indeed, for $g \in \mathrm{GL}(k,\mathbb{C})$ we have

$$(\sigma_\tau)^\sharp(bg) = \tau^\sharp((bg)^\vee) = \tau^\sharp(b^\vee g^{-1}) = \tau^\sharp(b)|\det g^{-1}|^r = (\sigma_\tau)^\sharp(b)|\det g|^{-r}\,.$$

Hence, $\sigma_\tau \in \Gamma_{-r}(V)$.

Ad 3.: If $c \in R_xV$, $d \in R_xZ$ then $b = (c,d) \in R_x(V \oplus Z)$. The isomorphism $\delta_r(V) \otimes \delta_r(Z) \to \delta_r(V \oplus Z)$ is given by $\sigma \otimes \tau \mapsto \sigma \cdot \tau$ where

$$(\sigma \cdot \tau)^\sharp(b) := \sigma^\sharp(c)\tau^\sharp(d).$$

This is well defined, since for another choice of (c',d') (instead of $(c,d) = b$) there are unique $g \in \mathrm{GL}(k,\mathbb{C})$ and $h \in \mathrm{GL}(m,\mathbb{C})$ ($m = \dim Z$) such that $c' = cg$, $d' = dh$. Therefore, with

$$G = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}$$

[62]The results of this proposition may be described by stating that the δ_r (or Δ_r), $r \in \mathbb{R}$ define a family of functors satisfying certain natural properties.

we obtain

$$\begin{aligned}(\sigma \cdot \tau)^\sharp(bG) &= (\sigma \cdot \tau)^\sharp(cg, dh) \\ &= \sigma^\sharp(cg)\tau^\sharp(dh) \\ &= \sigma^\sharp(c)|\det g|^r \tau^\sharp(d)|\det h|^r \\ &= (\sigma^\sharp(c)\tau^\sharp(d))|\det G|^r\end{aligned}$$

Now $\sigma \cdot \tau$ has to be extended to all of $R_x(V \oplus Z)$ by the transformation rule $(\sigma \cdot \tau)^\sharp(bG) = (\sigma \cdot \tau)^\sharp(b)|\det G|^r$ for all $G \in \mathrm{GL}(k+m, \mathbb{C})$ to become an r-density on V. The induced map is an isomorphism of line bundles.

Moreover, 3. is a special case of 4.

Ad 4.: We cannot apply 3. directly, since W is not necessarily isomorphic to the direct sum $V \oplus Z$, although this is locally true. We omit the base point "$_x$" in the following.

Let $(Z_1, \ldots, Z_m)$ be a basis of Z, let $c = (X_1, \ldots, X_k)$ be a basis of V and let $b = (X_1, \ldots, X_k, Y_1, \ldots Y_m)$ a basis of W such that $p(Y_j) = Z_j$ with respect to the projection $p : W \to Z$ in the exact sequence. Let $\sigma \in \delta_{-r}(V)$, $\varepsilon \in \delta_r(W)$. Define $\nu = \nu(\sigma, \varepsilon)$ by

$$\nu^\sharp(Z_1, \ldots, Z_m) := \varepsilon^\sharp(X_1, \ldots, X_k, Y_1, \ldots Y_m)\sigma^\sharp(X_1, \ldots X_k) = \varepsilon^\sharp(b)\sigma^\sharp(c)\,.$$

The choice of another basis $b' = (X'_1, \ldots, X'_k, Y'_1, \ldots Y'_m)$ such that $(X'_1, \ldots, X'_k)$ is in $R(V)$ and $p(Y'_j) = Z_j$, $j = 1, \ldots, m$, satisfies $c' = ch$ with a unique $h \in \mathrm{GL}(k, \mathbb{C})$ and $Y'_j = Y_j + X_\beta d^\beta_j$ for suitable $d^\beta_j \in \mathbb{C}$, $d = (d^\beta_j)$. As a consequence, b' is of the form $b' = bg$, where

$$g = \begin{pmatrix} h & d \\ 0 & 1 \end{pmatrix}$$

with $\det g = \det h$. Hence,

$$\varepsilon^\sharp(b')\sigma^\sharp(c') = |\det g|^r|\det h|^{-r}\varepsilon^\sharp(b)\sigma^\sharp(c) = \varepsilon^\sharp(b)\sigma^\sharp(c)\,,$$

and $\nu^\sharp(Z_1, \ldots, Z_m)$ is well-defined. In order to show that ν is an r-density let $g \in \mathrm{GL}(m, \mathbb{C})$. With the notation $z := (Z_1, \ldots, Z_k)$ the following transformation property holds true: $\nu^\sharp(zg) = \varepsilon^\sharp(X_1, \ldots, X_k, (Y_1, \ldots Y_m)g)\sigma^\sharp(c) = |\det g|^r \varepsilon^\sharp(X_1, \ldots, X_k, Y_1, \ldots Y_m)\sigma^\sharp(c) = |\det g|^r \nu^\sharp(z)$, so that $\nu \in \delta_r(Z)$. Finally, it is straightforward to check that the map

$$\delta_r(W) \otimes \delta_{-r}(V) \to \delta_r(Z)\,,\ \sigma \otimes \varepsilon \mapsto \nu = \nu(\sigma, \varepsilon)$$

is an isomorphism of line bundles. □

Remark 12.8. If we choose a fixed nowhere vanishing r-density $\epsilon \in \Gamma_r(W)$ in the last part of the proof above, the definition of $\nu \in \Gamma_r(Z)$ leads directly to the definition of a line bundle isomorphism $\nu : \delta_{-r}(V) \to \delta_r(Z)\,,\ \sigma \mapsto \nu(\sigma)$, by setting

$$\nu(\sigma)^\sharp(z) := \varepsilon^\sharp(b)\sigma^\sharp(c)\,.$$

This kind of isomorphisms of density bundles will be used later in several occasions.

12.2 Integration of Densities

Densities occur in the context of integration on a manifold X. A traditional definition is the following. Let X be a manifold of dimension m. A density is a rule that assigns to each chart $q : U \to V$ of a differentiable atlas of X a function $g_q \in \mathcal{E}(V, \mathbb{R})$, such that for any other chart $\bar{q} : \bar{U} \to \bar{V}$ with the following transition property is satisfied: $U \cap \bar{U} \neq \emptyset$

$$g_q = \left| \det \left(\frac{\partial \bar{q}}{\partial q} \right) \right| g_{\bar{q}} \, .$$

The concept agrees essentially with the notion of a 1-density $u \in \Delta_1(TX^{\mathbb{C}})$ in the sense of Definition 12.1 whenever u is real-valued, i.e. $u(b) \in \mathbb{R}$ for $b \in R(TX)$[63]. For such a density u, the assignment

$$g_q^u = u \left(\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n} \right) =: u \left(\frac{\partial}{\partial q} \right) \in \mathcal{E}(V)$$

for a chart $q : U \to V$ defines a density in the traditional way, since

$$u \left(\frac{\partial}{\partial q} \right) = u \left(\frac{\partial \bar{q}}{\partial q} \frac{\partial}{\partial \bar{q}} \right) = \left| \det \frac{\partial \bar{q}}{\partial q} \right| u \left(\frac{\partial}{\partial \bar{q}} \right) ,$$

by the definition of a 1-density u.

Conversely, a traditional density (g_q) defines a 1-density u through

$$u(x) := g_q(q(x)) |dq^1 \wedge \ldots \wedge dq^m| \, , \; x \in U \, .$$

u is well-defined because of the transition property, and the map $u \mapsto (g_q^u)$ is one-to-one.

In the following we recall how integral of a density on a general manifold X.

Definition 12.9 (Integration)**.** Let $w : R(TX^{\mathbb{C}}) \to \mathbb{C}$ be a 1-density. Leaving aside convergence problems the integral $\int_X w \in \mathbb{C}$ is defined as follows. With respect to a chart $q : U \to V$ of X , $V \subset \mathbb{R}^m$ open, the integral $\int_U w$ of w over U is

$$\int_U w := \int_V w \left(\frac{\partial}{\partial q} \right) dq \, ,$$

where dq denotes Lebesgue measure on $V \subset \mathbb{R}^m$ and

$$w \left(\frac{\partial}{\partial q} \right) (q) = w \left(\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^m} \right) (q) = w \left(\frac{\partial}{\partial q^1} |_x , \ldots , \frac{\partial}{\partial q^m} |_x \right)$$

with $q(x) = q \in V$, $x \in U$. The result is independent of the chart because of the change of variables formula for the Lebesgue measure dq and the transformation property of

[63]which means that u is a (real) 1-density on TX.

w: Both change with the absolute value of the Jacobian, but in an inverse manner to each other. When $\bar{q} : U \to \bar{V}$ is another chart and $q = q(\bar{q}) : V \to \bar{V}$ the induced change of coordinates, we have (schematically)

$$dq = \left|\det\left(\frac{\partial q}{\partial \bar{q}}\right)\right| d\bar{q}$$

and

$$w\left(\frac{\partial}{\partial q}\right) = w\left(\frac{\partial}{\partial \bar{q}}\frac{\partial \bar{q}}{\partial q}\right) = w\left(\frac{\partial}{\partial \bar{q}}\right)\left|\det\left(\frac{\partial \bar{q}}{\partial q}\right)\right| = w\left(\frac{\partial}{\partial \bar{q}}\right)\left|\det\left(\frac{\partial q}{\partial \bar{q}}\right)\right|^{-1}.$$

Hence,

$$\begin{aligned}\int_V w\left(\frac{\partial}{\partial q}\right) dq &= \int_{\bar{V}} w\left(\frac{\partial}{\partial q}\right)\left|\det\left(\frac{\partial q}{\partial \bar{q}}\right)\right| d\bar{q} \\ &= \int_{\bar{V}} w\left(\frac{\partial}{\partial \bar{q}}\right)\left|\det\left(\frac{\partial q}{\partial \bar{q}}\right)\right|^{-1}\left|\det\left(\frac{\partial q}{\partial \bar{q}}\right)\right| d\bar{q} \\ &= \int_{\bar{V}} w\left(\frac{\partial}{\partial \bar{q}}\right) d\bar{q}\,.\end{aligned}$$

The integral is extended to all of X by a partition of unity subordinated to a covering (U_j) with charts $q_j : U_j \to V_j$.

The integral is well-defined, even if X is not oriented. Recall, that the integration of a volume form η on an oriented manifold X is similarly defined: In that case an orientation is chosen meaning that we have an atlas with positive coordinate changes f_{ij} i.e. with $\det Df_{ij} > 0$. Then locally

$$\int_{U_j} \eta = \int_{V_j} \eta\left(\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^m}\right) dq\,.$$

As a consequence

$$\int_X \eta = \int_X |\eta|\,.$$

Remark 12.10. With the aid of densities we not only have a natural integration on a manifold X but we also can define natural L^p-spaces $L^p(X)$ of $1/p$-densities, where $p \in \mathbb{R}$, $1 \le p < \infty$: On the space $\Delta_{1/p}(TX^{\mathbb{C}})_c$ of $1/p$-densities u on X with compact support in X one defines the norm

$$\|u\|_p := \left(\int_X |u|^p\right)^{\frac{1}{p}},$$

and $L^p(X)$ as the completion of the normed space $(\Delta_{1/p}(TX^{\mathbb{C}})_c, \|\ \|_p)$.

In particular, we obtain the natural Hilbert space $L^2(X)$ as the (completion of the) space of half-densities u on X such that $\int_X |u|^2 < \infty$. The scalar product is $\langle u, v\rangle = \int_X \bar{u}v$. This Hilbert space is sometimes called the canonical Hilbert space of the manifold X.

We conclude the section with some remarks and results on orientable manifolds. These results are not needed for for the half-density quantization.

Definition 12.11 (Orientation Bundle). Let X be an m-dimensional manifold with the frame bundle $R(TX)$ which is a principal fibre bundle with structure group $\mathrm{GL}(m,\mathbb{R})$. The ORIENTATION BUNDLE is the real(!) line bundle $O = O(X)$ associated to $R(TX)$ using the representation $\sigma : \mathrm{GL}(m,\mathbb{R}) \to \mathrm{GL}(1,\mathbb{R}) = \mathbb{R}^\times$, $g \mapsto \operatorname{sgn}\det g$, i.e. with respect to the left action $\mathrm{GL}(m,\mathbb{R}) \times \mathbb{C} \to \mathbb{C}$, $(g,z) \mapsto (\operatorname{sgn}\det g)z$.

Another description of the orientation bundle is the following: Let (U_j) be an open covering of X with differentiable charts $\varphi_j : U_j \to V_j$. Denote $f_{ij} := \varphi_i \circ \varphi_j^{-1}$: $\varphi_j(U_{ij}) \to \varphi_i(U_{ij})$ the change of coordinates. Let O' be the line bundle which arises by glueing the $U_i \times \mathbb{C}$ with respect to the cocycle $s_{ij} := \operatorname{sgn}\det Df_{ij}$. O' is isomorphic to O. Indeed, with the aid of the given charts φ_j the frame bundle $R(TX)$ has as transition functions the $g_{ij} := Df_{ij}$[64]. Hence, the transition functions of the associated line bundle O (see Section D.3) are $\sigma(g_{ij}) = \operatorname{sgn}\det g_{ij} = s_{ij}$, and the two bundles are naturally equivalent (note, that $s_{ij}^{-1} = s_{ij}$).

Proposition 12.12. *X is orientable if and only if O is trivial.*

Proof. If X is orientable, then, by definition, there exists an atlas of the differentiable structure such that the changes of charts (coordinate changes) f_{ij} are all positive in the sense that there Jacobians $g_{ij} := Df_{ij}$ have positive determinants. Therefore the $s_{ij} = \operatorname{sgn}\det g_{ij}$ are all 1. This implies that O is trivial, a trivialization is induced by the section t given by local functions $t_j(x) = 1 , x \in U_j , j \in I$, respecting the section condition (S): $t_i = s_{ij}t_j$. Conversely, if O has a trivialization there is an atlas such that the transition functions can be chosen to be $s_{ij} = 1 = \operatorname{sgn}\det Df_{ij}$ and this means that the Df_{ij} have positive determinants. □

Notice, that for an orientable manifold X an orientation is given by a global nowhere vanishing section $t \in \Gamma(X,O)$.

Remark 12.13. If, in the case of $r = 1$, one changes the above definition for the bundle of r-densities by replacing the representation $\rho_1(g) = |\det g|^{-1}$ with the representation $\tau(g) = (\det g)^{-1}$, one obtains the line bundle $\Lambda^m T^*X^{\mathbb{C}}$ of complex m-forms on X up to isomorphism (cf. (50)). This line bundle is also called the determinant bundle of the cotangent bundle T^*X or the canonical bundle of X (see Section 15.1 for the concept of a canonical line bundle).

Proposition 12.14. *$\delta_1(TX) \cong \Lambda^m(T^*X) \otimes O$ as real vector bundles.*

Proof. T^*X is the dual $TX^\vee$ of TX. Let (U_j) be an open covering of X with charts $q_j : U_j \to V_j$ of the differentiable structure. The Jacobians $g_{ij} := Df_{ij}$ of the coordinate

[64] since the g_{ij} are the transition functions of the tangent bundle TX.

changes $f_{ij} := q_i \circ q_j^{-1}|_{q_j(U_i \cap U_j)}$ are transition functions defining the structure on the tangent bundle. Hence, the transition functions of $\Lambda^m(T^*X)$ are the $\det g_{ij}^{-1}$, those of $\delta_1(X)$ are the $|\det g_{ij}|^{-1}$, and those of the orientation bundle are $\operatorname{sgn}\det g_{ij}$. The isomorphism now follows from

$$|\det g_{ij}|^{-1} = \det g_{ij}^{-1} \cdot \operatorname{sgn}\det g_{ij} .$$

□

Corollary 12.15. *X is oriented if and only if there exists a volume form, i.e. a nowhere vanishing m-form.*

12.3 Partial Connection

We need the concept of partial connection on the line bundle of densities $\delta_r(P)$ on a polarization P. We formulate in this section a general approach suitable for densities on polarizations on an arbitrary symplectic manifold which can be generalized to half-forms (cf. Chapter 15). The partial connection will be used in Chapter 14. In the next chapter, Chapter 13, we need only a simplified version of the partial connection.

Let (M, ω) be a symplectic manifold of dimension $2n$, and let $P \subset TM^{\mathbb{C}}$ be a complex polarization. We want to define a partial connection on the line bundle $\delta_r(P)$ of r-densities on P (and similarly on $\delta_r(D^{\mathbb{C}})$, cf. end of this section, where $D = P \cap \overline{P} \cap TM$). The partial connection ∇_X will only be defined for vector fields $X \in \Gamma(M, P) \subset \mathfrak{V}(M)$, and not for general vector fields on M.

Let $U \subset M$ be an open subset U in M and let $\xi_j \in \Gamma(U, P)$, $1 \leq j \leq n$, local vector fields such that $(\xi_1(a), \ldots, \xi_n(a))$ is a basis of P_a for all $a \in U$. Then $\xi := (\xi_1, \ldots, \xi_n)$ yields a local section $\xi : U \to R(P)$ of the frame bundle $R(P)$ of P, which is also called a frame field.

For instance, ξ_j could be generated by n independent functions $F_1, F_2, \ldots, F_n \in \mathcal{E}(U, \mathbb{C})$ as the Hamiltonians $\xi_j = X_{F_j}$. In that case we call ξ a Hamiltonian frame field. It is clear that to every $a \in M$ there exists an open neighbourhood $U \subset M$ such that there exists a Hamiltonian frame field $\xi \in \Gamma(U, R(P))$.

Given a frame field ξ we define an r-density

$$\sigma_\xi \in \Gamma(U, \delta_r(P)) \ \text{ by } \ \sigma_\xi(a) := [(\xi(a), 1)] , \, a \in U .$$

It is easy to see that σ_ξ is the unique section $U \to \delta_r(P)$ with $\sigma_\xi^\sharp \circ \xi = 1$. In particular, σ_ξ is nowhere vanishing. Any other section $\sigma \in \Gamma(U, \delta_r(P))$ has a presentation as

$$\sigma = (\sigma^\sharp \circ \xi)\, \sigma_\xi ,$$

since for $\sigma = f\sigma_\xi$ and $a \in U$:

$$\sigma^\sharp(\xi)(a) = (f\sigma_\xi)^\sharp(\xi)(a) = f(a)\sigma_\xi^\sharp(\xi)(a) = f(a) .$$

Now let ξ be a Hamiltonian frame field. The connection to be defined shall have the property that the covariant derivatives of r-densities $\sigma \in \Gamma(U, \delta_r(P))$ should vanish when the function $\sigma^\sharp \circ \xi : U \to \mathbb{C}$ is covariantly constant along P in the sense that $L_X(\sigma^\sharp \circ \xi) = 0$ for all $X \in \Gamma(U, P)$. This requirement leads to the following definition.

Definition 12.16. Let let ξ be a Hamiltonian frame field. For a vector field $X \in \Gamma(U, P)$ and $\sigma = (\sigma^\sharp \circ \xi)\, \sigma_\xi \in \Gamma(U, \delta_r(P))$ one defines

$$\nabla_X \sigma := L_X(\sigma^\sharp \circ \xi)\, \sigma_\xi\,.$$

In particular, $\nabla_X \sigma_\xi = 0$.

Lemma 12.17. *The definition $\nabla_X \sigma$, where $X \in \Gamma(U, P)$ and $\sigma_\xi \in \Gamma(U, \delta_r(P))$, is independent of the Hamiltonian frame field ξ and extends to all of M.*

Proof. Any other Hamiltonian frame field $\xi' : U \to R(P)$ is of the form $\xi' = \xi g$ where $g : U \to \mathrm{GL}(n, \mathbb{C})$ is a smooth map. Because of

$$\sigma_{\xi g} = \sigma^\sharp_{\xi g}(\xi)\sigma_\xi \quad \text{and} \quad 1 = \sigma_{\xi g}(\xi g) = |\det g|^r \sigma_{\xi g}(\xi)$$

we have $\sigma_{\xi g} = |\det g|^{-r}\sigma_\xi$. Moreover, $\sigma^\sharp(\xi g) = |\det g|^r \sigma^\sharp(\xi)$, since $\sigma^\sharp$ is an r-density. Comparing now $L_X(\sigma^\sharp \circ \xi)\, \sigma_\xi$ with

$$\begin{aligned} L_X(\sigma^\sharp \circ \xi g)\, \sigma_{\xi g} &= L_X(\sigma^\sharp \circ \xi g)\, |\det g|^{-r}\sigma_\xi \\ &= L_X((\sigma^\sharp \circ \xi)|\det g|^r)\, |\det g|^{-r}\sigma_\xi \\ &= L_X(\sigma^\sharp \circ \xi)\, \sigma_\xi + (\sigma^\sharp \circ \xi) L_X(|\det g|^r)\, |\det g|^{-r}\sigma_\xi \end{aligned}$$

it remains to show $L_X(|\det g|^r) = 0$ in order to obtain

$$L_X(\sigma^\sharp \circ \xi)\, \sigma_\xi = L_X(\sigma^\sharp \circ \xi g)\, \sigma_{\xi g}.$$

It is enough to show $L_X g^\beta_\alpha = 0$ for all $1 \le \alpha, \beta \le n$, and $X \in \Gamma(U, P)$, i.e. the components g^β_α of the map $g = (g^\beta_\alpha) : U \to \mathrm{GL}(n, \mathbb{C})$ are "covariantly constant along P". For the moment, let us assume that X is Hamiltonian: $X = X_F \in \Gamma(U, P)$. Then

$$[X, \xi'_\alpha] = 0 \quad \text{and} \quad [X, \xi_\beta] = 0.$$

Using the Lie derivative[65] of vector fields, this is the same as

$$L_X \xi'_\alpha = 0 \quad \text{and} \quad L_X \xi_\beta = 0. \tag{51}$$

As a consequence

$$0 = L_X \xi'_\alpha = L_X(\xi_\beta g^\beta_\alpha) = (L_X \xi_\beta) g^\beta_\alpha + \xi_\beta (L_X g^\beta_\alpha) = \xi_\beta (L_X g^\beta_\alpha)\,.$$

[65] see Proposition A.24 for the Lie derivative $L_X Y = [X, Y]$ of a vector field Y along a vector field X and some of its properties.

This implies $L_X g_\alpha^\beta = 0$, since ξ is basis. Since a general X is locally a sum $X = \sum h_\kappa X^\kappa$ with Hamiltonian X^κ we finally obtain

$$L_X g_\alpha^\beta = \sum h_\kappa L_{X^\kappa} g_\alpha^\beta = 0\,.$$

We have shown, that the local definition of $\nabla_X s$ is independent of the frame field ξ. Therefore, the local definitions on U and U' with $U \cap U' \neq \emptyset$ agree on $U \cap U'$ and define ∇_X on $U \cup U'$, and furthermore, on all of M. □

The partial connection has all the usual linearity and derivation properties a connection should have. The following properties are easy to prove.

Lemma 12.18. *Whenever $\sigma \in \Gamma(M, \delta_r(P))$ and $X, Y \in \Gamma(M, P)$, the following equations hold for functions $f \in \mathcal{E}(M)$*

1. $\nabla_X(f\sigma) = (L_X f)\sigma + f\nabla_X\sigma$.
2. $\nabla_{fX+Y}\sigma = f\nabla_X\sigma + \nabla_Y\sigma$.
3. $\nabla_X\nabla_Y\sigma + \nabla_Y\nabla_X\sigma = \nabla_{[X,Y]}\sigma$.

In particular, the last equation means that the partial connection is flat.

The case D

In an analogous way there is a partial connection on $D^{\mathbb{C}}$: For $\sigma \in \Gamma_r(D^{\mathbb{C}})$ and $X \in \Gamma(M, P)$ we obtain $\nabla_X\sigma \in \Gamma_r(D^{\mathbb{C}})$. The definition works for vector fields $X \in \Gamma(M, P + \overline{P}) = \Gamma(M, E^{\mathbb{C}})$:

Let $U \subset M$ be an open subset for which there exists a Hamiltonian frame field $\xi \in \Gamma(U, D^{\mathbb{C}})$. Define $\sigma_\xi \in \Gamma(U, \delta_r(D^{\mathbb{C}})$ by the property $\sigma_\xi^\sharp(\xi) = 1$. Any $\sigma \in \Gamma(U, \delta_r(D^{\mathbb{C}}))$ is of the form $\sigma = \sigma^\sharp(\xi)\sigma_\xi$.

Definition 12.19. For a vector field $X \in \Gamma(U, E^{\mathbb{C}})$ and $\sigma \in \Gamma(U, \delta_r(D^{\mathbb{C}}))$ one defines

$$\nabla_X\sigma := L_X(\sigma^\sharp \circ \xi)\sigma_\xi\,.$$

Lemma 12.20. *The definition $\nabla_X\sigma$, where $X \in \Gamma(U, E^{\mathbb{C}})$ and $\sigma_\xi \in \Gamma(U, \delta_r(D^{\mathbb{C}}))$, is independent of the Hamiltonian frame field ξ and extends to all of M.*

Proof. This Lemma has essentially the same proof as Lemma 12.17. □

Observation 12.21. The connection is partial, because the condition (51) is crucial for the construction.

12.4 Partial Lie Derivative

We want to compare the partial connection just defined with a partial Lie derivative on r-densities $\sigma \in \Gamma_r(P)$. We need both derivatives for the formulation of the Half-Density Quantization.

The Lie derivative $L_X : \Gamma_r(P) \to \Gamma_r(P)$ will be defined only for those vector fields $X \in \mathfrak{V}(M)$ which preserve P, i.e. for which $[X, Y] = L_X Y \in \Gamma(M, P)$ holds for all $Y \in \Gamma(M, P)$. Notice, that the vector fields $X \in \Gamma(M, P)$ preserve P since P is involutive. But in general, the set of P-preserving vector fields is strictly larger than $\Gamma(M, P)$ as the following example shows.

Example 12.22. Let us have a look at the harmonic oscillator with phase space $M = T^*\mathbb{R}^n = \mathbb{C}^n$ and holomorphic polarization P generated by the vector fields

$$\frac{\partial}{\partial \bar{z}_j}$$

(see Example 10.23). The Hamiltonian vector field X_H of the energy $H = \frac{1}{2} z_j \bar{z}_j$ is preserving P but is not in P:

$$X_H = i\left(\sum z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\right) \notin \Gamma(M, P) \text{ and } \left[X_H, \frac{\partial}{\partial \bar{z}_k}\right] = i\frac{\partial}{\partial \bar{z}_k} \in \Gamma(M, P)\,.$$

Moreover, the set of P-preserving vector fields is not an $\mathcal{E}(M)$-module it is merely a module over the ring of polarized functions. In fact, $\bar{z}_k X_H$ does not preserve P.

As a consequence the partial Lie derivative L_X will be defined for a larger class of vector fields X than the previously considered vector fields $X \in \Gamma(M, P)$ for which the partial connection ∇_X has been defined. This is important when defining the quantum operator for observables F, since the directly quantizable observables F are precisely those for which X_F preserves P (cf. Definition 10.17), and this implies that $L_{X_F}\sigma$ is well-defined for $\sigma \in \Gamma_r(P)$.

We know that X preserves P if and only the local flow $(\Phi_t^X)_{t\in\mathbb{R}}$ of X satisfies

$$T\Phi_t^X(P) = P\,,\ t \in \mathbb{R}\,.$$

Let us abbreviate $\Phi_t = \Phi_t^X$ and $T_a\Phi_t(b) = (T_a\Phi_t(b_1), \ldots, T_a\Phi_t(b_n))$, where $b = (b_1, \ldots, b_n) \in R_a(P)$ is a frame. For $\sigma \in \Gamma(M, \delta_r(P))$ the pull-back $\Phi_t^*\sigma$ of σ with respect to Φ_t is given by $(\Phi_t^*\sigma)^\sharp(b) = \sigma^\sharp(T_a\Phi_t(b))$.

Definition 12.23. For $X \in \mathfrak{V}(M)$ preserving P the (partial) Lie derivative L_X : $\Gamma_r(P) \to \Gamma_r(P)$ is defined for $\sigma \in \Gamma_r(P)$ and $b \in R_a(P)$ by

$$(L_X\sigma)^\sharp(b) := \frac{d}{dt}\Big|_{t=0} (\Phi_t^*\sigma)^\sharp(b) = \frac{d}{dt}\Big|_{t=0} \sigma^\sharp(T_a\Phi_t(b))\,.$$

Notice, that the Lie derivative is defined in analogy to the Lie derivative of a differential form (cf. (72)) using the pullback of forms, see Definition A.33.

Proposition 12.24. *Let $\sigma_\xi \in \Gamma(U, \delta_r(P))$ be the local section corresponding to a Hamiltonian frame field $\xi \in \Gamma(U, R(P))$ with $(\sigma_\xi)^\sharp(\xi) = 1$, as above. For each Hamiltonian $X \in \Gamma(M, P)$ the Lie derivative $L_X\sigma_\xi$ vanishes.*

Proof. There exists a unique $g_t \in \mathcal{E}(U, \mathrm{GL}(n, \mathbb{C}))$ such that $T\Phi_t(\xi) = \xi g_t$, in the sense of

$$(T_a\Phi_t)\xi((a) = \xi(\Phi_t(a))g_t(a)\,,\ a \in U\,,$$

for the local flow $\Phi_t^X = \Phi_t$ of X. For a section $\sigma \in \Gamma(U, \delta_r(P))$ this definition of g_t leads to

$$\sigma^\sharp(T\Phi_t(\xi)) = \sigma^\sharp(\xi(\Phi_t)g_t) = |\det g_t|^r\sigma^\sharp(\xi(\Phi_t)).$$

Therefore, $L_X\sigma$ is given by

$$\begin{aligned}(L_X\sigma)^\sharp(\xi(a)) &= \frac{d}{dt}\Big|_{t=0}\sigma^\sharp\left(T_a\Phi_t(\xi(a))\right)\\ &= \frac{d}{dt}\Big|_{t=0}|\det g_t(a)|^r\sigma^\sharp\left(\xi(\Phi_t(a))\right)\\ &= \frac{d}{dt}\Big|_{t=0}|\det g_t(a)|^r\sigma^\sharp(\xi(a)) + |\det g_0(a)|^r\frac{d}{dt}\Big|_{t=0}\sigma^\sharp\left(\xi(\Phi_t(a))\right)\,.\end{aligned}$$

In the case of $\sigma = \sigma_\xi$ the second term vanishes immediately, since $\sigma_\xi^\sharp(\xi) = 1$. The first term vanishes, as well: Differentiating $T\Phi_t(\xi) = \xi g_t$ gives

$$\left(\frac{d}{dt}T\Phi_t\right)(\xi) = \left(\frac{d}{dt}\xi\right)g_t + \xi\frac{d}{dt}g_t = \xi\frac{d}{dt}g_t\,,$$

which implies

$$\xi\frac{d}{dt}g_t\Big|_{t=0} = \left(\frac{d}{dt}T\Phi_t\Big|_{t=0}\right)(\xi) = ([X,\xi_1],\dots,[X,\xi_n]) = 0\,.$$

As a consequence

$$\frac{d}{dt}g_t\Big|_{t=0} = 0 \ \text{ and, hence }\ \frac{d}{dt}|g_t|^r\Big|_{t=0} = 0\,,$$

so that the first term vanishes. □

The partial Lie derivative has all the usual linearity and derivation properties a Lie derivative should have. In particular,

$$L_X(f\sigma) = L_Xf\sigma + fL_X\sigma$$

for a scalar function $f \in \mathcal{E}(M)$.

Corollary 12.25. *For a Hamiltonian vector field* $X \in \Gamma(M,P)$ *the partial connection* ∇_X *and the partial Lie derivative* L_X *agree on* $\Gamma_r(M, \delta_r(P))$*:*

$$\nabla_X \sigma = L_X \sigma \,.$$

Proof. A general section σ has locally the form $\sigma|U = f\sigma_\xi \in \Gamma(U, \delta_r(P))$. By definition of the partial connection $\nabla_X \sigma = (L_X f)\sigma_\xi$. And $L_X\sigma = (L_X f)\sigma_\xi + fL_X\sigma_\xi = (L_X f)\sigma_\xi$, since $L_X\sigma_\xi = 0$. □

This is not true for all vector fields $X \in \Gamma(M,P)$ as the following example shows.

Example 12.26. In case of the vertical polarization P on $M = T^*\mathbb{R}$ the vector field

$$X = p\frac{\partial}{\partial p} \in \Gamma(M,P)$$

satisfies $L_X|dp| = |dp|$ but $\nabla_X|dp| = 0$.

Proof. $\nabla_X|dp| = 0$ by definition, since $\sigma_\xi = |dp|$ for $\xi = \frac{\partial}{\partial p}$. To determine $L_X|dp|$ we see that the flow of X is $\Phi_t = (q, pe^t)$. Now, a basis b of P is $b = \{\xi\}$, i.e. $b \in R(P)$. Inserting

$$|dp|^\sharp(T\Phi_t(b) = |dp|\left(T\Phi_t\left(\frac{\partial}{\partial p}\right)\right) = |dp|\left(e^t\frac{\partial}{\partial p}\right) = |e^t| = e^t$$

into

$$L_X|dp|(b) = \frac{d}{dt}\bigg|_{t=0}|dp|(T\Phi_t)(b) = \frac{d}{dt}\bigg|_{t=0} e^t = 1 = |dp|\,,$$

finally yields $L_X|dp| = |dp|$. □

Note, that for the Hamiltonian vector field

$$X_F = q\frac{\partial}{\partial q} + p\frac{\partial}{\partial p}$$

of $F = pq$ we obtain the same result

$$L_{X_F}|dp| = |dp|$$

with the same calculation now using the flow $\Phi_t = (e^t q, e^t p)$.

The following example will be used to determine the Half-Density Quantization of the harmonic oscillator.

Example 12.27. In case of the holomorphic polarization P on $M = T^*\mathbb{R} \cong \mathbb{C}$ the Hamiltonian vector field

$$X_H = i\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right)$$

of the energy of the harmonic oscillator (see Example 12.22) satisfies

$$L_{X_H}|d\bar{z}| = 0\,.$$

Morever, for the vector field

$$X = i\bar{z}\frac{\partial}{\partial \bar{z}}$$

we obtain the same result $L_X|d\bar{z}| = 0$.

Proof. The flow is $\Phi_t(z) = e^{it}z$, and for the basis

$$\left\{\frac{\partial}{\partial \bar{z}}\right\}$$

of P we obtain

$$|d\bar{z}|\left(T\Phi_t\left(\frac{\partial}{\partial \bar{z}}\right)\right) = |d\bar{z}|\left(e^{it}\left(\frac{\partial}{\partial \bar{z}}\right)\right) = |e^{it}| = 1\,.$$

As a consequence

$$L_{X_H}|d\bar{z}|\left(\frac{\partial}{\partial \bar{z}}\right) = \frac{d}{dt}\Big|_{t=0}|d\bar{z}|\left(T\Phi_t\left(\frac{\partial}{\partial \bar{z}}\right)\right) = \frac{d}{dt}\Big|_{t=0}1 = 0\,,$$

and $L_X|d\bar{z}| = 0$. □

Not all Lie derivatives L_X annihilate $|d\bar{z}|$:

Example 12.28. Again in case of the holomorphic polarization P on $M = T^*\mathbb{R} \cong \mathbb{C}$ the vector field

$$X = +\bar{z}\frac{\partial}{\partial \bar{z}} \in \Gamma(M, P)$$

satisfies

$$L_X|d\bar{z}| = |d\bar{z}|\,.$$

Proof. With the flow $\Phi_t(z) = e^t z$ we get

$$L_X|d\bar{z}|\left(\frac{\partial}{\partial \bar{z}}\right) = \frac{d}{dt}\Big|_{t=0}|d\bar{z}|\left(T\Phi_t\left(\frac{\partial}{\partial \bar{z}}\right)\right) = \frac{d}{dt}\Big|_{t=0}|d\bar{z}|\left(e^t\frac{\partial}{\partial \bar{z}}\right) = \frac{d}{dt}\Big|_{t=0}|e^t| = 1\,,$$

hence $L_X|d\bar{z}| = |d\bar{z}|$. □

Observation 12.29. The various partial connections $\nabla^r := \nabla$ on $\delta_r(P)$, $r \in \mathbb{R}$, as well as the partial Lie derivatives L_X are compatible with each other- For instance, given $\sigma \in \Gamma_r(P)$, $\tau \in \Gamma_s(P)$ and $\sigma\tau = \sigma \otimes \tau \in \Gamma_{r+s}(P)$ the following holds:

$$\nabla_X^{r+s}(\sigma\tau) = (\nabla_X^r \sigma)\tau + \sigma(\nabla_X^s \tau)\,,$$

$$L_X(\sigma\tau) = (L_X\sigma)\tau + \sigma(L_X\tau)\,,$$

In particular $\nabla_X^{2s}(\tau^2) = 2\nabla_X^s\tau$ and $L_X(\tau^2) = 2L_X\tau$.

Remark 12.30. The partial connection ∇_X on $\delta_r(D^{\mathbb{C}})$ for $X \in \Gamma(M, E^{\mathbb{C}})$ agrees with the Lie derivative L_X if X is Hamiltonian as in the case of P, see Corollary 12.25. However, the Lie derivative L_X can be extended to vector fields X preserving $E^{\mathbb{C}}$ as in the case of P.

Summary

13 Half-Density Quantization of the Momentum Phase Space

On the way to the half-density quantization in general we consider in this chapter the special case of a cotangent bundle $M = T^*Q$ with the vertical polarization P. This case is sufficiently important to deserve a special attention and, moreover, it is less complicated than the general case, so that the procedure is easier to explain.

So let $M = T^*Q$ be the cotangent bundle of an n-dimensional manifold Q, with its standard symplectic form $\omega = -d\lambda = dq \wedge dp$ and with the vertical polarization $P \subset TM^{\mathbb{C}}$. Recall that the vertical polarization is $P = D^{\mathbb{C}}$ where $D = \operatorname{Ker} T\tau^*$ is the vertical bundle (or distribution) with respect to the natural projection $\tau^* : T^*Q \to Q$. Of course, P is reducible, we have $M/D \cong Q$. We shall denote the projection onto the quotient manifold M/D by $\pi : M \to M/D$, which is essentially $\tau^* : T^*Q \to Q$ with fibres $\cong \mathbb{R}^n$. With respect to local coordinates $q = (q^1, \dots, q^n) : U \to V \subset \mathbb{R}^n$ on an open subset $U \subset Q$ we have the bundle chart

$$(q^1, \dots, q^n, p_1, \dots p_n) : T^*U \to V \times \mathbb{R}^n$$

on $T^*U = T^*Q|_U \cong V \times \mathbb{R}^n$ with its generalized momenta p_j. A basis (over $\mathcal{E}(U)$) for the vertical polarization $P_U = P|_U$ is given by

$$\left(\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}\right) = (X_{q^1}, \dots, X_{q^n}) \,.$$

In particular, the above basis is a basis of the distribution D_U, now over $\mathcal{E}(U, \mathbb{R})$.

The prequantum line bundle L is trivializable and we set $L = M \times \mathbb{C}$. The Hermitian structure H on L is the induced structure given in terms of the standard section $s_1(a) = (a, 1)\,,\ a \in M$, through $H(s_1, s_1) = 1$. Finally, we choose the connection determined by the Liouville form $\lambda = p_j dq_j$, i.e.

$$\nabla_X f s_1 = (L_X f - 2\pi i \lambda(X) f)\, s_1 \,,$$

for a general section $s = f s_1$ of L with $f \in \mathcal{E}(M)$ and X a vector field on M[66].

The local connection form $\alpha = -\lambda = -p_j dq^j$ is adapted to P: $\alpha(X) = 0$ for every $X \in P$. In particular, s_1 is polarized, i.e. $\nabla_X s_1 = 0$ for all $X \in \Gamma(M, P)$.

Observe, that the covariant derivative of $f s_1$ along P, i.e. $\nabla_X f s_1\,,\ X \in \Gamma(M, P)$, is essentially the usual directional derivative L_X on the scalar f on M:

$$\nabla_{X_j \frac{\partial}{\partial p_j}} f s_1 = \left(L_{X_j \frac{\partial}{\partial p_j}} f\right) s_1 = \left(X_j \frac{\partial f}{\partial p_j}\right) s_1 \,,$$

since $\nabla_X s_1 = 0$.

[66]other connections are possible by choosing other potentials α of ω leading to the same results.

As a consequence, the space of polarized sections is

$$\Gamma_{\nabla,P}(M,L) = \{fs_1 \mid \frac{\partial f}{\partial p_j} = 0\,,\ j = 1,\ldots,n\}\,,$$

and can be viewed to be the space $\mathcal{E}(Q)$ of functions on Q. The representation space can be constructed on the basis of these functions. Since, in general, there does not exist a natural volume form on Q we work with 1-densities on Q. Here the preceding chapter comes into action! The representation space turns out to be the square integrable 1-densities on the configuration space Q!

After these preliminaries, we describe the half-density quantization of T^*Q with respect to the vertical polarization in the following three sections.

13.1 Descend of Densities

In order to obtain suitable densities on $TQ^{\mathbb{C}}$ we relate the densities on $TQ^{\mathbb{C}}$ with densities on $P = D^{\mathbb{C}}$. We use the quotient bundle $Z^D := TM^{\mathbb{C}}/P \to M$ and see that the tangent map $T\pi : TM^{\mathbb{C}} \to TQ^{\mathbb{C}}$ satisfies $P = \operatorname{Ker} T\pi$ and therefore induces a map $T : Z^D \to TQ^{\mathbb{C}}$ which is fibrewise an isomorphism. Z^D is the pull-back of $TQ^{\mathbb{C}}$ with respect to π, $Z^D = \pi^*(TQ^{\mathbb{C}})$, and it is essentially the complexified tangent bundle $TQ^{\mathbb{C}}$ but lifted to a bundle over $T^*Q = M$. The following commutative diagram illustrates these properties

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & TM^{\mathbb{C}} & \longrightarrow & Z^D & \longrightarrow & 0 \\ & & & & {\scriptstyle T\pi}\downarrow & \swarrow {\scriptstyle T} & & & \\ & & & & TQ^{\mathbb{C}} & & & & \end{array}$$

The natural map $T : Z^D \to TQ^{\mathbb{C}}$ enables us to lift the densities ν on $TQ^{\mathbb{C}}$ to densities τ on Z^D, as we explain in the following.

Every r-density $\nu \in \Gamma(Q, \delta_r(TQ^{\mathbb{C}}))$ can be lifted to become an r-density $\tau = \tau_\nu \in \Gamma(M, \delta_r(Z^D)) = \Gamma_r(Z^D)$: In terms of the functions ${\tau_\nu}^\sharp$ and $\nu^\sharp$ one sets

$${\tau_\nu}^\sharp(Y_1,\ldots,Y_n)(a) := \nu^\sharp(T_a\pi(X_1),\ldots,T_a\pi(X_n))$$

for any basis $(Y_1,\ldots,Y_n) \in R_a(Z^D)$ and $X_1,\ldots,X_n \in T_aM^{\mathbb{C}}$ with $X_j + P = Y_j\,,\ j = 1,2,\ldots,n$. It is easy to check that τ_ν transforms appropriately and thus t_ν is an r-density. In particular, we see that $\nu \mapsto \tau_\nu$ defines a homomorphism, the lift $L : \Gamma_r(T(Q)^{\mathbb{C}}) \to \Gamma_r(Z^D)$.

But not every r-density on Z^D is such a lift, in fact, $\Gamma_r(Z^D) = \Gamma(M, \delta_r(Z^D))$ is a module over $\mathcal{E}(M)$ while $\Gamma_r(TQ^{\mathbb{C}}) = \Gamma(Q, \delta_r(TQ^{\mathbb{C}})) \cong \Delta_r(TQ^{\mathbb{C}})$ is a module over $\mathcal{E}(Q)$. With respect to any nowhere vanishing $\nu_1 \in \Gamma_r(TQ^{\mathbb{C}})$ and its lift $\tau_1 \in \Gamma_r(Z^D)$ each $\tau \in \Gamma_r(Z^D)$ has the unique representation $\tau = \lambda\tau_1$ with a factor $\lambda \in \mathcal{E}(M)$ which may depend on p.

We are essentially interested to determine those densities τ on Z^D which descend to a density ν on $TQ^{\mathbb{C}}$, in other words, which are a lift of a ν. To obtain a characterizing property we introduce the partial connection ∇^δ on the line bundle $\delta := \delta_r(Z^D)$.

Definition 13.1. For τ_1 as above (lift of non-vanishing $\nu_1 \in \Gamma_r(TQ^{\mathbb{C}})$) every $\tau \in \Gamma_r(Z^D) = \Gamma(M,\delta)$ has the form $\tau = f\tau_1$ with a unique $f \in \mathcal{E}(M)$ and we set

$$\nabla^\delta_X \tau = \nabla^\delta_X (f\tau_1) := (L_X f)\tau_1$$

for $X \in \Gamma(M,P)$.

Lemma 13.2. *This definition is independent of* τ_1.

Proof. For another τ_2 which is the lift of a non-vanishing ν_2 there is a $\lambda \in \mathcal{E}(Q)$ such that $\nu_1 = \lambda\nu_2$. It follows that $\tau_1 = h\tau_2$ with $h := \lambda \circ \pi \in \mathcal{E}(M)$. Now, for $\tau = f\tau_1 = fh\tau_2$ we have

$$(L_X(fh))\tau_2 = ((L_X f)h + fL_X h)\,\tau_2 = (L_X f)h\tau_2 = (L_X f)\tau_1\,,$$

because of $L_X h = 0$ for $X \in \Gamma(M,P)$, which shows the assertion. □

∇^δ is called partial connection, since it is only defined for $X \in \Gamma(M,P)$.

The partial connection is used to formulate the following simple criterion.

Lemma 13.3. *$\tau \in \Gamma_r(Z^D)$ is the lift of an r-density $\nu \in \Gamma_r(TQ^{\mathbb{C}})$ on $TQ^{\mathbb{C}}$, and hence descends to ν, if and only if $\nabla^\delta_X \tau = 0$ for all $X \in \Gamma(M,P)$, i.e. if τ is covariantly constant along $D^{\mathbb{C}} = P$.*

Proof. Let σ_1 be a lift of a nowhere vanishing ν_1, as above. A general $\sigma \in \Delta_r(Z^D)$ can be expressed as $\sigma = f\sigma_1$ with $f \in \mathcal{E}(M)$. Then $\nabla^\delta_X \sigma = L_X f \sigma_1$. We conclude

$\nabla^\delta_X \sigma = 0$ for all $X \in \Gamma(M,P)$

$\iff L_X f = 0$ for all $X \in \Gamma(M,P)$

$\iff f$ does not depend on the vertical direction

$\iff f$ has the form $f = \lambda \circ \pi$ with $\lambda \in \mathcal{E}(Q)$

$\iff \sigma$ is the lift of $\lambda\nu_1$, where $f = \lambda \circ \pi$ with $\lambda \in \mathcal{E}(Q)$ □

13.2 Representation Space

We now come to the representation space of the half-quantization of $M = T^*Q$ using densities. As explained at the beginning of the preceding chapter, the main aspect is to replace the line bundle L with a line bundle $L \otimes \delta$, as we describe in the following.

Construction 13.4 (Representation Space of T^*Q)**.** The new line bundle on $M = T^*Q$ is $L \otimes \delta$, where $\delta := \delta_{1/2}(Z^D)$[67] and where the corresponding new connection is $\nabla \otimes \nabla^\delta$, with ∇^δ the (partial) connection just defined.

A section $\psi = s \otimes \tau \in \Gamma(M, L \otimes \delta)$[68] is covariantly constant along D (or P) (also called a polarized section) if $(\nabla \otimes \nabla^\delta_X)(s \otimes \tau) = 0$ for all $X \in \Gamma(M, D)$, and this is equivalent to

$$\nabla_X s = 0 \quad \text{and} \quad \nabla^\delta_X \tau = 0\,.$$

In particular, when $s \otimes \tau$ is covariantly constant along P, τ descends to a $1/2$-density $\nu = \nu(\tau)$ on $TQ^{\mathbb{C}}$ according to Lemma 13.3. As a result we obtain for two such polarized sections $\psi = s \otimes \tau, \psi' = s' \otimes \tau'$, the 1-density

$$H(s, s')\bar{\nu}\nu' = H(s, s')\bar{\nu}(\tau)\nu(\tau')$$

on $TQ^{\mathbb{C}} \cong T(M/D)^{\mathbb{C}}$ as a section of $\delta_1(TQ^{\mathbb{C}})$, where $\nu' = \nu(\tau')$ lifts to τ'. In this way one defines the scalar product through integration of the density $H(s, s')\bar{\nu}\nu'$ over $Q = M/D$:

$$\langle \psi, \psi' \rangle = \langle s \otimes \tau, s' \otimes \tau' \rangle := \int_{M/D} H(s, s')\bar{\nu}^\sharp \nu'^\sharp \tag{52}$$

on the vector space H_P of polarized sections with compact support. Finally, the completion $\mathbb{H}^\delta_P = \mathbb{H}^\delta(M, L, P)$ of H_P is the representation space we wanted to construct.

Note that we could have used the line bundle $\delta_{-1/2}(P)$ instead of $\delta_{1/2}(Z^D)$, see Remark 13.10, as it will be done in the general case in the next chapter.

In a more concrete picture of the representation space the basic elements of the Hilbert space are functions on Q which are integrated by suitable half-densities on the space Q, as we show in the following:

Fix a nowhere vanishing positive $1/2$-density ν_1 on Q which exists according to Lemma 12.5. Then $\nu := \overline{\nu_1}\nu_1 = \nu_1^2$ is a positive 1-density on Q. For instance, when Q has a natural volume form ε, we can take $\nu_1 = \sqrt{|\varepsilon|}$ and it follows $\nu = |\varepsilon|$.

Let τ_1 be the uniquely defined lift of ν_1 in $\Gamma_{1/2}(M, Z^D)$. τ_1 is a positive half-density on Z^D. Any polarized section $s \otimes \tau$ in $L \otimes \delta$, $\delta = \delta_{1/2}(Z^D)$, can be expressed as

$$s \otimes \tau = f s_1 \otimes g \tau_1 = \phi s_1 \otimes \tau_1$$

with $f, g \in \mathcal{E}(Q)$ and $\phi = fg$[69].

[67] $\delta = \delta_{-1/2}(P)$ is an alternative choice, as we will see below and in the next chapter.

[68] Notice, that every global section $\psi \in \Gamma(M, L \otimes \delta)$ has the form $\psi = s \otimes \tau$ for suitable $s \in \Gamma(M, L)$ and $\tau \in \Gamma(M, \delta)$, since δ is trivial.

[69] More precisely, there are $f, g \in \mathcal{E}(Q)$ with $\phi = fg$ such that $\tau = (f \circ \pi)s_1 \otimes (g \circ \pi)\tau_1 = \phi \circ \pi s_1 \otimes \tau_1 = \phi$.

Therefore, the scalar product for polarized sections $\psi = s \otimes \tau = \phi s_1 \otimes \tau_1$, $\psi' = s' \otimes \tau' = \phi' s_1 \otimes \tau_1 \in \Gamma(M, L \otimes \delta)$, as defined in (52) can be written as

$$\langle \psi, \psi' \rangle = \langle s \otimes \tau, s' \otimes \tau' \rangle = \int_Q \bar{\phi}\phi' \overline{\nu_1}^\sharp \nu_1^\sharp = \int_Q \bar{\phi}\phi' \nu^\sharp \,,$$

which is

$$\langle \psi, \psi' \rangle = \langle \phi, \phi' \rangle = \int_Q \bar{\phi}\phi' \nu^\sharp \,.$$

It is easy to show that this integral is independent of the choice of ν_1.

As a result, we can identify $\mathbb{H}_P^\delta$ with the Hilbert space of functions ϕ on Q such that $\int_Q |\phi|^2 \nu^\sharp < \infty$, or – which is essentially the same – the canonical Hilbert space $L^2(Q)$ of Q (cf. Remark 12.10).

13.3 Quantum Operators

To finish the case of the momentum phase space $T^*Q = M$ the main step is to determine the quantum operators associated to the quantizable classical observables. Regarding the prequantization – as developed in Chapter 7 – one is tempted to define the quantum operator in $\mathbb{H}_P^\delta$ assigned to a quantizable classical observable $F \in \mathcal{E}(M)$ by the preliminary definition

$$q^\delta(F)\psi := \left(-\frac{i}{2\pi} (\nabla \otimes \nabla^\delta)_{X_F} + F \right) \psi = (q(F)s) \otimes \tau - \frac{i}{2\pi} s \otimes \nabla^\delta_{X_F} \tau \,, \tag{53}$$

where $\psi = s \otimes \tau$ is a polarized section of $L \otimes \delta$, and $q(F)$ is the prequantum operator.

However, the partial connection operator ∇_X^δ is defined only for vector fields $X \in \Gamma(M, P)$. Let us have a look at these vector fields. When $X_F \in P$ we know that $(\nabla \otimes \nabla^\delta)_{X_F} \psi = 0$, since ψ is polarized. In this way we only get multiplication operators as quantum operators, and the whole effort of introducing connections on line bundles and polarizations looks superfluous, in particular, there would be no geometric quantization.

To overcome this problem, we shall define a partial Lie derivative operator $L_X : \Gamma_r(Z^D) \to \Gamma_r(Z^D)$ which for vector fields X on M such that L_X is well-defined at least for the case of $X = X_F$, $F \in \mathcal{E}(M)$, where, in addition, F is directly quantizable (cf. Section 10.3). Under these assumption, L_X can be used to define the quantum operator. The notation L_X for this operator is justified since it is related to the Lie derivative, although it is not defined directly as a Lie derivative. As a reminder, F is called directly quantizable or simply quantizable, if $\nabla_{X_F} s$ is polarized for polarized sections s. This is equivalent to X_F preserving P, i.e. to the property that $[X_F, Y] \in \Gamma(M, P)$ for all $Y \in P$. The Lie algebra of all quantizable classical observables is denoted by $\mathfrak{R}_P$.

In order to define the operator L_X we recall the concept of a Lie derivative of n-forms on an n-dimensional manifold Q and its divergence: The Lie derivative $L_X\varepsilon$ of an n-form $\varepsilon \in \Gamma(Q, \Lambda^n TQ)$ is given by

$$L_X\varepsilon = di_X\varepsilon + i_X d\varepsilon = di_X\varepsilon\,,$$

since $d\varepsilon = 0$. The divergence of X with respect to ε as the unique $\mathrm{div}_\varepsilon(X) \in \mathcal{E}(Q)$ with $L_X\varepsilon = \mathrm{div}_\varepsilon(X)\varepsilon$ where $\varepsilon \neq 0$, otherwise $\mathrm{div}_\varepsilon(X) = 0$. For instance, locally, for coordinates q^j in an open subset U of Q, when $X = X^j\partial_j$ and $\varepsilon = dq^1 \wedge \ldots \wedge dq^n \in \Gamma(U, \Lambda^n(TQ))$ we have the familiar expression

$$\mathrm{div}_\varepsilon(X) = \sum_1^n \frac{\partial X^j}{\partial q^j}\,.$$

A general n-form $\eta \in \Gamma(U, \Lambda^n(TQ))$ can be written as $\eta = h\varepsilon$ with $h \in \mathcal{E}(U)$. Then $L_X\eta = (L_X h)\varepsilon + hL_X\varepsilon = (L_X h + h\,\mathrm{div}_\varepsilon(X))\varepsilon$, and we deduce

$$h\,\mathrm{div}_\eta(X) = L_X h + h\,\mathrm{div}_\varepsilon(X)\,.$$

Using the local flow $\Phi_t^X : Q_t \to Q_{-t}$ of the vector field X there is the alternative description of the Lie derivative by the formula

$$L_X\varepsilon(\xi_1, \ldots, \xi_n) = \frac{d}{dt}\varepsilon\left(T\Phi_t^X(\xi_1), \ldots, T\Phi_t^X(\xi_n)\right)\Big|_{t=0}\,,$$

when $(\xi_1, \ldots, \xi_n)$ is a basis of T_qQ, $q \in Q$, see (72).

For the Lie derivative of densities, we take the last equation as the definition.

Definition 13.5. For $\nu \in \Gamma_r(TQ^{\mathbb{C}})$ define

$$(L_X\nu)^\sharp(\xi_1, \ldots, \xi_n) := \frac{d}{dt}\left(\nu^\sharp\left(T\Phi_t^X(\xi_1), \ldots, T\Phi_t^X(\xi_n)\right)\right)\Big|_{t=0}\,.$$

We obtain the induced divergence div_ν through $L_X\nu = \mathrm{div}_\nu(X)\nu$.[70].

For instance, locally, for coordinates q^j in an open subset U of Q, when $X = X^j\partial_j$ and $\nu = |dq^1 \wedge \ldots \wedge dq^n|^r \in \Gamma(U, \delta_r(TQ^{\mathbb{C}})$

$$\mathrm{div}_\nu(X) = r\sum_1^n \frac{\partial X^j}{\partial q^j}\,.$$

We now want to define $L_{X_F}\tau$ for $1/2$-densities $\tau \in \Gamma_{1/2}(Z^P)$ for the special vector fields $X = X_F$, $F \in \mathfrak{R}_P$, by lifting the Lie derivative $L_X : \Gamma_{1/2}(TQ^{\mathbb{C}}) \to \Gamma_{1/2}(TQ^{\mathbb{C}})$ just

[70] div_ν is not a one form, in general, since it is not $\mathcal{E}(Q)$-linear; it does not define a connection!

defined. Let $\tau_1 \in \Gamma_{1/2}(Z^D)$ be the lift of a non-vanishing $\nu_1 \in \Gamma_{1/2}(TQ^{\mathbb{C}})$, as before. For a general $\tau \in \Gamma_{-1/2}(Z^D)$, $\tau = g\tau_1$ with $g \in \mathcal{E}(M)$ we set:

$$L_{X_F}\tau := (L_{X_F}g + \mathrm{div}_{\nu_1}(T\pi(X_F))g)\,\tau_1\,.$$

It is easy to see that this definition does not depend on the choice of τ_1.

In case of $X_F \in \Gamma(M,P)$, the definition entails $L_{X_F}\tau_1 = 0$, since $T\pi(X_F) = 0$. Hence, for general $\tau = g\tau_1$:

$$L_{X_F}(g\tau_1) = (L_{X_F}g)\tau_1.$$

Comparison: Recall that for $X_F \in \Gamma(M,P)$ the partial connection is defined by $\nabla^\delta_{X_F}(g\tau_1) = (L_{X_F}g)\tau_1$, for a non-vanishing lifted τ_1 (cf. Definition 13.1). Comparing with the last equation it follows that the partial connection ∇^δ_X for $X = X_F \in \Gamma(M,P)$ agrees with L_X just defined. Hence, L_{X_F} for general $F \in \mathfrak{R}_P$ can be considered to be an extension of the partial connection ∇_{X_F}, $X_F \in \Gamma(M,P)$ to mor general Hamiltonian vector fields X_F by L_{X_F}.

We now can define the quantum operator $q^\delta(F)$ on the polarized sections $s \otimes \tau \in \Gamma_P(M, L\otimes\delta)$ improving our first attempt in (53):

Definition 13.6 (Quantum Operator)**.** For directly quantizable $F \in \mathcal{E}(M)$ the Quantum Operator is

$$\boxed{q^\delta(F)(s\otimes\tau) := (q(F)s)\otimes\tau - \frac{i}{2\pi}s\otimes L_{X_F}\tau\,,}$$

where $q(F)$ is the prequantum operator. Hence, the quantum operator also has the form

$$q^\delta(F)(s\otimes\tau) := -\frac{i}{2\pi}\left(\nabla_{X_F}s\otimes\tau + s\otimes L_{X_F}\tau\right) + Fs\otimes\tau\,,$$

An additional motivation for this definition is explained later in the context of defining the quantum operator as an infinitesimal generator for the general case in Definition 14.8.

Main Result

Summarizing the results of this chapter we have the following.

Theorem 13.7. *The constructed half-density quantization for the cotangent bundle $M = T^*Q$ with its vertical polarization P yields the representation space $\mathbb{H}^\delta_P$ as described above and is a full geometric quantization in the following sense:*

1. *The quantization map $q^\delta : \mathfrak{R}_P \to \mathcal{S}(\mathbb{H}^\delta_P)$ is $\mathbb{R}$-linear and satisfies* (D1) *and* (D2)*, now for the new representation space, the well-defined Hilbert space $\mathbb{H}^\delta_P \cong L^2(Q)$*[71]

[71]This means that the Dirac conditions are satisfied for any set $\mathfrak{o}$ of classical observables contained in $\mathfrak{R}_P$.

2. If X_F is complete, $F \in \mathfrak{R}_P$, then $q^\delta(F)$ is self-adjoint.

The description of half-density quantization of the momentum phase space $M = T^*Q$ with its vertical polarization is now completed.

But we want to make the formula for $q^\delta(F)$ slightly more explicit in our special case $M = T^*Q$ regarding the fact, that the quantum operators act essentially on functions $\phi \in \mathcal{E}(Q)$. Let s_1 be the usual global section of L and ν_1 a nowhere vanishing $\frac{1}{2}$-density on Q with its unique lift $\tau_1 \in \Gamma_{1/2}(Z^D)$. Then every section ψ of $L \otimes \delta$ has the form $\phi s_1 \otimes \tau_1$. The quantum operator is

$$\begin{aligned} q^\delta(F)(\phi s_1 \otimes \tau_1) &= (q(F)\phi s_1) \otimes \tau_1 - \frac{i}{2\pi}\phi s_1 \otimes \operatorname{div}_{\nu_1}(T\pi(X_F))\tau_1 \\ &= \left(q(F) - \frac{i}{2\pi}\operatorname{div}_{\nu_1}(T\pi(X_F))\right)\phi s_1 \otimes \tau_1 \,. \end{aligned}$$

This means acting on $\mathcal{E}(M)$, or more precisely on the functions with compact support, the operator is

$$\boxed{q^\delta(f) = q(F) - \frac{i}{2\pi}\operatorname{div}_{\nu_1}(T\pi(X_F))}\,,$$

and the additional term induced by the half-density quantizaion in comparison to the prequantization is apparent in this form. Let us study the additional term in the case of a simple phase space:

Example 13.8 (Simple phase space)**.** When the configuration space Q is an open subset of $\mathbb{R}^n$ with the standard volume form $\varepsilon = dq^1 \wedge \dots dq^n = dq$ we get a formula for the quantum operators $q^\delta(F)$ which is more explicit. Here, F is a directly quantizable observable on the simple phase space $M = T^*Q$.

Let us begin by calculating the prequantum operator $q(F)$. F has the form $F = A + B^j p_j$ with functions $A, B^j \in \mathcal{E}(Q)$.

The Hamiltonian vector field is

$$X_F = B^k \frac{\partial}{\partial q^k} - \left(\frac{\partial A}{\partial q^k} + \frac{\partial B^j}{\partial q^k} p_j\right)\frac{\partial}{\partial p_k}$$

in global coordinates q^k of Q and the induced p_j. Applied to the functions $\phi \in \mathcal{E}(Q)$ only the first term of X_F is relevant. Moreover, $\lambda(X_F) = B^j p_j$. Prequantization then yields for $s = \phi s_1$:

$$\begin{aligned} q(F)s &= \left(-\frac{i}{2\pi}\nabla_{X_F} + F\right)\phi s_1 \\ &= \left(-\frac{i}{2\pi}L_{X_F}\phi - \mathrm{B}^j p_j\phi + F\phi\right)s_1 \\ &= \left(-\frac{i}{2\pi}B^j\frac{\partial}{\partial q^j}\phi + A\phi\right)s_1 . \end{aligned}$$

Acting on $\phi \in \mathcal{E}(Q)$ this means

$$\boxed{q(F) = A - \frac{i}{2\pi}B^j\frac{\partial}{\partial q^j}\,.}$$

To bring the additional term into a simpler form we use the density $\nu := |\varepsilon|$, also denoted as $|dq|$, and the positive half-form $\nu_1 := \sqrt{\nu} = |dq|^{1/2} \in \Gamma_{1/2}(TQ)$. From $L_X\nu_1^2 = 2\nu_1 L_X\nu_1$ and $L_X\nu = \operatorname{div}_\varepsilon(X)\nu$ we receive

$$L_X\nu_1 = \frac{1}{2}\operatorname{div}_\varepsilon(X)\nu_1$$

for $X \in \mathfrak{V}(Q)$. We have mentioned already that $\operatorname{div}(X) = \operatorname{div}_\varepsilon(X)$ has the familiar form

$$\operatorname{div}(X) = \sum\frac{\partial X^k}{\partial q^k}\,.$$

Furthermore, $T\pi(X_F) = B^j\partial_j =: B$, and so

$$L_B\nu_1 = \frac{1}{2}\operatorname{div}(B)\nu_1 = \frac{1}{2}\sum\frac{\partial B^k}{\partial q^k}\nu_1\,.$$

As a consequence

$$\boxed{q^\delta(A + B^j p_j) = \left(A - \frac{i}{4\pi}\sum\frac{\partial B^k}{\partial q^k}\right) - \frac{i}{2\pi}B^j\frac{\partial}{\partial q^j}\,.}$$

This formula reveals the impact of L_{X_F} resp. of $\frac{1}{2}\operatorname{div}(B)$ as the essential part of the additional term. In comparison to the prequantization of $T^*\mathbb{R}^n$ (cf. Example 7.18) without half-density quantization by simply restricting to those functions which are independent of the variable p_j and using the Lebesgue volume on Q we see that half-density quantization imposes an additional term in the quantization, namely multiplication with

$$-\frac{i}{4\pi}\sum\frac{\partial B^k}{\partial q^k} = -\frac{i}{2\pi}\frac{1}{2}\operatorname{div}(B)\,. \tag{54}$$

Restricting to the special cases $F = A = q^j$ and $F = B^j = p_j$ we obtain the known quantized observables

$$\boxed{q^\delta(q^j) = q^j\,,\ q^\delta(p_j) = -\frac{i}{2\pi}\frac{\partial}{\partial q^j}}\,.$$

The additional term is 0 in these cases.

For a single particle, or a particle system, one wants to quantize the energy H of the system in order to complete the Schrödinger picture. However, an energy like $H = g^{ij}p_ip_j - V(q)$ is not directly quantizable when one uses the vertical polarization. The rules of Geometric Quantization are too restrictive to treat this important case. They have to be extended to cover this case and more. We come back to the quantization of the energy after having introduced the so-called BKS-pairing in the context of half-form quantization.

Observation 13.9. The additional term resembles operator ordering, a concept which is not part of Geometric Quantization in a direct way. For instance, in case of $M = T^*\mathbb{R}$ and $F = qp$ we see from the above formulas

$$q(qp) = -\frac{i}{2\pi}q\frac{\partial}{\partial q}\,,$$

and

$$q^\delta(qp) = -\frac{i}{4\pi} - \frac{i}{2\pi}q\frac{\partial}{\partial q}\,.$$

With the notation

$$Q := q^\delta(q) = q \ \text{ and } \ P := q^\delta(p) = -\frac{i}{2\pi}\frac{\partial}{\partial q}\,,$$

this amounts to

$$q^\delta(qp) = QP\,,$$
$$q^\delta(qp) = -\frac{i}{4\pi} + QP\,,$$

with $-{}^{i}/_{4\pi}$ as the additional term. Inserting the canonical commutaion relation

$$PQ - QP = -\frac{i}{2\pi}$$

we deduce

$$q^\delta(qp) = \frac{1}{2}(PQ + QP)\,.$$

Similarly,

$$q^\delta(q^2p) = -\frac{i}{2\pi}Q + Q^2P = PQ^2 - QPQ + Q^2P\,.$$

For the typical component of the angular momentum $F = q^k p_j - q^j p_k$ (here $M = T^*\mathbb{R}^n, n > 1$ and $j \neq k$) the additional terms cancels out and we get the familiar quantum operator

$$q^\delta(F) = Q^k P_j - Q^j P_k$$

agreeing with the prequantum operator.

13.4 The Case of a Real Polarization

Before we study the case of a general polarization in the next chapter, let us consider the case of a real reducible polarization, i.e. $P = \overline{P}$ and therefore $D^{\mathbb{C}} = P \cap \overline{P} = P = \overline{P}$. This case is close to the case of a cotangent bundle $M = T^*Q$ with vertical polarization and can be treated in the same way.

Step 1: The quotient bundle $Z^D = TM^{\mathbb{C}}/P$ is essentially the complexified tangent bundle $T(M/D)^{\mathbb{C}}$ and the r-densities on $T(M/D)^{\mathbb{C}}$ can be lifted to r-densities on Z^D.

Moreover, a section $\tau \in \Gamma_r(Z^D)$ is a lift of an r-density (i.e. descends to) $\nu \in \Gamma_r(T(M/D)^{\mathbb{C}})$ if and only if $\nabla^\delta_X \tau = 0$ for the partial connection ∇ on $\delta_r(Z^D)$, defined as in Definition 13.1.

Step 2: L will be replaced with

$$L \otimes \delta\,,\ \delta := \delta_{1/2}(Z^D)\,,$$

and ∇ will be replaced with $\nabla \otimes \nabla^\delta$.

As before, any two sections $\psi = s \otimes \tau, \psi' = s' \otimes \tau' \in \Gamma(M, L \otimes \delta)$, which are polarized, i.e. on which $\nabla \otimes \nabla^\delta$ vanishes, induce on M/D the 1-density

$$H(s, s')\overline{\nu}\nu'\,,$$

where τ (resp. τ') is the lift of ν (resp. ν').

This leads to the scalar product

$$\langle \psi, \psi' \rangle := \int_{M/D} H(s, s')\overline{\nu}^\sharp \nu'^\sharp$$

on the space of polarized sections $H_P \subset \Gamma(M, L \otimes \delta)$ with compact support, Finally, the representation space $\mathbb{H}^\delta_P = \mathbb{H}^\delta(M, L, P)$ is the completion of H_P with respect to the norm given by the scalar product.

Step 3:

$$q^\delta(F)(s \otimes \tau) := (q(F)s) \otimes \tau - \frac{i}{2\pi} s \otimes L_{X_F}\tau\,.$$

We conclude this chapter with the following remark which shows how the $1/2$-density bundle $\delta = \delta_{1/2}(Z^D)$ can be replaced by $-1/2$-densities on P. This remark has no impact to the half-density quantization of T^*Q which we just have carried through, but it prepares the use of P instead of Z^D in the subsequent considerations for the general case of a complex polarization P given on a symplectic manifold (M,ω).

Proposition 13.10. *There is a natural isomorphism $\tau_P : \delta_{-r}(P) \to \delta_r(Z^D)$ induced by the exact sequence*

$$0 \longrightarrow D^{\mathbb{C}} \longrightarrow TM^{\mathbb{C}} \longrightarrow Z^D = TM^{\mathbb{C}}/D^{\mathbb{C}} \longrightarrow 0\,.$$

Recall $P = D^{\mathbb{C}}$. And we are back in the case of the momentum phase space $M = T^*Q$.

Proof. We know this from Proposition 12.7. With the r-density $\varepsilon = \varepsilon_\omega$ on $TM^{\mathbb{C}}$ defined by

$$(\varepsilon_\omega^\sharp)_a(b) := |\omega_a^n(b)|^r\,,\; b \in R_a(TM^{\mathbb{C}})\;, a \in M,$$

and with the notation $[Z] := Z + P \in Z_a^D$ for $Z \in T_aM^{\mathbb{C}}$, an r-density $\tau_P(\rho)$ on Z^D can be defined for $\rho \in \Gamma_{-r}(P)$ by

$$(\tau_P(\rho))^\sharp([Z_1],\dots,[Z_n]) := \varepsilon^\sharp(X_1,\dots,X_n,Z_1,\dots,Z_n)\rho^\sharp(X_1,\dots,X_n)\,,$$

where $b = (X_1,\dots,X_n,Z_1,\dots,Z_n) \in R_a(TM^{\mathbb{C}})$ with $x = (X_1,\dots,X_n) \in R_a(P)$ and $z = (Z_1,\dots,Z_n)$ such that $([Z_1],\dots,[Z_n]) \in R_a(Z^D)$. This is well-defined for fixed $(X_1,\dots,X_n)$. When $\tilde{Z}_j$ with $[Z_j] = [\tilde{Z}_j]$, we have $\varepsilon^\sharp(X_1,\dots,X_n,Z_1,\dots,Z_n) = \varepsilon^\sharp(X_1,\dots,X_n,\tilde{Z}_1,\dots,\tilde{Z}_n)$, since the matrix g with $bg = \varepsilon^\sharp(X_1,\dots,X_n,\tilde{Z}_1,\dots,\tilde{Z}_n)$ satisfies $\det g = 1$. The definition is independent of the choice of the frame x. Any other basis $y = (Y_1,\dots,Y_n) \in R(P)$ of P_a has the form $y = xg$ for a suitable $g \in \mathrm{GL}(n,\mathbb{C})$, and we conclude

$$\begin{aligned}\varepsilon^\sharp(xg,z)\rho^\sharp(xg) &= \varepsilon^\sharp(Y_1,\dots,Y_n,Z_1,\dots,Z_n)\rho^\sharp(Y_1,\dots,Y_n)\\ &= \varepsilon^\sharp(x,z)|\det g|^r\rho^\sharp(x)|\det g|^{-r}\\ &= \varepsilon^\sharp(X_1,\dots,X_n,Z_1,\dots,Z_n)\rho^\sharp(X_1,\dots,X_n)\,.\end{aligned}$$

Moreover, $\tau(\rho)$ is an r-density on Z^D since

$$(\tau(\rho))^\sharp([z]) := \varepsilon^\sharp(x,zg)\rho^\sharp(x) = \varepsilon^\sharp(x,g)|\det g|^r\rho^\sharp(x) = (\tau(\rho))^\sharp([z])|\det g|^r\,.$$

(cf. proof of Proposition 12.7, part 4.) □

Note, that in the case of $M = T^*\mathbb{R}^n$ the isomorphism has the following simple and explicit form: With the notation $|dp| = |dp_1 \wedge \ldots \wedge dp_n| \in \Gamma(M,\delta_1(P))$ and with the lift τ_1 of a suitable non-vanishing r-density ν_1 on $TQ^{\mathbb{C}}$ the above isomorphism $\tau : \delta_{-r}(P) \to \delta_r(Z^D)$ is given by

$$f|dp|^{-r} \mapsto f\tau_1\,.$$

Moreover, when we choose $\nu_1 := |dq|^r$, $dq = d1^1 \wedge \ldots \wedge dq^n$ and the lift τ_1 is denoted by the same symbol the isomorphism is simply

$$f|dp|^{-r} \mapsto f|dq|^r .$$

As a consequence, the crucial half-densities $\tau, \tau' \in \Gamma(M, \delta_{1/2}(Z^D))$ tensored to the sections $s, s' \in \Gamma(M, L)$ of the original prequantum bundle L could also be considered as to be $-1/2$-densities ρ, ρ' on P with $\tau = \tau(\rho)$, $\tau' = \tau(\rho')$. The partial connection on $\delta_{1/2}(Z^D)$ in this interpretation will correspond to a partial connection on $\delta_{-1/2}(P)$ which has been defined in the preceding chapter (see Definition 12.16).

Summary

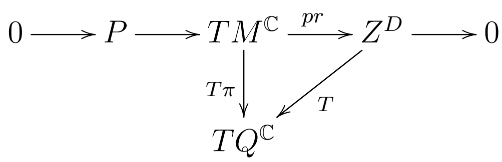
14 Half-Density Quantization in General

We now come to the construction of the representation space in case of a general complex polarization in order to achieve the half-density quantization in general. Let (M, ω) be a quantizable symplectic manifold with prequantum bundle (L, ∇, H) and let P be a complex polarization with real part D, i.e. D is the distribution $D = P \cap \overline{P} \cap TM$ of rank $k = \dim_{\mathbb{R}} D$. It is assumed throughout this section, that M/D exists as a $(2n-k)$-dimensional differentiable quotient manifold such that the quotient map $\pi : M \to M/D$ is a submersion (i.e. the complex polarization P is reducible, according to Definition 9.15).

In the following, we generalize the three construction steps described in the preceding chapter for the vertical polarization on the momentum phase space $M = T^*Q$ to the case of an arbitrary reducible complex polarization P on a general symplectic manifold (M, ω). This is the content of the next three sections. First, we have to generate half-densities on the quotient manifold M/D which are induced by the polarization P. Then the representation space will be determined and finally the quantum operators are defined.

14.1 Descend of Densities

We use again the quotient bundle $Z^D := TM^{\mathbb{C}}/D^{\mathbb{C}} \to M$ which is essentially the (lifted) tangent bundle $T(M/D)^{\mathbb{C}}$: Since the quotient map $\pi : M \to M/D$ is a submersion by definition, the tangent map $T\pi : TM \to T(M/D)$ induces a vector bundle homomorphism $T : Z^D \to T(M/D)^{\mathbb{C}}$ which is fibrewise an isomorphism, i.e. the restrictions $T_a : Z^D_a \to T_{\pi(a)}(M/D)^{\mathbb{C}}$ are vector space isomorphisms. Notice, that the action of T is essentially that of $T\pi$: When $pr : TM^{\mathbb{C}} \to Z^D = TM^{\mathbb{C}}/D^{\mathbb{C}}$ is the natural projection, we have $T\pi = T \circ pr$. Altogether there is the following commutative diagram:



In particular, we obtain a bundle morphism $R(Z^D) \to R(T(M/D)^{\mathbb{C}})$ which we denote again by T with isomorphisms $T_a : R_a(Z^D) \to R_a(T(M/D)^{\mathbb{C}})$, $a \in M$. For vectors $Z_j \in T_a M^{\mathbb{C}}$, $j = 1 \ldots, 2n-k$, such that $([Z_1], \ldots, [Z_{2n-k}])$ is a basis of Z^D_a ($[Z_j] := Z_j + D_a$), T_a is given by

$$T_a([Z_1], \ldots, [Z_{2n-k}]) := (T_a\pi(Z_1), \ldots T_a\pi(Z_{2n-k})) = (T_a([Z_1]), \ldots, T_a([Z_{2n-k}]))\,.$$

With the aid of T one can lift the r-densities on M/D, i.e. the r-densities on $T(M/D)^{\mathbb{C}}$, to r-densities on Z^D as we have done in Chapter 13.

Let us describe this lifting in detail for a given section $\nu \in \Gamma_r(M/D) = \Gamma(M/D, \delta_r(T(M/D)^{\mathbb{C}}))$: For Z_j as above the lifted r-density τ is defined by

$$\tau_a^\sharp([Z_1], \dots, [Z_{2n-k}]) := \nu_{\pi(a)}^\sharp(T_a\pi(Z_1), \dots T_a\pi(Z_{2n-k}))\,.$$

This yields a unique r-density $L(\nu) = \tau \in \Gamma_r(Z^D) = \Gamma(M, \delta_r(Z^D))$ with the property $L(\nu)^\sharp = \nu^\sharp \circ T$. The lifting defines an $\mathcal{E}(M/D)$-homomorphism $L : \Gamma_r(T(M/D)^{\mathbb{C}}) \to \Gamma_r(Z^D)$.

We describe how suitable $-r$-densities on P descend to r-densities on M/D, and apply the result to the case $r = 1/2$. To achieve this goal we first discuss which densities on Z^D are lifted densities and then consider isomorphisms $\delta_{-r}(P) \to \delta_r(Z^D)$.

The lifted densities can be characterized by a partial connection (as in the case of $M = T^*Q$, cf. Chapter 13):

Definition 14.1. Let $\nu_1 \in \Gamma(M/D, T(M/D)^{\mathbb{C}})$ be a nowhere vanishing r-density on M/D (recall that all density line bundles are trivial according to Lemma 12.5, and consequently admit nowhere vanishing sections), and let $\tau_1 \in \Gamma_r(Z^D) = \Gamma(M, \delta_r(Z^D))$ its lift: $\tau_1 := L(\nu)$. Any section $\tau \in \Gamma_r(Z^D)$ has the form $\tau = f\tau_1$ with a function $f \in \mathcal{E}(M)$. Now, for $X \in \Gamma(M, D^{\mathbb{C}})$ we define

$$\nabla_X \tau := (L_X f)\tau_1\,.$$

This definition is independent of the choice of ν_1 and yields a well-defined partial connection for $X \in \Gamma(M, D^{\mathbb{C}})$. This can be shown as in the case of $M = T^*Q$ in Section 13.1.

Lemma 14.2. *$\tau \in \Gamma_r(Z^D)$ is a lift of an r-density ν on M/D if and only if $\nabla_X \tau = 0$ for all $X \in \Gamma(M, D)$.*

Proof. A general $\tau \in \Gamma_r(Z^D)$ has the form $\tau = f\tau_1$ with $f \in \mathcal{E}(M)$. $\nabla_X\tau = (L_X f)\tau_1 = 0$, i.e. $L_X f = 0$, for all $X \in \Gamma(M, D^{\mathbb{C}})$ is equivalent to f having the form $f = h \circ \pi$ with $h \in \mathcal{E}(M/D)$. And $f\tau_1$ is the lift of $h\nu_1$ if and only if $f = h \circ \pi$. □

To proceed in the construction of the half-density quantization we want to determine a line bundle isomorphism $\delta_{-r}(P) \to \delta_r(Z^D)$ respecting the partial connections. Recall from Proposition 12.7 that the exact sequences

$$0 \longrightarrow D^{\mathbb{C}} \longrightarrow TM^{\mathbb{C}} \longrightarrow Z^D \longrightarrow 0$$

$$0 \longrightarrow D^{\mathbb{C}} \longrightarrow P \longrightarrow P/D^{\mathbb{C}} \longrightarrow 0$$

induce natural isomorphisms

$$\delta_r(Z^D) \cong \delta_r(TM^{\mathbb{C}}) \otimes \delta_{-r}(D^{\mathbb{C}})\,,$$

$$\delta_{-r}(D^{\mathbb{C}}) \cong \delta_r(P/D^{\mathbb{C}}) \otimes \delta_{-r}(P)\,,$$

and as a combination

$$\delta_r(Z^D) \cong \delta_r(TM^{\mathbb{C}}) \otimes \delta_r(P/D^{\mathbb{C}}) \otimes \delta_{-r}(P)\,.$$

As a result, there exist natural isomorphisms (see Remark 12.8)

$$\Gamma_{-r}(D^{\mathbb{C}}) \to \Gamma_r(Z^D)$$

and

$$\Gamma_{-r}(P) \to \Gamma_r(Z^D)\,.$$

In the following, we work with explicit isomorphisms $\tau_D : \Gamma_{-r}(D^{\mathbb{C}}) \to \Gamma_r(Z^D)$, resp. $\tau_P : \Gamma_{-r}(P) \to \Gamma_r(Z^D)$ (cf. Proposition 14.3), and use them for the construction of the representation space. Later we show that the construction is essentially independent of the isomorphism used (see Observation 14.7). And we will see in Remark 14.15, that using τ_D or τ_P in the construction will lead to the same representation space up to unitary equivalence[72], and thus to the same geometric quantization.

The main technical result is the following proposition. The proof is rather elementary, but due to the importance of the result in the construction of the representation space we will present all details of the proof.

Proposition 14.3. *There exists a natural line bundle isomorphism*

$$\tau_P : \delta_{-r}(P) \to \delta_r(Z^D)$$

with

$$\nabla_X \circ \tau_P = \tau_P \circ \nabla_X$$

for all $X \in \Gamma(M, D^{\mathbb{C}})$. *In particular, the following diagram is commutative*

$$\begin{array}{ccc} \Gamma_{-r}(P) & \xrightarrow{\tau_P} & \Gamma_r(Z^D) \\ {\scriptstyle\nabla_X}\downarrow & & {\scriptstyle\nabla_X}\downarrow \\ \Gamma_{-r}(P) & \xrightarrow[\tau_P]{} & \Gamma_r(Z^D) \end{array}$$

Here, the partial connections ∇_X *are defined previously:* ∇_X *on* $\delta_r(P)$ *in Definition 12.16 and* ∇_X *on* $\delta_r(Z^D)$ *in Definition 14.1.*

The isomorphism τ_P *can be understood as an isomorphism of line bundles with connection as in Definition 8.5, but here the connections are only partial.*

---

[72] In the literature τ_P is preferred.

Proof. We introduce the following notation: Let

$$\beta = (X_1, \dots, X_k, Y_{k+1} \dots, Y_n, Z_1 \dots, Z_k, Z_{k+1} \dots, Z_n,)$$

be a basis of $T_aM^{\mathbb{C}}$ such that

$$\begin{aligned}
&1.\ \xi^D := (X_1, \dots, X_k) \text{ is a basis of } D_a,\\
&2.\ \xi\ := (X_1, \dots, X_k, Y_{k+1}, \dots, Y_n) \text{ is a basis of } P_a,\\
&3.\ \xi'\ := (X_1, \dots, X_k, Z_{k+1}, \dots, Z_n) \text{ is a basis of } \overline{P}_a,\\
&4.\ \gamma\ := (Y_{k+1}, \dots, Y_n, Z_1 \dots Z_n) \text{ determines a basis } T_a\pi(\gamma)\\
&\quad \text{of } T_{\pi(a)}M/D^{\mathbb{C}}.
\end{aligned} \tag{55}$$

For $Z \in T_aM^{\mathbb{C}}$ let us denote $[Z] := Z + D_a^{\mathbb{C}} \in Z_a^D$, the image of the projection $T_aM^{\mathbb{C}} \to TM_a^{\mathbb{C}}/D_a^{\mathbb{C}} = Z_a^D$. Then the image of $\gamma := (Y_{k+1}, \dots, Y_n, Z_1 \dots Z_n)$ is a basis $[\gamma] = ([Y_{k+1}], \dots, [Y_n], [Z_1] \dots [Z_n])$ of Z_a^D. Moreover, with $\eta := (Y_{k+1}, \dots, Y_n)$ we obtain a basis $[\eta]$ of $P_a/D^{\mathbb{C}}$ and we see that $(\eta, \bar{\eta}) := (Y_{k+1}, \dots, Y_n, \bar{Y}_{k+1}, \dots, \bar{Y}_n)$ induces a basis $([\eta], [\bar{\eta}])$ of $E_a/D_a^{\mathbb{C}}$.

Let $\varepsilon := |\omega^n|^r$ be the r-density on $TM^{\mathbb{C}}$ induced by the symplectic form ω. And let $\theta^\sharp([\eta]) := |\omega^{n-k}|^{r/2}(\eta, \bar{\eta})$ define a corresponding r-density θ on $P/D^{\mathbb{C}}$. Then for $\rho \in \Gamma_{-r}(D^{\mathbb{C}})$ we set:

$$\tau_P(\rho)^\sharp([\gamma]) := \tau_P(\rho)^\sharp([Y_{k+1}], \dots, [Y_n], [Z_1] \dots [Z_n]) := \varepsilon^\sharp(\beta)\theta^\sharp([\eta])\rho^\sharp(\xi)\,. \tag{56}$$

$\tau_P(\rho) \in \Gamma_r(Z^D)$ is a well-defined r-density as we have shown in similar situations before:

For an alternative choice $\tilde{\beta} = (\tilde{X}_1, \dots, \tilde{X}_k, \tilde{Y}_{k+1}, \dots, \tilde{Y}_n, \tilde{Z}_1 \dots \tilde{Z}_n) \in R_a(T_aM^{\mathbb{C}})$ to β satisfying the analogue of (55) we denote

$$\begin{aligned}
\tilde{\xi}^D &:= (\tilde{X}_1, \dots, \tilde{X}_k),\\
\tilde{\xi}\ &:= \tilde{X}_1, \dots, \tilde{X}_k, \tilde{Y}_{k+1}, \dots, \tilde{Y}_n),\\
\tilde{\eta}\ &:= (\tilde{Y}_{k+1}, \dots, \tilde{Y}_n),\\
\tilde{\gamma}\ &:= (\tilde{Y}_{k+1}, \dots, \tilde{Y}_n, \tilde{Z}_1 \dots \tilde{Z}_n).
\end{aligned} \tag{57}$$

Then there exists a unique $G \in \mathrm{GL}(2n, \mathbb{C})$ with $\tilde{\beta} = \beta G$ such that G can be written as a block matrix

$$G = \begin{pmatrix} d & * & * \\ 0 & h & * \\ 0 & 0 & k \end{pmatrix}$$

with $d \in \mathrm{GL}(k, \mathbb{C}), h \in \mathrm{GL}(n-k, \mathbb{C})$ and $k \in \mathrm{GL}(n, \mathbb{C})$. Moreover, $\tilde{\xi}^D = \xi d$, $\tilde{\eta} = \eta h$, $\tilde{\xi} = \xi H$ and $\tilde{\gamma} = \gamma g$, where H is the block matrix

$$H := \begin{pmatrix} d & * \\ 0 & h \end{pmatrix}$$

and g is the block matrix

$$g := \begin{pmatrix} h & * \\ 0 & k \end{pmatrix} . \tag{58}$$

Then $\tau_P^\sharp([\tilde{\gamma}])$ as in (56) satisfies

$$\begin{aligned}
\tau_P^\sharp([\gamma]g) &= \tau_P^\sharp([\tilde{\gamma}]) = \varepsilon^\sharp(\tilde{\beta})\theta^\sharp([\tilde{\eta}])\rho^\sharp(\tilde{\xi}) \\
&= \varepsilon^\sharp(\beta G)\theta^\sharp([\eta]h)\rho^\sharp(\xi H) \\
&= \varepsilon^\sharp(\beta)|\det G|^r\theta^\sharp([\eta])|\det h|^r\rho^\sharp(\xi)|\det H|^{-r} \\
&= \varepsilon^\sharp(\beta)|\det d|^r|\det h|^r|\det k|^r\theta^\sharp([\eta])|\det h|^r\rho^\sharp(\xi)|\det d|^{-r}|\det h|^{-r} \\
&= \tau_P^\sharp([\gamma])|\det g|^r ,
\end{aligned}$$

since $\det G = \det d \det h \det k$, $\det H = \det d \det h$ and $\det g = \det h \det k$. Consequently, there is the following transformation rule.

$$\tau_P^\sharp([\gamma]g) = \tau_P^\sharp([\gamma])|\det g|^r = \tau_P^\sharp([\gamma])|\det G|^r|\det d|^{-r} , \tag{59}$$

for the special G, g, d as above.

In order to check whether the definition (56) is independent of the choice of the representatives Z of the classes $[Z] = T_aM^{\mathbb{C}}/D_a^{\mathbb{C}}$ we assume ξ^D to be fixed and $[\tilde{\gamma}] = [\gamma]$. In that case all elements of the diagonal of G are 1. Hence $\det G = 1 = \det d = \det g$, and the equation (59) yields

$$\tau_D^\sharp([\tilde{\gamma}]) = \tau_D^\sharp([\gamma]) .$$

Hence the term $\tau_P(\rho)^\sharp([\gamma])$ is well-defined when the basis $\xi^D = (X_1, \ldots, X_k)$ of D_a is fixed.

Next we show that the expression (56) is independent of the choice of the basis $\xi^D \in R_a(D^{\mathbb{C}})$. Any other basis $\tilde{\xi}^D$ can be described as $\tilde{\xi}^D = \xi^D d$ with a unique $d \in \mathrm{GL}(k, \mathbb{C})$. With $h = 1$ and $k = 1$ in our block matrix G we have $\det d = \det G$ and we obtain from (59)

$$\tau_P^\sharp([\tilde{\gamma}]) = \tau_P^\sharp([\gamma]g) = \varepsilon^\sharp(\beta)|\det G|^r\theta^\sharp([\eta])\rho^\sharp(\xi)|\det d|^{-r} = \varepsilon^\sharp(\beta)\theta^\sharp([\eta])\rho^\sharp(\xi) = \tau_P^\sharp([\gamma]) .$$

Thus, (56) is independent of the choice of ξ^D.

As a result, $\tau_P(\rho)^\sharp([\gamma])$ is well-defined.

Let us confirm that $\tau_P(\rho)$ determines an r-density on Z^D. Let $\tilde{\beta}$ an alternative choice of a frame for $T_aM^{\mathbb{C}}$ according to (57). We can require $\tilde{\xi}^D = \xi^D$ since the expression (56) is independent of ξ^D as we have just shown. There is a unique $g \in \mathrm{GL}(2n-k, \mathbb{C})$ such that $\tilde{\gamma} = \gamma g$ and which can be written as a block matrix as in (58).

With G as the block matrix defined by $\tilde{\beta} = \beta G$ we obtain immediately from (59)

$$\tau_P^\sharp([\gamma]g) = \tau_P^\sharp([\gamma])|\det g|^r .$$

As a consequence, fixing a $\beta = (\xi^D, \gamma)$ with (55) the assignment

$$\tau_P^\sharp([\gamma]g) := \tau_P^\sharp([\gamma])|\det g|^r \,, \quad g \in GL(2n-k, \mathbb{C})$$

for general $g \in \mathrm{GL}(2n-k, \mathbb{C})$ coincides with just proven transformation property for special g with (58), and thus defines a well-defined r-density on Z^D.

We have shown that $\tau_P(\rho)$ is an r-density and defines an isomorphism of line bundles $\tau_P : \delta_{-r}(P) \to \delta_r(Z^D)$.

Finally, let $\rho \in \Gamma(M, \delta_{-r}(D^{\mathbb{C}}))$. We have to show $\nabla_X \tau_P(\rho) = \tau_P(\nabla_X \rho)$ for all $X \in \Gamma(M, D^{\mathbb{C}})$. It is enough to show this locally. Let $a \in M$ and let U be an open neighbourhood of a for which there exists a frame field $\beta : U \to R(TM^{\mathbb{C}})$, $\beta = (\xi_1, \ldots, \xi_k, \eta_{k+1}, \ldots, \eta_n, \zeta_1, \ldots, \zeta_n)$, which satisfies (55) in each point of U. In particular $\gamma := (\eta_{k+1}, \ldots, \eta_n, \zeta_1, \ldots, \zeta_n,)$ generates Z^D, i.e. $[\gamma] := ([\eta_{k+1}], \ldots, [\eta_n], \zeta_1, \ldots, [\zeta_n,])$ is a frame field for Z^D, and $T\pi(\gamma)$ is a frame field of $T(M/D)^{\mathbb{C}}$. Finally, by changing ξ^D we can assume $\varepsilon^\sharp(\beta)\theta^\sharp(\eta) = 1$ (replace ξ^D with $\lambda\xi^D$, where $\lambda \in \mathbb{C}$ satisfying $\lambda^{-k} = \varepsilon^\sharp(\beta)\theta^\sharp(\eta)$).

From Section 12.3 we know that there is a unique density $\rho_1 = \rho_\xi \in \Gamma_{-r}(U, P)$ with $\rho_1(\xi) = 1$ and $\nabla_X(\rho_1) = 0$ for all $X \in \Gamma(M, D^{\mathbb{C}})$ (even for all $X \in \Gamma(M, P)$). Moreover, any $\rho \in \Gamma_{-r}(U, P)$ is of the form $\rho = \rho^\sharp(\xi)\rho_1$ with $\nabla_X \rho = L_X(\rho^\sharp(\xi))\rho_1$. Hence

$$(\nabla_X \rho)^\sharp(\xi) = L_X(\rho^\sharp(\xi)) \,.$$

In the same way there exists a unique density $\tau_1 : U \to \delta_r(Z^D)$ with $\tau_1^\sharp([\gamma]) = 1$ with $\nabla_X(\tau_1) = 0$ for the partial connection ∇_X , $X \in \Gamma(M, D^{\mathbb{C}})$. Any $\tau = f\tau_1 \in \Gamma(U, Z^D)$ has the representation $\tau = \tau^\sharp([\gamma])\tau_1$ and satisfies $\nabla_X \tau = L_X(\tau^\sharp([\gamma]))\tau_1$. In particular,

$$\tau_P(\rho) = \tau_P(\rho)^\sharp([\gamma])\tau_1 = \varepsilon^\sharp(\beta)\theta^\sharp(\eta)\rho^\sharp(\xi)\tau_1 = \rho^\sharp(\xi)\tau_1 \,.$$

It follows $\tau_P(\rho_1) = \tau_1$ since $\tau_P(\rho_1)([\gamma]) = \rho^\sharp(\xi)\tau_([\gamma]) = 1$. In addition we obtain

$$\nabla_X \tau_P(\rho) = L_X(\rho^\sharp(\xi))\tau_1 \,.$$

Replacing in the last term the expression $L_X(\rho^\sharp(\xi))$ by $(\nabla_X \rho)^\sharp(\xi)$ completes the proof of the proposition due to the following identities

$$\nabla_X \tau_P(\rho) = (\nabla_X \rho)^\sharp(\xi)\tau_1 = \varepsilon^\sharp(\beta)\theta^\sharp(\eta)(\nabla_X \rho)^\sharp(\xi)\tau_1 = \tau_P(\nabla_X \rho) \,.$$

□

Observation 14.4. Let $\tau_1 \in \Gamma(M, \delta_r(Z^D))$ be a nowhere vanishing (now global!) lift of a nowhere vanishing r-density $\nu_1 \in \Gamma_r(T(M/D)^{\mathbb{C}})$. The result of the preceding proposition implies that there exists a nowhere vanishing density $\rho_1 \in \Gamma(M, \delta_1(P))$ such that $\nabla_X \rho_1 = 0$ for all $X \in \Gamma(M, D^{\mathbb{C}})$, namely $\rho_1 := (\tau_P)^{-1}(\tau_1)$. Conversely, the existence of such a ρ_1 yields the result of the proposition, by simply defining

$$\tau_P : \delta_{-r}(P) \to \delta_r(Z^D) \;, \; f\rho_1 \mapsto f\tau_1 \,.$$

Lemma 14.5. *Any other line bundle isomorphism* $\tau : \delta_{-r}(P) \to \delta_r(Z^D)$ *has the form* $\tau = f\tau_P$ *with* $f \in \mathcal{E}(M)$*. Moreover,* τ *satisfies*

$$\nabla_X \circ \tau = \tau \circ \nabla_X$$

for all $X \in \Gamma(M, D)$ *if and only if* $f = h \circ \pi$ *for a suitable* $h \in \mathcal{E}(M/D)$*.*

Proof. The first statement is clear. Moreover, when $\tau = f\tau_P$ satisfies $\nabla_X \circ \tau = \tau \circ \nabla_X$ for all $X \in \Gamma(M, D)$, then $\nabla_X(f\tau_P(\rho_1)) = \nabla_X(f\tau_1) = L_X f\tau_1 = \tau(\nabla_X(\rho_1) = 0$, hence $L_X f = 0$ and $f = h \circ \pi$, $h \in \mathcal{E}(M/D)$. And viceversa. □

14.2 Representation Space

The construction of the representation space can now be carried through as in Chapter 13 using directly the bundle $\delta_{1/2}(Z^D)$. However, we are interested in describing the representation space using the density bundle $\delta_{-1/2}(P)$ (and $\delta_{-1/2}(D^{\mathbb{C}})$, see below in Section 14.4).

Construction 14.6 (Representation Space)**.** We replace L with

$$L \otimes \delta \,,\; \delta := \delta_{-1/2}(P)\,,$$

and ∇ with $\nabla \otimes \nabla^\delta$, where ∇^δ is the partial connection on $\delta = \delta_{-1/2}(P)$.

Let $s \otimes \rho \in \Gamma(M, L \otimes \delta_{-1/2}(P))$ be polarized[73]. Then $\nabla_X s = 0$ and $\nabla_X \rho = 0$ for all $X \in \Gamma(M, D^{\mathbb{C}})$[74]. According to Proposition 14.3 $\nabla_X(\tau_P(\rho)) = 0$. This implies that there is a unique $\nu = \nu(\rho) \in \Gamma_{1/2}(T(M/D)^{\mathbb{C}})$ (cf. Lemma 14.2), such that $\tau_P(\rho)$ is the lift of ν. As a consequence, any two polarized sections $\psi = s \otimes \rho, \psi' = s' \otimes \rho' \in \Gamma(M, L \otimes \delta)$, determine the 1-density

$$H(s, s')\bar{\nu}\nu'$$

on M/D (with $\nu' = \nu(\rho')$). This 1-density on M/D defines the scalar product

$$\langle \psi, \psi' \rangle := \int_{M/D} H(s,t)\bar{\nu}^\sharp \nu'^\sharp$$

and the pre Hilbert space

$$H_P = \{\psi \in \Gamma(M, L \otimes \delta)\psi \ \text{ polarited } \text{ and} \, \langle \psi, \psi \rangle < \infty\}$$

of polarized sections with with finite integral. The completion of H_P with respect to the induced norm

$$\|\psi\| = \sqrt{\langle \psi, \psi \rangle}$$

is the representation space $\mathbb{H}_P^\delta = \mathbb{H}^\delta(M, L, P)$ which we wanted to construct.

[73] Every section $\psi \in \Gamma(M, L \otimes \delta)$ can be written in the form $\psi = s \otimes \rho$, since there exists a nowhere vanishing global section ρ of δ.

[74] ρ can be chosen to be polarized and nowhere vanishing, hence $\nabla_X(s \otimes \rho) = (\nabla_X s) \otimes \rho + s \otimes \nabla_X \rho = 0$ if and only if $\nabla_X s = 0$.

Observation 14.7. Using a different isomorphism

$$\tau = f\tau_P : \Gamma_{-1/2}(P) \to \Gamma_{-1/2}(Z^D)$$

with the property $\nabla_X \circ \tau = \tau \circ \nabla_X$ for all $X \in \Gamma(M, D^{\mathbb{C}})$ the scalar has the form $f = h \circ \pi$, cf. Lemma 14.5. The use of τ instead of τ_P leads to the representation space $\mathbb{H}^\delta(\tau)$ which is unitarily equivalent to $\mathbb{H}^\delta$ by the unitary map

$$\mathbb{H}^\delta(\tau) \to \mathbb{H}^\delta\,,\ \phi \mapsto h\phi\,.$$

14.3 Quantum Operator

Before we state the main result of this section in the theorem below the quantum operator q^δ has to be defined. Let (L, ∇, H) be a prequantum line bundle on the symplectic manifold (M, ω) with complex polarization P and with the additional line bundle $\delta := \delta_{-1/2}(P)$. For every directly quantizable observable $F \in \mathcal{E}(M, \mathbb{R})$[75] the local flow (Φ_t^F) of the Hamiltionian vector field X_F on M induces – with the aid of the naturally lifted vector field Z_F on $L^\times$ – the local one-parameter group (ρ_t^F) of transformations, $\rho_t^F : \Gamma(M_t, L) \to \Gamma(M_{-t}, L)$, where the impact of the connection is already included as we have deduced in Lemma 7.12. This local one-parameter group (ρ_t^F) has as its infinitesimal generator the prequantum operator

$$q(F)s = \frac{i}{2\pi}\frac{d}{dt}\rho_t^F(s)\Big|_{t=0}.$$

See Proposition 7.13. Moreover, the local flow (Φ_t^F) of the Hamiltonian vector field X_F, given by the diffeomorphisms $\Phi_t^F : M_t \to M_{-t}$, induces a pull-back $(\Phi^F)_t^* : \Gamma(M_t, \delta) \to \Gamma(M_{-t}, \delta)$, a local one-parameter group, which we denote again by $\Phi_t^F = (\Phi^F)^*$. We know

$$L_{X_F}\sigma = \frac{d}{dt}\Phi_t^F\sigma\Big|_{t=0} = \frac{d}{dt}(\Phi_t^F)^*\sigma\Big|_{t=0}$$

by definition of the Lie derivative L_{X_F}.

The two local one-parameter groups (ρ_t^F) and $\Phi_t^F)$ on the sections of L resp. δ define a local one-parameter group

$$\kappa_t^F : \Gamma(M_t, L \otimes \delta) \to \Gamma(M_{-t}, L \otimes \delta).$$

Locally, we set $\kappa_t^F(\varphi) = \rho_t^F s \otimes \Phi_t^F \sigma$ for $\varphi = s \otimes \sigma$.

Definition 14.8. The (half-density) quantum operator $q^\delta(F)$ for directly quantizable F is defined by

$$\boxed{q^\delta(F)\varphi := \frac{i}{2\pi}\frac{d}{dt}\kappa_t^F(\varphi)\Big|_{t=0},}$$

where φ is a section of $\Gamma(M, L \otimes \delta)$.

[75] Only real observables are relevant.

Lemma 14.9. *For sections* $\varphi = s \otimes \sigma$

$$q^\delta(F)(s \otimes \sigma) = q(F)s \otimes \sigma - \frac{i}{2\pi} s \otimes L_{X_F}\sigma$$
$$= -\frac{i}{2\pi}\left((\nabla_{X_F} s + 2\pi i F s) \otimes \sigma + s \otimes L_{X_F}\sigma\right).$$

Proof.

$$q^\delta(F)\varphi = q^\delta(F)(s \otimes \sigma) = \frac{i}{2\pi}\frac{d}{dt}\rho_t^F s \otimes \Phi_t^* \sigma\Big|_{t=0}$$
$$= \frac{i}{2\pi}\frac{d}{dt}\rho_t^F s\Big|_{t=0} \otimes \sigma + s \otimes \frac{i}{2\pi}\frac{d}{dt}\Phi_t^*\sigma\Big|_{t=0}$$
$$= q(F)s \otimes \sigma - s \otimes \frac{i}{2\pi} L_{X_F}\sigma$$

□

Certainly, the formula

$$\boxed{q^\delta(F)(s \otimes \sigma) = q(F)s \otimes \sigma - \frac{i}{2\pi} s \otimes L_{X_F}\sigma} \qquad (60)$$

can also serve as a definition of the quantum operator.

Observe, that $P_t := (\Phi^F)_t^*(P) = P$, when X_F is complete, and this implies that the corresponding representation spaces $\mathbb{H}_{P_t}$ agree with $\mathbb{H}_P$. Therefore, κ_t^F and $q^\delta(F)$ is defined on $\mathbb{H}_P$ and $q^\delta(F)$ is an operator in $\mathbb{H}_P$.

Lemma 14.10. *$q(F)^\delta$ is the infinitesimal generator of the one-parameter group (κ_t^F) of unitary operators.*

In particular, when X_F is complete, $q^\delta(F)$ is self-adjoint. Summerizing:

MAIN RESULT

Theorem 14.11 (Half-Density Quantization)**.** *Let (L, ∇, H) be a prequantum line bundle on the symplectic manifold (M, ω) with a reducible complex polarization P. The half-density quantization for P has as its quantum operators $q^\delta : \mathfrak{R}_P \to \mathcal{S}(\mathbb{H}^\delta)$ the maps*

$$q^\delta(F)(s \otimes \rho) = q(F)s \otimes \rho - \frac{i}{2\pi} s \otimes L_{X_F}\rho\,,$$

for polarized sections $s \otimes \rho \in \Gamma(M, L \otimes \delta)$, where $q(F)$ is the prequantum operator.

Moreover, half-density quantization is a full geometric quantization in the following sense:

1. *$q^\delta(F)$ is $\mathbb{R}$-linear and satisfies* (D1) *and* (D2)*, now for the new representation space $\mathbb{H}^\delta = \mathbb{H}^\delta(M, L, P)$ (resp. $\mathbb{H}^\delta(M, L, D)$, see below).*

2. *If for $F \in \mathfrak{R}_P$, the vector field X_F is complete, then $q^\delta(F)$ is self-adjoint.*

Let us consider the special case $D = 0$, i.e. P is a Kähler polarization. Then $M = M/D$. In particular, there is no need to search for a natural measure or volume form, we have the natural volume form ω^n on M.

Example 14.12. Let $M = T^*\mathbb{R}^n = \mathbb{C}^n$ be the simple phase space (see Examples 9.16, 10.9, 10.23) in complex coordinates $z_j = p_j + iq^j$ and with symplectic form $\omega = \frac{1}{2}i \sum d\bar{z}_j \wedge dz_j$. The holomorphic polarization P is given by

$$P := \operatorname{span}_{\mathcal{E}(M)} \left\{ \frac{\partial}{\partial \bar{z}_j} \;\middle|\; 1 \le j \le n \right\}.$$

The prequantum bundle is the trivial bundle $L = M \times \mathbb{C}$ with connection given by

$$\alpha = \frac{i}{2} \sum \bar{z}_j dz_j\,.$$

The additional line bundle $\delta = \delta_{-1/2}(P)$ is generated by the $-1/2$-density $|d\bar{z}|^{-1/2}$, where $d\bar{z}$ is the n-form $d\bar{z} = d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_n$.

In order to describe the transformation $\tau_P : \delta_{-1/2}(P) \to \delta_{1/2}(M)$ (note that $Z^D = M$) let $\rho \in \Gamma_{-1/2}(P)$ be defined by

$$\rho^\sharp(X_1, \ldots, X_n) := (|\omega^n|^{-1/4})(X_1, \ldots, X_n, \overline{X_1} \ldots, \overline{X_n})\,,$$

for a basis $(X_1, \ldots, X_n) \in P_a$. Up to a constant ρ coincides with $|d\bar{z}|^{-1/2}$. The corresponding $\tau_P(\rho)$ according to (56) is

$$\tau_P(\rho) = |\omega^n|^{\frac{1}{2}} \in \Gamma_{\frac{1}{2}}(M)\,.$$

A general section $\psi \in \Gamma(M, L \otimes \delta)$ has the form $\psi = f s_e \otimes \rho$ with $f \in \mathcal{E}(M)$ and

$$s_e(a) := \left(a, \exp\left(-\sum_{j=1}^n \frac{\pi}{2} \bar{z}_j z_j \right) \right) = \exp(-\frac{\pi}{2}\bar{z}z)(a, 1)$$

(with $\bar{z}z := \|z\|^2$), as in Example 10.9.

Since $\nabla_X s_e := 2\pi i \alpha(X) s_e$, for $X \in \mathfrak{V}(M)$, and hence

$$\nabla_X f s_e \otimes \rho = (L_X f) s_e \otimes \rho$$

for $X \in \Gamma(M, P)$, the polarized sections of the line bundle $L \otimes \delta$ are the sections $\psi = f s_e \otimes \rho$ with

$$\frac{\partial}{\partial \bar{z}_j} f = 0\,,\ j = 1, \ldots n\,.$$

Consequently, they are in one to one correspondence to the holomorphic functions f on $\mathbb{C}^n$ and the representation space $\mathbb{H}_P^\delta$ of the half-density quantization is – according to the construction in Section 14.2 – essentially the Hilbert space

$$\mathbb{H}_P^\delta = \{f \in \mathcal{O}(\mathbb{C}^n) \mid \int_{\mathbb{C}^n} \bar{f} f \exp(-\pi \bar{z} z) < \infty\}\,.$$

This is the Bargmann-Fock space $\mathbb{F}$, the reduced representation space $\mathbb{H}_P$ of the uncorrected quantization as in Example 10.9. As a consequence, the quantum operators q and q^δ are both defined on $\mathbb{F}$ with

$$q^\delta(F)(s \otimes \rho) = q(F)(s) \otimes \rho - \frac{i}{2\pi} s \otimes L_{X_F}\rho\,. \tag{61}$$

We are interested to know to which extend $q(F)$ and $q^\delta(F)$ differ for directly quantizable observables F which amounts to determine $L_{X_F}\rho$.

We restrict to real observables, i.e. F has the form $F = A + \bar{D}^k z_k + D^k \bar{z}_k + \sum C_j \bar{z}_j z_j$ with real constants A, C_j and complex constants D^k. We concentrat on $A = 0 = D^k$ and $C_j = \frac{1}{2}$. Then F is the energy $F = H = \frac{1}{2}\bar{z}z$ of the harmonic oscillator. We have seen $Ł_{X_H}|d\bar{z}| = 0$ in the case of $n = 1$ (cf. Example 12.27) and this result holds true for arbitrary n. But then we have

$$L_{X_H}\rho = 0\,.$$

As a result, the additional term of the quantum operator $q^\delta(H)$ in (61) is zero, and the half-density quantization does not change the Kähler quantization $q(H)$ we had discussed before in Chapter 10 without using half-densities (see Remark 10.8). In particular, the half-density quantization does not resolve the problem of the shift in the spectrum of the quantized energy (cf. Example 10.23).

For complex observables F the partial Lie derivative does not always annihilate $|d\bar{z}|$ as the Example 12.28 shows. In general, for $F = iC(z)\bar{z}z$, a non-zero imaginary part of $C(z)$ will contribute a non-zero factor γ: $L_{X_F}|d\bar{z}| = \gamma|d\bar{z}|$.

The half-form correction, however, will lead to a correct quantization in the case of the harmonic oscillator as we show in the next chapter, Chapter 15, on half-form quantization.

14.4 Other Constructions

Instead of $\delta_{-1/2}(P)$ one can also use the line bundle $\delta_{1/2}(Z^D)$ as we have done in the preceding chapter in the case of the momentum space $M = T^*Q$. Moreover, one can take the line bundle $\delta = \delta_{-1/2}(D^{\mathbb{C}})$ as the additional line bundle, when $D \neq 0$, i.e. when P is not Kähler. The general message is that these other choices lead to the same half-density quantizations up to unitary equivalence.

Before we go into details needed for $\delta = \delta_{-1/2}(D^{\mathbb{C}})$ we look at the case of a momentum phase space $M = T^*Q$ and compare the half-density quantization of this chapter with the quantization described in the preceding chapter.

Proposition 14.13. *Let P be the vertical polarization on $M = T^*Q$. The constructions using $\delta = \delta_{1/2}(Z_D)$ or $\delta = \delta_{-1/2}(P)$ lead to the same half-density quantization.*

Proof. We use the line bundle isomorphism $\tau_P : \delta_{-1/2}(P) \to \delta_{1/2}(Z^D)$ introduced in Proposition 14.3 which induces an isomorphism

$$\Gamma(M, L \otimes \delta_{-1/2}(P)) \to \Gamma(M, L \otimes \delta_{1/2}(Z^D))$$

respecting the connections. For polarized sections $\psi = s \otimes \rho, \psi' = s' \otimes \rho'$ of $L \otimes \delta_{-1/2}(P)$ the scalar product of $\mathbb{H}_P$ (with respect to $\delta_{-1/2}(P)$) is

$$\langle \psi, \psi' \rangle = \int_Q H(s, s') \nu(\rho)^\sharp \nu(\rho')^\sharp \,.$$

(Recall $M/D \cong Q$.) For the corresponding sections $s \otimes \tau_P(\rho), \psi' = s' \otimes \tau_P(\rho')$ of $L \otimes \delta_{1/2}(Z^D)$ the scalar product of $\mathbb{H}_P$ (with respect to $\delta_{1/2}(Z^D)$) is

$$\langle s \otimes \tau_P \rho, s' \otimes \tau_P \rho' \rangle := \int_Q H(s, s') \bar{\nu}^\sharp \nu'^\sharp \,,$$

according to (52), where $\nu = \nu(\tau_P(\rho))\,,\ \nu' = \nu(\tau_P(\rho'))$. This establishes a unitary equivalence between the two representation spaces.

For the quantum operators we look at the local situation and assume $Q \subset \mathbb{R}^n$. The quantizable classical variables are the $F = A + B^j p_j$ with $A, B^j \in \mathcal{E}(Q)$. Then the Hamiltonian of F is

$$X_F = B^j \frac{\partial}{\partial q^j} - \frac{\partial F}{\partial q^j} \frac{\partial}{\partial p_j} \,.$$

The quantum operators agree on the sections s of L. So one only has to show that the additional terms agree, as well.

In case of $\delta = \delta_{-1/2}(P)$ the partial Lie derivative L_{X_F} on $\delta_1(P)$, executed on $|dp| \in \Gamma(M, \delta_1(P))$ yields

$$L_{X_F} |dp| = \operatorname{div}(X_F) |dp| = \left(\frac{\partial^2 F}{\partial q^j \partial p_j} \right) |dp| = -\frac{\partial B^j}{\partial q^j} |dp| \,.$$

Hence,

$$\boxed{L_{X_F} |dp|^{-1/2} = \frac{1}{2} \frac{\partial B^j}{\partial q^j} |dp|^{-1/2} \,.}$$

As a consequence, the additional term is

$$-\frac{i}{2\pi}\frac{1}{2}\frac{\partial B^j}{\partial q^j}\,.$$

In Chapter 12 the Hamiltonian vector field $X = X_F$ leads to the operator L_{X_F} on sections τ of $\delta_r(Z^D)$ by $L_{X_F}\tau = \mathrm{div}_\nu(T\pi(X_F))\tau$, where τ is a lift of ν. Note, that

$$T\pi(X_F) = B^j\frac{\partial}{\partial q^j}\,.$$

The Lie derivative L_X of densities on Q is determined by $L_X|dq|\,,\ |dq| \in \Gamma(Q, \delta_1(TQ^{\mathbb{C}}))$:

$$L_X|dq| = \frac{\partial X^j}{\partial q^j}|dq|\ ,\quad \text{for}\ \ X = X^j\frac{\partial}{\partial q^j} \in \mathfrak{V}(Q)\,.$$

In particular,

$$L_{X_F}|dq| = \frac{\partial B^j}{\partial q^j}|dq|\,.$$

This implies

$$\boxed{L_{X_F}|dq|^{1/2} = \frac{1}{2}\frac{\partial B^j}{\partial q^j}|dq|^{1/2}\,.}$$

Therefore, the additional term is the same as above:

$$-\frac{i}{2\pi}\frac{1}{2}\frac{\partial B^j}{\partial q^j}\,.$$

□

We now want to show that in the case of $D \neq 0$ one can use $\delta = \delta_{-1/2}(D^{\mathbb{C}})$ instead of $\delta_{-1/2}(P)$. The construction using is similar to the one described in the preceding sections. One starts with $Z^D = TM^{\mathbb{C}}/D^{\mathbb{C}}$ and considers the lifting of densities from $T(M/D)^{\mathbb{C}}$ to Z^D. In Section 14.1 we recall that densities on Z^D are lifts of densities on $T(M/D)^{\mathbb{C}}$ if they are covariantly constant along D (cf. Lemma 14.2). As a further part of obtaining a suitable descend one proves the analogue of Proposition 14.3, namely

Proposition 14.14. *There exists a natural line bundle isomorphism*

$$\tau_D : \delta_{-r}(D^{\mathbb{C}}) \to \delta_r(Z^D)$$

with

$$\nabla_X \circ \tau_D = \tau_P \circ \nabla_X$$

for all $X \in \Gamma(M, D^{\mathbb{C}})$.

The definition of τ_D is simpler than that of τ_P: With the notation of the proof of Proposition 14.3 we set for $\sigma \in \Gamma_{-r}(D^{\mathbb{C}})$:

$$\tau_D^\sharp([\gamma]) := \varepsilon^\sharp(\beta)\sigma^\sharp(\xi^D)\,.$$

This result is used in the construction of the representation space on the basis of $\delta = \delta_{-1/2}(D\mathbb{C})$ along the same lines as in the case of $\delta_{-1/2}(P)$. The outcome is the representation space $\mathbb{H}^\delta = \mathbb{H}^\delta(M, L, D)$ induced by polarized sections of $L \otimes \delta$. Moreover, the quantum operator is defined the same way and leads to a corresponding main theorem, now based on $D^{\mathbb{C}}$ and $\delta = \delta_{-1/2}(D^{\mathbb{C}})$.

Remark 14.15. The representation spaces $\mathbb{H}^\delta(M, L, D)$ and $\mathbb{H}^\delta(M, L, P)$ are unitarily equivalent.

One advantage of using P instead of D – as is done in the literature – is, that the construction (based on the definition of τ_P) has a natural interpretation also for $D = 0$.

Another advantage of using P instead of D is that a pairing between $\mathbb{H}_P$ and $\mathbb{H}_{P'}$ for different complex polarizations can be defined in a natural way as we explain in the next section.

14.5 Half-Density Pairing

In general, given two different polarizations P, P' on a quantizable manifold with prequantum bundle, one should be able to compare the geometric quantization induced by P with the corresponding geometric quantization induced by P'. In particular, the resulting representation spaces $\mathbb{H}_P$ and $\mathbb{H}_{P'}$ should be closely related. In an ideal case they should be unitarily equivalent by a natural isomorphism and the quantum operators should be intertwined by this unitary equivalence.

The results of the preceding section can be used to formulate a pairing between representation spaces $\mathbb{H}_P$ and $\mathbb{H}_{P'}$ as a natural relation between these Hilbert paces.

Definition 14.16. Let P, P' be complex polarizations on the symplectic manifold (M, ω). They are called COMPATIBLE with each other, when the following three conditions are satisfied.

C1 The intersection $D := P \cap \overline{P'} \cap TM$ is an integrable distribution.

C2 D is reducible in the sense that the quotient M/D exists as a manifold such that the projection $\pi : M \to M/D$ is a submersion.

C3 $E := (P + \overline{P'}) \cap TM$ is integrable.

P is called TRANSVERSAL to P' if $P \cap \overline{P'} = \{0\}$, i.e. if $D = 0$.

Transversal polarizations are compatible to each other.

It is easy to see, that P is compatible to itself if it is reducible and if $E = (P + \overline{P}) \cap TM$ is integrable, i.e. if P is strongly reducible.

Note, that $\overline{P'}$ is a complex polarization, which we sometimes write $Q := \overline{P'}$ in the following in order to simplify notation.

In order to construct the pairing we consider the quotient vector bundle $Z^D := TM^{\mathbb{C}}/D^{\mathbb{C}}$, as before. Recall that Z^D is essentially the tangent bundle $T(M/D)^{\mathbb{C}}$ of the quotient M/D, and $Z^D = TM^{\mathbb{C}}$ if $D = 0$.

The polarization P resp. $Q = \overline{P'}$ has its real distribution $D_P := P \cap \overline{P} \cap TM$ resp. $D_Q = Q \cap \overline{Q} \cap TM = \overline{P}' \cap P' \cap TM$. Moreover, the distribution D as in C1 is the intersection $D_P \cap D_Q$: By definition $D = P \cap Q \cap TM$ contains the intersection $D_P \cap D_Q = P \cap \overline{P} \cap Q \cap \overline{Q} \cap TM$. Since for $X \in D = P \cap Q \cap TM$ we have $\overline{X} = X \in \overline{P} \cap \overline{Q} \cap TM$ we see $X \in P \cap Q \cap \overline{P} \cap \overline{Q} \cap TM$, hence

$$D \subset D_P \cap D_Q = P \cap \overline{P} \cap Q \cap \overline{Q} \cap TM\,.$$

Along the same arguments as in the proof of Proposition 14.3 we obtain

Proposition 14.17. *Let P be a complex polarization on (M, ω) and let $D \subset D_P$ be a distribution such that M/D exists as quotient manifold where the quotient map $M \to M/D$ is a submersion. Then there is a natural line bundle isomorphism*

$$\tau_P : \delta_{-r}(P) \to \delta_r(Z^D) \quad \text{with} \quad \nabla_X \circ \tau_P = \tau_P \circ \nabla_X$$

for all $X \in \Gamma(M, D^{\mathbb{C}}) \subset \Gamma(M, P)$. Here ∇_X are the partial connections on $\delta_{-r}(P)$ resp. on $\delta_r(Z^D)$.

Observation 14.18. Any isomorphism satisfying the compatibility condition $\nabla_X \circ \tau = \tau \circ \nabla_X$ for all $X \in \Gamma(M, D^{\mathbb{C}})$ is of the form $f\rho_1 \to f\tau_1$, where τ_1 is the lift of a nowhere vanishing $\nu_1 \in \Gamma(M/D, \delta_r(T(M/D)^{\mathbb{C}})$ and where ρ_1 is a nowhere vanishing section of $\delta_{-r}(P)$ satisfying $\nabla_X \rho_1 = 0$ for all $X \in \Gamma(M, D^{\mathbb{C}})$. We know that ν_1 and τ_1 exist. And ρ_1 can locally be defined by

$$\rho_1^\sharp(\xi) := (\varepsilon^\sharp(\beta)\theta^\sharp(\eta))^{-1}\tau_1^\sharp([\gamma])\,,$$

where the notation is the one used in the proof of Proposition 14.3.

Now we come to the construction of the pairing on the basis of two compatible polarizations P and $P' = \overline{Q}$. Let $\psi = s \otimes \rho \in \Gamma(M, L \otimes \delta)$[76] resp. $\psi = s' \otimes \rho' \in \Gamma(M, L \otimes \delta')$ global sections (where $\delta' := \delta_{-\frac{1}{2}}(Q)$) which are polarized, i.e.

$$(\nabla \otimes \nabla^\delta)_X \psi = 0 \;\text{ for all }\; X \in \Gamma(M, P)\,,\; (\nabla \otimes \nabla^{\delta'})_X \psi' = 0 \;\text{ for all }\; X \in \Gamma(M, Q)\,.$$

[76] Every global section can be written in this form.

Then

$$\nabla_X s = 0 \text{ for all } X \in \Gamma(M,P)\,,\ \nabla_X s' = 0 \text{ for all } X \in \Gamma(M,Q)\,,$$

and

$$\nabla^\delta_X \rho = 0\,,\ \nabla^\delta_X \rho' = 0 \text{ for all } X \in \Gamma(M, D^{\mathbb{C}})\,.$$

As a consequence, s and s' are constant on the leaves of D and the result of Proposition 14.17 implies that $\tau_P(\rho) \in \Gamma_{1/2}(Z^D)$ resp. $\tau_Q(\rho') \in \Gamma_{1/2}(Z^D)$ are lifts of suitable half-densities on M/D which we denote by $\nu = \nu(\rho)$ resp. $\nu' = \nu(\rho')$. Altogether, the sections ψ, ψ' induce a 1-density

$$(\psi, \psi') := H(s,t)\bar{\nu}^\sharp \nu'^\sharp$$

on M/D, which defines a natural sesquilinear map

$$B_{P,Q} : H_P \times H_Q \to \mathbb{C}\,,\ (\psi,\psi') \mapsto B_{P,Q}(\psi,\psi') := \int_{M/D} (\psi,\psi') = \int_{M/D} H(s,t)\overline{\nu}^\sharp \nu'^\sharp\,.$$

Recall, that $H_P = H_P^\delta$ resp. $H_Q = H_Q^{\delta'}$ [77] is the space of polarized sections of $L \otimes \delta_{-1/2}(P)$ resp. $L \otimes \delta_{-1/2}(Q)$ with compact support. We can also take the larger spaces of secctions where the integal is finite.

To make the formula for the pairing $B_{P,Q}$ more concrete let us assume, that L is trivial and has a global nowhere vanishing section $s_1 \in \Gamma(M,L)$, a situation which is locally true. Moreover, let ρ_1 resp. ρ'_1 be global nowhere vanishing sections of $\delta = \delta_{-\frac{1}{2}}(P)$ resp. $\delta' = \delta_{-\frac{1}{2}}(Q)$ with $\nabla_X \rho_1 = 0$ for all $X \in \Gamma(M,P)$ resp. $\nabla_X \rho'_1 = 0$ for all $X \in \Gamma(M,Q)$. Then the polarized sections $\psi \in \Gamma(M, L \otimes \delta), \psi' \in \Gamma(M, L \otimes \delta')$ can be written in the form $\psi = \phi s_1 \otimes \rho_1$ resp. $\psi' = \phi' s_1 \otimes \rho'_1$ with functions $\phi, \phi' \in \mathcal{E}(M/D)$. Moreover, $\tau_P(\rho_1)$ resp. $\tau_Q(\rho'_1)$ is the lift of a half-density ν_1 resp. ν'_1 in $\Gamma(M/D, \delta_{\frac{1}{2}}(T(M/D)^{\mathbb{C}}))$. As a result, the density (ψ,ψ') has the form $(\psi,\psi') = \bar{\phi}\phi'\bar{\nu}_1^\sharp \nu_1'^\sharp$ which implies

$$B(\psi,\psi') = \int_{M/D} \bar{\phi}\phi'\bar{\nu}_1^\sharp \nu_1'^\sharp\,. \tag{62}$$

Note that ν_1 and ν'_1 can be chosen to be $\mu^{1/2}$ with an everywhere positive 1-density μ on M/D[78]. With this choice the formula simplifies to

$$B(\psi,\psi') = \int_{M/D} \bar{\phi}\phi'\mu^\sharp\,.$$

Moreover, B is non-degenerated.

We have defined a natural pairing $B = B_{P,Q}$ on $H_P \times H_Q$ which is directly induced by the construction of the half-density representation spaces. To obtain a map $T = T_{P,Q} : H_P \to H_Q$ from this pairing the continuity of B would be helpful.

[77] In the following we omit the upper indices δ, δ' for convenience.

[78] Such a μ exists according to Proposition 12.5.

Lemma 14.19. *When B is partially continuous, i.e. B is continuous in each variable, there exists a linear map $T : H_P \to \mathbb{H}_Q$ such that for each $\psi \in H_P$*

$$B(\psi, \psi') = \langle T\psi, \psi' \rangle$$

for all $\psi' \in H_Q$.

Proof. In fact, fixing $\psi \in H_P$ defines a continuous and linear map

$$H_Q \to \mathbb{C}\ ,\ \psi' \mapsto B(\psi, \psi')\,,$$

which can be extended to $\mathbb{H}_Q$ as a continuous linear functional $B_\psi : \mathbb{H}_Q \to \mathbb{C}$. By the Riesz representation theorem the continuous linear functional B_ψ can be represented by a unique $T\psi \in \mathbb{H}_Q$ in the sense that $B(\psi, \psi') = \langle T\psi, \psi' \rangle$ for all $\psi' \in \mathbb{H}_Q$. The map $T : H_P \to \mathbb{H}_Q$ is linear and injective. □

When B is continuous, i.e

$$B(\psi, \psi') \leq C\,\|\psi\|\,\|\psi'\|\ ,\ (\psi, \psi') \in H_P \times H_Q$$

for a constant C, then the map $T = T_{P,Q}$ in the preceding lemma is continuous as well and can be continued to all of $\mathbb{H}_P$ yielding a continuous and bijective $T : \mathbb{H}_P \to \mathbb{H}_Q$. However, T will not be unitary, in general.

There seems to be no general method to show the continuity of B. One of the problems to implement such a method originates in the fact, that the norms of H_P resp. H_Q are, in general, not related to the definition of the form B: The three distributions $D = P \cap Q \cap TM$, $D_P := P \cap \overline{P} \cap TM$, $D_Q := Q \cap \overline{Q} \cap TM$ are, in general different form each other. Therefore, the integration over M/D, which is used to define B, is not related to the integration over M/D_P, which is used to define the norm on H_P and not related to the integration over M/D_Q, which is used to define the norm of H_Q , $Q = \overline{P'}$.

Summarizing, we cannot be sure that the pairing is defined on $\mathbb{H}_P \times \mathbb{H}_Q$ in a reasonable way, nor that it is continuous. However, in important cases T turns out to be well-defined and unitary as we see in the following proposition.

Proposition 14.20 (Fourier Transform)**.** *Let $M = T^*\mathbb{R}^n$ be the momentum phase space (simple case) with standard symplectic form $\omega = dq^j \wedge dp_j$ and prequantum bundle (L, ∇, H), where $L = M \times \mathbb{C}$, ∇ given by the connection form $-p_j dq^j$ and H the constant Hermitian metric on $M \times \mathbb{C}$. In the case of the vertical polarization P and the horizontal polarization $P' = Q$ the natural pairing $B_{P,Q}$ is continuous on $\mathbb{H}_P \times \mathbb{H}_Q$, and the corresponding map $T_{P,Q}$ is unitary. T is the Fourier transform up to a scaling factor.*

Proof. P is transversal to Q and $Z^D = TM^{\mathbb{C}}$, since $D = P \cap Q \cap TM = \{0\}$. The connection can be written in the form

$$\nabla_X f s_1 = \big(L_X f - 2\pi i\, p_j dq^j(X) f\big)\, s_1$$

for a general section $s = f s_1$, $f \in \mathcal{E}(M)$, of L. Here, s_1 is the special section $s_1(a) = (a, 1)$, $a \in M$, as before.

Let dp denote the n-form $dp := dp_1 \wedge dp_2 \wedge \ldots \wedge dp_n \in \Gamma(M, \Lambda^n(P^\vee))$ with the densities $|dp| = |dp^1 \wedge dp^2 \wedge \ldots \wedge dp^n| \in \Gamma(M, \delta_1(P))$ and $|dp|^{-1/2} \in \Gamma(M, \delta_{-1/2}(P))$. Any $-1/2$-density of P is of the form $f|dp|^{-1/2}$ with $f \in \mathcal{E}(M)$. Hence, a general section $\psi = s \otimes \rho \in \Gamma(M, L \otimes \delta)$ with $\delta = \delta_{-1/2}(P)$ can be written as

$$\psi = \phi s_1 \otimes |dp|^{-1/2}$$

with $\phi \in \mathcal{E}(M)$. The generating section $|dp|^{-1/2}$ is polarized with respect to P, since

$$|dp|^{-1/2}\Big(\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_1}\Big) = 1\,,$$

so that $\nabla_X(|dp|^{-1/2}) = 0$ for $X \in \Gamma(M, P)$. As a consequence, the section $\psi = \phi s_1 \otimes |dp|^{-1/2} \in \Gamma(M, L \otimes \delta)$ is polarized if and only if ϕs_1 is polarized and this is equivalent to

$$\frac{\partial}{\partial p^j}\phi = 0\,,\ j = 1 \ldots, n\,.$$

Hence, we have recovered the result of Construction 13.4 that the space of polarized sections is

$$\Gamma_{\nabla\otimes\nabla^\delta, P}(M, L \otimes \delta) = \{ f s_1 \otimes |dp|^{-1/2} \mid f = f(q) \in \mathcal{E}(\mathbb{R}^n)\}\,,$$

and $\mathbb{H}^\delta_P(M)$ can be identified with $L^2(\mathbb{R}^n, d\lambda(q)$.

In the same way, $dq := dq^1 \wedge dq^2 \wedge \ldots \wedge dq^n \in \Gamma(M, \Lambda^n(Q^\vee))$ induces the polarized section $|dq|^{-1/2} \in \Gamma(M, \delta_{-1/2}(Q))$. And a section $\psi' = \phi' s_1 \otimes |dq|^{-1/2} \in \Gamma(M, L \times \delta')$, where $\delta' = \delta_{-1/2}(Q)$ and $\phi' \in \mathcal{E}(M)$, is polarized if an only if

$$\frac{\partial}{\partial q^j}\phi' - 2\pi i p_j \phi' = 0\,,\ j = 1 \ldots, n\,.$$

The general solution of this system of differential equations is

$$\phi'(q,p) = f'(p) e^{2\pi i p_j q^j}\,,\ (q,p) \in T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n\,,$$

with an arbitrary function $f' = f'(p) \in \mathcal{E}(\mathbb{R}^n, \mathbb{C})$.

Hence, the space of polarized sections with respect to the horizontal polarization Q is

$$\Gamma_{\nabla\otimes\nabla^{\delta'}, Q}(M, L \otimes \delta') = \{ f' e^{2\pi i p_j q^j} s_1 \otimes |dq|^{-1/2} \mid f' = f'(p) \in \mathcal{E}(\mathbb{R}^n)\}\,,$$

and $\mathbb{H}_P^\delta(M)$ can be identified with $\{f'e^{2\pi p_j q^j} \mid f' \in L^2(\mathbb{R}^n, d\lambda(p))\} \cong L^2(\mathbb{R}^n)$.

The induced 1-density (ψ, ψ') on $T^*M^{\mathbb{C}} = Z^D$ – which leads to the form $B_{P,Q} : H_P \times H_Q \to \mathbb{C}$ according to the construction explained above (see (62)) – is

$$(\psi, \psi') = \bar{\phi}\phi'|dq||dp| = \bar{\phi}\phi'|dq \wedge dp|\,.$$

Hence,

$$\begin{aligned} B_{P,Q}(\psi, \psi') &= \int_M (\psi, \psi') = \int_M \bar{\phi}\phi' dq \wedge dp \\ &= \int_M \overline{f(q)} f'(p) e^{2\pi i p_j q^j} dq \wedge dp \\ &= \int_{\mathbb{R}^n} \left(\overline{\int_{\mathbb{R}^n} f(q) e^{-2\pi i p q} dq} \right) f'(p) dp\,. \end{aligned}$$

$B = B_{P,Q}$ is continuous, since it is bounded:

$$|B(\psi, \psi')| \leq \int |f||f'| \leq \sqrt{\int |f|^2) \int |f'|^2} = \|\psi\|\|\psi'\|$$

for $(\psi, \psi') \in H_P \times H_Q$. Therefore, B can be extended uniquely to $\mathbb{H}_P \times \mathbb{H}_Q$ as a continuous bilinear form. Furthermore, $T : \mathbb{H}_P \to \mathbb{H}_Q$ defined by $T\psi = Tfs_1 \otimes |dq|^{-1/2}$ with

$$Tf(p) := \int_{\mathbb{R}^n} f(q) e^{-2\pi i p_j q^j}\, dq\,,$$

satisfies

$$B(T\psi, \psi') = \int_{\mathbb{R}^n} \overline{Tf} f' dp = \langle Tf, f' \rangle = \langle T\psi, \psi' \rangle$$

for all $(\psi, \psi') \in \mathbb{H}_P \times \mathbb{H}_Q$.

T is linear, continuous and bijective. Since $\langle Tf, Tf \rangle = \langle f, f \rangle$ the map T is even unitary. Up to a scaling factor $c > 0$, T is the Fourier transform. Altogether we have a natural unitary mapping $T : \mathbb{H}_P \to \mathbb{H}_Q$ between the representation spaces $\mathbb{H}_P$ and $\mathbb{H}_Q$. □

Remark 14.21. T intertwines the quantum operators $Q_j = q^\delta(q_j), P_j = q^\delta(p_j)$, resp. $Q'^j = q^{\delta'}(q_j), P'_j = q^{\delta'})p_j)$ in the sense that $T \circ Q^j = Q'^j \circ T$ resp. $T \circ P_j = P'_j \circ T$. In other words, T intertwines the representations

$$q := q^\delta \ \text{ resp. } \ q' := q^{\delta'}$$

of the algebra $\mathfrak{o} := \{a + b^j p_j + c_k q^k \mid a, b^j, c_k \in \mathbb{R}\}$, a subalgebra of the Poisson algebra $(\mathbb{R}^{2n}, \mathbb{R})$. This means that the following diagram is commutative

$$\begin{array}{ccc} \mathbb{H}_P & \xrightarrow{T} & \mathbb{H}_Q \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q'} \\ \mathbb{H}_P & \xrightarrow[T]{} & \mathbb{H}_Q \end{array}$$

In fact, it can be shown that $P'_j = p_j$ and

$$Q'^j = \frac{i}{2\pi}\frac{\partial}{\partial p_j}\,.$$

Hence

$$T \circ Q^k(f)(p) = \int q^k f(q) e^{-2\pi i p_j q^j}\, dq$$

and

$$\begin{aligned} Q'^k T(f)(p) &= \frac{i}{2\pi}\frac{\partial}{\partial p_k}\int f(q) e^{-2\pi i p_j q^j}\, dq \\ &= \frac{i}{2\pi}\int f(q)(-2\pi i q^k) e^{2\pi i p_j q^j}\, dq \\ &= \int q^k f(q) e^{-2\pi i p_j q^j}\, dq = T \circ Q^k(f)(p)\,. \end{aligned}$$

As a consequence, $T \circ Q^k = Q'^k \circ T$. $T \circ P_j = P'_j \circ T$ can be shown in an analoguous manner.

Summary

15 Half-Form Quantization

In principle, half-form quantization is similar to half-density quantization. The differences are:

1. Half-form quantization corrects the shift of the eigenvalues in many cases, which is not achieved by the half-density approach. Recall, that the geometric quantization of the harmonic oscillater using the holomorphic polarization (i.e. Kähler quantization) leads to an incorrect model with a shift of the eigenvalues of the quantized energy operator, see Example 10.23, and that the half-density quantization does not change this, see Example 14.12. However, in the sketch of the half-form quantization for the harmonic oscillator in the Example 10.25 we arrived at the correct model.

2. Half-form quantization is only possible when a certain topological condition of the phase space (M, ω) (more precisely of the frame bundle $R(P)$ assigned to P) is satisfied, while half-density quantization works without any additional assumption.

3. The topological condition forces one to consider rather involved new structures, like the concept of a metalinear frame bundle as a special case of a metalinear structure for a principal fibre bundle with structure group $\mathrm{GL}(n, \mathbb{C})$ or the concept of a metaplectic structure. These structures make the half-form quantization less accessible, although from an elementary standpoint forms can be considered as to be more basic than densities.

In this chapter we present the half-form quantization without using the concept of a metalinear or metaplectic structure. Instead, we only need the existence of a square root line bundle S of a certain line bundle K, i.e. $S \otimes S \cong K$, where $K = K(P)$ is naturally induced by the polarization P, it is the canonical bundle of P.

In general, such a square root bundle S will not exist. And in case of existence there might be several inequivalent choices. Existence is guaranteed if a certain cohomology class induced by P vanishes. We give a detailed explanation of this result in the Section 15.5 below. Note, that a given metalinear structure for the frame bundle $R(P)$ of P always induces such a square root.

Interestingly enough, the condition which is necessary for the existence of a square root bundle of K is exactly the same condition which ensures the existence of a metalinear frame bundle associated to the frame bundle $R(P)$. A metalinear frame bundle immediately leads to a square root S and the metalinear structure allows one to define this bundle by transformation properties of the sections similar to the properties of half-densities. We come back to metalinear structures in the next chapter where also metaplectic structures are studied. A metaplectic structure on (M, ω) induces metalinear frame bundles for several different polarizations at once and thus opens the possibility of comparing polarizations and the corresponding half-density quantization in an efficient and elegant way, in particular in order to construct a natural pairing. We study all this in the next chapter.

15.1 Canonical Bundle of a Vector Bundle

In order to define half-forms of a polarization we introduce the notion of the canonical line bundle of a general complex vector bundle.

Definition 15.1. Let $V \to X$ be a complex vector bundle of rank k over an m-dimensional manifold X: Then $K(V) := \Lambda^k(V^\vee)$ is called the CANONICAL BUNDLE of V.

Note, that the canonical bundle of the tangent bundle $TX^{\mathbb{C}}$ on the m-dimensional manifold X is the line bundle of forms of top degree m on X: $\Lambda^m(T^*X^{\mathbb{C}}) \cong \Lambda^m((TX^{\mathbb{C}})^\vee) = K(TX^{\mathbb{C}})$. This line bundle $K(TX^{\mathbb{C}})$ is sometimes denoted by $K(X)$ and called the canonical bundle of X. For $\eta \in K(X)$ we have the standard integral of m-forms with respect to an orientation of X: $\int_X \eta$. Without orientation a reasonable general integration is only possible for densities.

Lemma 15.2. *Let $V \to X$ be a complex vector bundle of rank k with transition functions g_{ij} with respect to an open cover (U_j) of X. Then $\det g_{ij}^{-1}$ are suitable transition functions of the canonical bundle $K(V) = \Lambda^k(V^\vee)$. Moreover, $K(V)$ is (isomorphic to) the line bundle $R(V) \times_{\mathrm{GL}(k,\mathbb{C})} \mathbb{C}$ associated to the frame bundle $R(V)$ of V with respect to the representation $\rho = \det^{-1} : \mathrm{GL}(k,\mathbb{C}) \to \mathbb{C}^\times$. As a result, the global sections $\alpha \in \Gamma(M, K(V))$ can be identified with the smooth functions $\alpha^\sharp : R(V) \to \mathbb{C}$ on the frame bundle $R(V)$ of V satisfying the equivariance property*

$$\alpha^\sharp(bg) = (\det g)\, \alpha^\sharp(b)$$

for all $b \in R(V)$ and $g \in \mathrm{GL}(k,\mathbb{C})$ (for the notation bg see Section 12.1).

Proof. The dual vector bundle $V^\vee$ has the transition functions g_{ij}^{-1} which implies that $\det g_{ij}^{-1} = (\det g_{ij})^{-1}$ are transition functions of $\Lambda^k(V^\vee)$. From this result we can read off that $K(V)$ can also be defined as the line bundle $R(V) \times_\rho \mathbb{C}$ with respect to $\rho = \det^{-1}$, since, in general, the transition functions of such associated bundle are $\rho(g_{ij})$ (cf. D.9). Finally, the transformation property for $\alpha^\sharp$ is the general transformation property in case of associated bundles (cf. D.7) which we have encountered in a similar form in the case of r-densities (cf. 12.1). □

The correspondence $\alpha \mapsto \alpha^\sharp$ is very simple in the case of the canonical bundle: Each section $\alpha \in \Gamma(M, K(V)$ induces a family of maps $\alpha_x : V_x^k \to \mathbb{C}$, $x \in X$, and therefore, $\alpha^\sharp(b) := \alpha_x(b_1, \ldots, b_k)$ is well-defined for each basis $b = (b_1, \ldots, b_k)$ of V_x. In other words, $\alpha^\sharp$ is the restriction of α to $R(P)$.

As a result of the lemma, there is a similarity of sections α of $K(V)$ with 1-densities on V: The line bundle $\delta_1(V)$ of 1-densities on V is determined by transition functions $|\det g_{ij}|^{-1}$. And the sections $\mu \in \Gamma(M, \delta_1(V))$ are in bijective correspondence to maps $\mu^\sharp : R(V) \to \mathbb{C}$ with the equivariance property

$$\mu^\sharp(bg) = |\det g|\mu^\sharp(b)$$

for all $b \in R(V)$ and $g \in \mathrm{GL}(k, \mathbb{C})$, cf. Section 12.1. In particular, any $\alpha \in \Gamma(M, K(V))$ induces a 1-density $|\alpha| \in \Gamma(M, \delta_1(V))$.

Notation. The sections $\alpha \in \Gamma(X, K(V))$ are called 1-forms of V. Of course, this concept of a 1-form is different from the notion of a differential one form $\eta \in \mathcal{A}^1(X)$.

The transformation property of the canonical bundle $K(V)$ gives rise for introducing the following definition.

Definition 15.3. Let $V \to X$ be a complex vector bundle of rank k over an m-dimensional manifold X. Then for $\ell \in \mathbb{N}$: $K_\ell(V) := K(V)^{\otimes \ell}$[79], $K_{-\ell}(V) := K(V^\vee)^{\otimes \ell} \cong K_\ell(V)^\vee$.

Observation 15.4. Let g_{ij} transition functions of V, then $(\det g_{ij})^{-\ell}$ are suitable transition functions of $K_\ell(V)$ for $\ell \in \mathbb{Z}$. And the equivariance property of sections $\beta \in \Gamma(M, K_\ell(V)$ is

$$\beta^\sharp(bg) = (\det g)^\ell \beta^\sharp(b)$$

for $b \in R(V)$ and $g \in \mathrm{GL}(k, \mathbb{C})$. Moreover, $K_\ell(V)$ can also be described as the associated bundle $R(V) \times_{\mathrm{GL}(k,\mathbb{C})} \mathbb{C}$ with respect to the representation $\rho = \det^{-\ell}$.

Remark 15.5. A concept which is similar to the canonical bundle is the determinant line bundle which is simply the dual of the canonical bundle (up to isomorphism): For a vector bundle V of rank r we have the following isomorphisms

$$\det(V) := \Lambda^r(V) \cong (\Lambda^r(V^\vee)^\vee \cong K(V)^\vee = K_{-1}(V)\,.$$

In generalization to the sequence $K_\ell(V)_{(\ell \in \mathbb{Z})}$ of "canonical" bundles we intend to introduce $K_\ell(V)$ for half-integers $\ell \in \{\ell \mid 2\ell \in \mathbb{Z}\}$ or at least for $\ell = 1/2$ and $-1/2$. This means we need to require the existence of a square root S of $K(V)$ in order to define the new sequence by $K_\ell(V) := S^{2\ell}$ for $\ell \in \{\ell \mid 2\ell \in \mathbb{Z}\}$. Note, that S is a square root of $K(V) = K_1(V)$ if and only if $S^\vee$ is a square root of $K_{-1}(V)$:

Observation 15.6. One might be tempted to define a square root of $K(V)$ by simply requiring the transformation property

$$u(bg) = (\det g)^{1/2} u(b)\,,$$

[79] $W^{\otimes \ell}$ is the ℓ-fold tensor product $W \otimes W \otimes \ldots \otimes W$

for a function $u : R(V) \to \mathbb{C}$. Or equivalently, to define the square root of $K(V)$ through the transition functions $(\det g_{ij})^{-1/2}$, when g_{ij} are transition functions for V. But the square root $(\det h)^{-1/2}$ is not well-defined for general $h \in \mathcal{E}(M, GL(k,\mathbb{C}))$. In general, for a manifold M every given function $g \in \mathcal{E}(M,\mathbb{C})$ has a square root in $\mathcal{E}(M,\mathbb{C})$ if and only if M is simply connected.

Let us have a look at the related problem of describing a possible square root of a given holomorphic function $f : U \to \mathbb{C}$ on an open $U \subset \mathbb{C}$ in Complex Analysis: One uses a double covering $p : \widetilde{U} \to U$ of U on which a holomorphic $\widetilde{f} : \widetilde{U} \to \mathbb{C}$ is defined with $\widetilde{f} = f \circ p$ and such that $\widetilde{f}$ has a holomorphic square root $g : \tilde{U} \to \mathbb{C}$, i.e. $g^2 = \widetilde{f}$.

On the basis of this idea to obtain a square root of $K(V)$ one can consider a double cover of $\mathrm{GL}(k,\mathbb{C})$ – namely the metalinear group $\mathrm{ML}(n,\mathbb{C})$ – together with a suitable double cover $\widetilde{R}(V)$ of $R(V)$. This will be carried through in Section 16.1 of the next chapter.

We postpone the discussion of these problems to the next chapter. In the actual chapter we want to circumvent these issues and treat the half-form quantization under the hypothesis that we only have a line bundle S with $S \otimes S \cong K(P)$. Before that, let us comment the usage of the term "canonical bundle":

Remark 15.7. We have introduced the notion of canonical line bundle in Definition 15.1 which works for a general complex vector bundle. The usage in the literature is not completely uniform. In the context of Geometric Quantization the canonical bundle of a polarization $P \subset TM^{\mathbb{C}}$ is often defined as $\Lambda^n P^0$ where $P^0 \subset T^*M^{\mathbb{C}}$ is the polar or annihilator $P^0 := \{\mu \in T^*M^{\mathbb{C}} \mid \mu|_P = 0\}$. Since P is isomorphic to P^0 when P is a polarization[80] this definition amounts essentially to $\Lambda^n P^0 \cong \Lambda^n P \cong K_P^{-1}$. i.e. the canonical line bundle defined with the aid of P^0 is the inverse of the canonical bundle $K = K_P$ used in these Notes, and changing the definitions leads merely to interchanging K_ℓ and $K_{-\ell}$, $\ell \in \mathbb{Z}$, as well as $K_{1/2}$ and $K_{-1/2}$ in case of existence. We have introduced the general Definition 15.1 in accordance with the usage in Algebraic Geometry and Complex Analysis, where the canonical line bundle for a non-singular variety X is the line bundle of forms of top degree.

15.2 Need for a Square Root of a Line Bundle

This section is inserted into this chapter in order to motivate the search for a square root of $K(P)$ resp. $K_{-1}(P)$ for a polarization. The results are not needed in the following, so the reader can proceed immediately with the next two sections to learn how the square root is applied to achieve the half-form quantization.

Recall that a square root of a line bundle B over M is a line bundle C with $C^2 := C \otimes C = B$. We also speak of a square root when $S \otimes S \cong B$ with a fixed line bundle isomorphism $S \otimes S \to B$.

[80] with respect to the isomorphism $X \mapsto \omega(X, \ )$, $X \in P_a$

Our starting point is a prequantum line bundle (L, ∇, H) on a symplectic manifold (M, ω) with a reducible complex polarization P. The polarized sections $s \in \Gamma(M, L)$ are constant on the leaves of the distribution $D = P \cap \overline{P} \cap TM$ and consequently they define sections on the quotient manifold M/D with values in the induced line bundle $L|_{M/D}$. In particular, for two polarized sections $s, t \in \Gamma(M, L)$ the Hermitian structure H on L yields a function $H(s,t) \in \mathcal{E}(M/D)$. Such functions we intend to integrate in order to obtain the representation space (a Hilbert space) of the quantization. However, as we have stated already several times, there is, in general, no natural measure on M/D. Therefore, as a first possibility one studies densities on M/D which can be integrated without depending on an extra structure on M/D, e.g. like a volume. This approach leads to the half-density quantization considered in the preceding chapter.

If one wants to base the quantization on forms instead on densities, one sees that sections on $K_{-1}(P)$ can be directly composed to yield 2-densities on the quotient $Q := M/D$. We explain this in the case of a reducible real polarization P, i.e. $D^{\mathbb{C}} = P = \overline{P}$: Let $Z^D = TM^C/D^{\mathbb{C}} \cong \pi^*(T(Q)^{\mathbb{C}})$ be the pullback of $TQ^{\mathbb{C}}$ with respect to the quotient map $\pi : M \to M/D = Q$.

Lemma 15.8. *Let $P = \overline{P}$. There is a natural sesquilinear pairing*

$$\langle\ ,\ \rangle : K_{-1}(P) \times K_{-1}(P) \to \delta_2(Z^D)$$

defined by

$$\langle \alpha, \beta \rangle^\sharp([Z_1], \ldots, [Z_n]) := |\omega^n(X_1, \ldots, X_n, Z_1, \ldots, Z_n)|^2 \overline{\alpha}^\sharp(X_1, \ldots X_n)\beta^\sharp(X_1, \ldots, X_n)\,,$$

where $(\alpha, \beta) \in K_{-1}(P) \times K_{-1}(P)$ and $(X_1, \ldots, X_n, Z_1, \ldots, Z_n) \in R_a(TM^{\mathbb{C}})$ such that $(X_1, \ldots, X_n) \in R_a(P)$ and $([Z_1], \ldots, [Z_n]) \in R_a(Z^D)$ (recall $[Z] := Z + P_a \in {Z^D}_a$ for $Z \in T_a M^C$).

Proof. For a fixed $Z = (Z_1, \ldots, Z_n)$ the definition is independent of $X = (X_1, \ldots, X_n)$: For another basis X' of $R_a(P)$ there exists $g \in \mathrm{GL}(n, \mathbb{C})$ with $X' = Xg$; and

$$\begin{aligned} |\omega^n(Xg, Z)|^2 \overline{\alpha}^\sharp(Xg)\beta^\sharp(Xg) &= \big(|\det g|^2 |\omega^n(X,Z)|^2\big)\,(\overline{\det g})^{-1}\overline{\alpha}^\sharp(X)(\det g)^{-1}\beta^\sharp(X) \\ &= |\omega^n(X,Z)|^2 \overline{\alpha}^\sharp(X)\beta^\sharp(X)\,, \end{aligned}$$

Replacing Z by $\tilde{Z} = Z + Y, Y_j \in P$, the following holds for fixed X

$$|\omega^n|^2(X, \tilde{Z})\overline{\alpha}^\sharp(X)\beta^\sharp(X) = |\omega^n|^2(X, Z)\overline{\alpha}^\sharp(X)\beta^\sharp(X),$$

hence $\langle \alpha, \beta \rangle$ is well-defined. Moreover, for $g \in \mathrm{GL}(n, \mathbb{C})$

$$\begin{aligned} \langle \alpha, \beta \rangle^\sharp([Zg]) &= |\omega^n(X, Zg)|^2 \overline{\alpha}^\sharp(X)\beta^\sharp(X) \\ &= |\det g|^2 |\omega^n(X, Z)|^2 \overline{\alpha}^\sharp(X)\beta^\sharp(X) \\ &= |\det g|^2 \langle \alpha, \beta \rangle^\sharp([Z]) \end{aligned}$$

Therefore, $\langle \alpha, \beta \rangle$ is a 2-density. □

Disregarding at the moment the question of how to descend from Z^D to $Q = M/D$ (which has been settled in the preceding chapters) we see that (α, β) would lead to a suitable 1-density on M/D if one would be able to take square roots of the occurring objects. This amounts to take square roots of the involved line bundles, in particular of $K_{-1}(P)$.

Let us explain this idea: If S is a square root of $K_{-1}(P)$, i.e. $S \otimes S = K_{-1}(P)$, we obtain for sections $\alpha, \beta \in \Gamma(M, S)$ a natural 1-density $\langle \alpha, \beta \rangle_S \in \Gamma(M, \delta_1(Z^D))$ defined by

$$\langle \alpha, \beta \rangle_S^\sharp([Z_1], \dots, [Z_n]) = |\omega^n(X_1, \dots, X_n, Z_1, \dots, Z_n)| \overline{\alpha}^\sharp(X_1, \dots X_n) \beta^\sharp(X_1, \dots, X_n)\,,$$

as in the preceding lemma.

Furthermore, $\langle \alpha, \beta \rangle_S$ descends to a 1-density on $Q = M/D$, if the α, β are constant along D, i.e. if they are polarized. This is the starting point of forming a representation space from polarized sections of $L \otimes S$ as the first main step in half-form quantization.

15.3 Descend of Half-Forms

We present in this section a direct approach to the half-form quantization which avoids the use of metalinear or metaplectic structures on the vector bundles in question. In this way we obtain a straightforward and rather elementary construction similar to the half-density quantization. But we cannot obtain stronger results on isomorphisms of the relevant line bundles or relations between square roots, since we are not yet in the position to describe half-forms by transformation properties on the frame bundles. The metalinear structure will enable us to formulate the half-form condition via transformation properties (cf. next chapter).

In the following, we assume that there exists a square root S of $K_{-1}(P)$, i.e. a line bundle S with $S \otimes S = K_{-1}(P)$, or – more generally – a line bundle S with a fixed isomorphism $S \otimes S \to K_{-1}(P)$. S will be denoted also by $S = K_{-1/2}(P)$. The existence and uniqueness of such a line bundle will be discussed in the next section. A section in $K_{-1}(P)$ will be called a -1-P-form and correspondingly a section of $S = K_{-1/2}(P)$ will be called a $-1/2$-P-form. The term "half-form" will be used, in general, to denote sections of $K_{1/2}(V)$ or $K_{-1/2}(V)$ for a vector bundle, in particular for $P = V$.

As before, P is a reducible complex polarization on a symplectic manifold (M, ω) and we consider the quotient bundle $Z^D = TM^{\mathbb{C}}/D^{\mathbb{C}}$ which can be described as the pullback of $TQ^{\mathbb{C}}$ where $Q := M/D$ is the space of leaves of the distribution $D = TM \cap P \cap \overline{P}$ with its projection $\pi : M \to Q = M/D$. We intend to define a natural sesquilinear map (a pairing) $B_P : K_{-1/2}(P) \times K_{-1/2}(P) \to \delta_1(Z_D)$, as was explained in the preceding section. We need the following elementary result:

Lemma 15.9. *Any two $-1/2$-P-forms $\alpha, \alpha' \in \Gamma(M, K_{-1/2}(P))$ define a -1-density $\bar{\alpha}\alpha' \in \Gamma(M, \delta_{-1}(P))$.*

Proof. The squares $\alpha^2 = \alpha \otimes \alpha$, $\alpha'^2 = \alpha' \otimes \alpha'$ are in $\Gamma(M, K_{-1}(P))$, hence, the product $\beta := \bar{\alpha}^2 \alpha'^2 = (\bar{\alpha}\alpha')^2$ is a -2-density $\beta \in \Gamma(M, \delta_{-2}(P))$: Indeed, for $b \in R_a(P)$ and $g \in \mathrm{GL}(n, \mathbb{C})$ one has

$$\beta^\sharp(bg) = (\bar{\alpha}^2)^\sharp(bg)(\alpha'^2)^\sharp(bg) = (\overline{\det g})^{-1}(\bar{\alpha}^2)^\sharp(b)(\det g)^{-1}(\alpha'^2)^\sharp(b) = |\det g|^{-2}\beta^\sharp(b)\,.$$

Hence, $\bar{\alpha}\alpha' = \sqrt{\beta}$ is a -1-density[81] on P. □

Another way to see that $\bar{\alpha}\alpha'$ is a 1-density is to determine the transition functions for the line bundle

$$\overline{K}_{-1/2}(P) \otimes K_{-1/2}(P) \ :$$

Let z_{ij} be transition functions for of $K_{-1/2}(P)$. Then $\bar{z}_{ij} z_{ij} = |z_{ij}|^2$ are transition functions of $\overline{K}_{-1/2}(P) \otimes K_{-1/2}(P)$. Since $|z_{ij}|^2 = |\det g_{ij}|$, they are also transition functions of of the density bundle $\delta_{-1}(P)$ according to Definition 12.2. Hence the two line bundles are isomorphic. In particular, the sections of $\overline{K}_{-1/2}(P) \otimes K_{-1/2}(P)$ are -1-densities.

In other words, the line bundle $\overline{K}_{-1/2}(P) \otimes K_{-1/2}(P)$ is trivial and isomorphic to the line bundle $\delta_{-1}(P)$. The choice of an isomorphism $\overline{K}_{-1/2}(P) \otimes K_{-1/2}(P) \to \delta_{-1}(P)$ determines a non-degenerate sesquilinear pairing $K_{-1/2}(P) \times K_{-1/2}(P) \to \delta_{-1}(P)$ and vice versa.

We know from Proposition 14.3 that there exists a natural isomorphism

$$\tau_P : \delta_{-1}(P) \to \delta_1(Z^D)$$

preserving the partial connections.

Corollary 15.10. *We thus obtain a natural pairing map*

$$\langle\ ,\ \rangle_P : K_{-1/2}(P) \times K_{-1/2}(P) \to \delta_1(Z^D)\ ,\ \mapsto \langle \alpha, \alpha' \rangle_P := \tau_P(\bar{\alpha}\alpha')\,,$$

where $\bar{\alpha}\alpha'$ *is the* -1*-density from the previous lemma.*

Because of the central importance of the statement of the corollary, we recall the resulting definition of the pairing $\langle\ ,\ \rangle_P$ in detail: It is enough to define the map locally around each $a \in M$.

In a suitable open neighbourhood $U \subset M$ of $a \in M$ there exists a frame field

$$(\xi, \zeta) = (\xi_1, \dots, \xi_n, \zeta_1, \dots, \zeta_n) : U \to R(TM^{\mathbb{C}})$$

for $TM^{\mathbb{C}}$ such that

81 Here we notice a little disadvantage resulting from the decision to avoid the use of metalinear frames in this chapter: We cannot give a seemingly obvious proof by establishing the transformation property for $(\bar{\alpha}\alpha')^\sharp$ through inserting bg in $\bar{\alpha}^\sharp$ and in $\alpha'^\sharp$, since we cannot use a transformation property like $\alpha(bg) = (\det g)^{-1/2}\alpha(b)$. Such a transformation property does not hold. It is, however, satisfied on the appropriate double cover of $R(P)$ as we explain in Section 16.2 of the next chapter.

1. $\xi : U \to R(P)$ is a frame field for P,

2. $(\xi_1, \dots \xi_k) : U \to R(D^{\mathbb{C}})$ is a frame field for $D^{\mathbb{C}}$,

3. the $[\xi_j] := \xi_j + D^{\mathbb{C}} \in Z^D$ and $[\zeta_j] := \zeta_j + D^{\mathbb{C}} \in Z^D$ define a frame field $[\gamma] := ([\xi_{k+1}], \dots, [\xi_n], [\zeta_1], \dots, [\zeta_n]) : U \to R(Z^D)$ for Z^D, and

4. the $[\xi_j] \in P^{\mathbb{C}}/D^{\mathbb{C}}$ and their conjugates $[\bar{\xi_j}] := \bar{\xi_j} + D^{\mathbb{C}} \in \overline{P}^{\mathbb{C}}/D^{\mathbb{C}}$ yield a frame field $\eta := \underline{([\xi_{k+1}], \dots, [\xi_n], [\bar{\xi}_{k+1}], \dots [\bar{\xi}_n])} : U \to R(E^{\mathbb{C}}/D^{\mathbb{C}})$ for $E^{\mathbb{C}}/D^{\mathbb{C}}$. Here, $E = TM \cap (P + \overline{P})$ and therefore, $E^{\mathbb{C}} = P + \overline{P}$.

Finally, let θ be the 1-density on $E^{\mathbb{C}}/D^{\mathbb{C}}$ induced by $|\omega^{n-k}|^{1/2}$, i.e.

$$\theta^\sharp(\eta) = \left(|\omega^{n-k}(\xi_{k+1}, \dots, \xi_n, \bar{\xi}_{k+1}, \dots, \bar{\xi}_n)|\right)^{1/2} .$$

Definition 15.11. For every pair $(\alpha, \alpha') \in \Gamma(U, K_{-1/2}(P)) \times \Gamma(U, K_{-1/2}(P))$ we define $\langle \alpha, \alpha' \rangle_P \in \Gamma(U, \delta_1(Z_D))$ by

$$\langle \alpha, \alpha' \rangle_P^\sharp([\gamma]) := |\omega^n(\xi, \zeta)| \theta^\sharp(\eta)(\overline{\alpha}\alpha)^\sharp(\xi).$$

Proposition 15.12. *$\langle \alpha, \alpha' \rangle_P$ yields a well-defined 1-density in $\Gamma(M, \delta_1(Z^D))$. Moreover, it is a lift of a unique 1-density $\nu(\alpha, \alpha') \in \Gamma(Q, \delta_1(TQ^{\mathbb{C}}))$ on $Q = M/D$ if and only if it is polarized and this is the case if the -1-density $\bar{\alpha}\alpha'$ satisfies $\nabla_X(\bar{\alpha}\alpha') = 0$ for all $X \in \Gamma(M, D)$, where ∇ is the partial connection on $\delta_{-1}(P)$ (cf. Definition 12.16).*

Proof. To be completed in a similar way as before in the preceding chapter. □

Partial Connection

To exploit the last statement about describing the density $\langle \alpha, \alpha' \rangle_P$ as the lift of a density on $Q := M/D$, we need to define a partial connection also on our square root line bundle $S = K_{-1/2}(P)$. We follow the definition of the partial connection on $\delta_r(P)$ (see Section 12.3), but now for the line bundle $K_\ell(P)$, $2\ell \in \mathbb{Z}$.

We begin with $\ell \in \mathbb{Z}$. Locally, on a suitable (e.g. for example contractible) open subset $U \subset M$ there exists a Hamiltonian frame field $\xi : U \to R(P)$. This frame field determines a unique $\sigma_\xi \in \Gamma(U, K_\ell(P))$ with $\sigma_\xi^\sharp(\xi) = 1$.

Definition 15.13. Let $\ell \in \mathbb{Z}$. Every $\sigma \in \Gamma(U, K_\ell(P))$ has the form $\sigma = f\sigma_\xi$ with $f = \sigma^\sharp(\xi)$. For $X \in \Gamma(U, P)$ the partial connection is defined by

$$\nabla_X \sigma := (L_X f)\sigma_\xi = L_X\left(\sigma^\sharp(\xi)\right)\sigma_\xi .$$

In particular, $\nabla_X \sigma_\xi = 0$, i.e. σ_ξ is polarized. For $\ell \in \{\frac{1}{2}, -\frac{1}{2}\}$ the definition of ∇_X is given below in Corollary 15.15. See also Section 16.2.

The partial connection is well-defined for all $X \in \Gamma(M, P)$ and $\ell \in \mathbb{Z}$ (i.e. independent of the choice of the frame field ξ), and satisfies the properties of a flat connection.

This can be shown in the same way as for the partial connection on $\delta_r(P)$ in Section 12.3.

Moreover, an analogous partial connection is given on the conjugate canonical bundles $\overline{K}_\ell(P) := \{\bar{\rho} \mid \rho \in K_\ell(P)\}$ which can be defined also by

$$\nabla_X \bar{\rho} := \overline{\nabla_X \rho}.$$

Lemma 15.14. *All these partial connections are compatible to each other, i.e. for instance*

$$\nabla_X(\sigma\rho) = (\nabla_X \sigma)\rho + \sigma \nabla_X \rho$$

for $\sigma \in \Gamma(M, K_m(P)), \rho \in \Gamma(M, K_\ell(P))$ *where* $\sigma\rho = \sigma \otimes \rho \in \Gamma(M, K_{m+\ell}(P))$.

Proof. Given the Hamiltonian frame field ξ the sections $\sigma_\xi \in \Gamma(U, K_m(P))$, $\rho_\xi \in \Gamma(U, K_\ell(P))$ and $\tau_\xi \in \Gamma(U, K_{m+\ell}(P))$ are determined and they satisfy $\tau_\xi = \sigma_\xi \rho_\xi$. For general sections $\sigma = f\sigma_\xi, \rho = g\rho_\xi$ and $\tau = \sigma\rho = fg\tau_\xi$ we have

$$\begin{aligned}\nabla_X \tau =& L_X(fg)\tau_\xi = ((L_X f)g + fL_X g)\sigma_\xi \rho_\xi \\ =&(L_X f\sigma_\xi)g\rho_\xi + f\sigma_x i L_X g\rho_\xi \\ =&(\nabla_X \sigma)\rho_\xi + \sigma_x i \nabla_X \rho_\xi \,.\end{aligned}$$

□

Corollary 15.15. *In particular,* $\nabla_X \alpha^2 = 2\alpha \nabla_X \alpha$ *for* $\alpha \in K_\ell(P)$ *and for nowhere vanishing* α

$$\nabla_X \alpha = \frac{1}{2\alpha} \nabla_X \alpha^2 \,.$$

This result is used to define $\nabla_X = \nabla_X^S$ *on the root bundle* $S = K_{-1/2}$ *or* $S = K_{-1/2}$*: For* $\alpha \in \Gamma(U, S)$ *we set*

$$\nabla_X^S \alpha = \frac{1}{2\alpha} \nabla_X \alpha^2 \,.$$

The compatibility extends also for mixing the conjugate with non-conjugate sections of $K_\ell(P), K_m(P)$.

For instance, for $\sigma, \rho \in \Gamma(M, K_{-1/2}(P))$ the following holds:

$$\nabla_X(\bar{\sigma}\rho) = (\nabla_X \bar{\sigma})\rho + \bar{\sigma}(\nabla_X \rho). \tag{63}$$

Again there can be proven many compatibility conditions for all these partial connections, including the special connections ∇^S. For instance

$$\nabla_X(\bar{\alpha}\alpha') = (\nabla_X^S \bar{\alpha})\alpha' + \bar{\alpha}\nabla_X^S \alpha', \tag{64}$$

for any pair of sections $\alpha, \alpha' \in \Gamma(M, K_{-1/2}(P))$.

Lemma 15.16. *For polarized sections $\alpha, \alpha' \in \Gamma(M, K_{-1/2}(P))$ the density $\bar{\alpha}\alpha' \in \Gamma(M, \delta_{-1}(P))$ satisfies*

$$\nabla_X(\bar{\alpha}\alpha') = 0$$

for $X \in \Gamma(M, D)$. Here, ∇ is the partial connection on $\delta_1(P)$ defined in 12.16.

Proof. This follows immediately from (64). □

Collecting these results we obtain:

Corollary 15.17. *For any pair of polarized sections $\alpha, \alpha' \in \Gamma(M, K_{-1/2}(P))$ the density $\langle \alpha, \alpha' \rangle_P \in \Gamma(\delta_1(Z^D))$ is the lift of a unique* 1*-density $\nu(\alpha, \alpha') \in \Gamma(Q, \delta_1(TQ^{\mathbb{C}}))$.*

15.4 Representation Space and Quantum Operator

We are now in the position to describe the half-form quantization in full generality:

Construction 15.18 (Representation Space, Using $K_{-1/2}(P)$)**.** Let (L, ∇, H) be a prequantum line bundle on the symplectic manifold (M, ω) and let P be a reducible complex polarization on (M, ω). In addition, we assume that the line bundle $K_{-1}(P)$ has a the square root $S = K_{-1/2}(P)$.

As before, we replace L with

$$L \otimes S\,,\; S = K_{-1/2}(P)\,,$$

and ∇ with $\nabla \otimes \nabla^S$, where ∇^S is the partial connection on $S = K_{-1/2}(P)$ (see Definition 15.13).

Let $\psi = s \otimes \alpha, \psi' = s' \otimes \alpha' \in \Gamma(M, L \otimes S)$ polarized sections. Then s, s' are polarized, hence $\nabla_X H(s, s') = 0$ for all $X \in \Gamma(M, D^{\mathbb{C}})$, i.e. $H(s, s')$ is constant on the leaves of D and therefore descends to a smooth function on $Q = M/D$ which we denote by $H(s, s')$ again. Moreover, the sections α, α' of S are also polarized, which by Corollary 15.17 implies that the result $B_P(\alpha, \alpha') \in \Gamma(M, \delta_1(Z^D))$ of the pairing is a lift of a density $\nu = \nu(\alpha, \alpha') \in \Gamma(Q, \delta_1(TQ^{\mathbb{C}}))$.

As a consequence, we obtain the scalar product

$$\langle \psi, \psi' \rangle = \langle s \otimes \alpha, s' \otimes \alpha' \rangle := \int_{M/D} H(s,t)\nu(\alpha, \alpha')^\sharp\,.$$

The completion of

$$H_P = \{\psi \in \Gamma(M, L \otimes S) \mid \psi \text{ polarized and } \langle \psi, \psi \rangle < \infty\}$$

with respect to the induced norm is the representation space $\mathbb{H}_P^S = \mathbb{H}^S(M, L, P)$ which we wanted to construct.

For the definition of the quantum operator we need, as in the case of half-density quantization, the concept of a partial Lie derivative

$$L_X : K_\ell(P) \to K_\ell(P)$$

for vector fields X preserving P, in particular for $\ell = -\frac{1}{2}$:

Definition 15.19. A vector field $X \in \Gamma(M,P)$ preserving P induces the flow $\Phi_t : M_t \to M_{-t}$ with $T\Phi_t(P_a) = P_a$, $a \in M_t$. Let $\ell \in \mathbb{Z}$. Then for $\alpha \in \Gamma(M, K_\ell(P))$ the natural partial Lie derivative

$$(L_X\alpha)^\sharp(\xi) := \frac{d}{dt}\alpha^\sharp(T\Phi_t(\xi))\Big|_{t=0},$$

$\xi \in R(P)$, is well-defined.

The definition can be transferred to $S = K_{-1/2}(P)$ to obtain $L_X^S = L_X : \Gamma(M,S) \to \Gamma(M,S)$, as well, by setting

$$L_X^S\alpha := \frac{1}{2\alpha}L_X(\alpha^2)$$

on $\{a \in M \mid \alpha_a \neq 0\}$[82]. Similarly, we can define the partial Lie derivative on general $K_\ell(P)$, $2\ell \in \mathbb{Z}$.

Definition 15.20. The quantum operator $q^S(F)$[83] for $F \in \mathfrak{R}_P$ on polarized sections $\varphi = s \otimes \alpha$ of $L \otimes S$ is defined by

$$\boxed{q^S(F)(s \otimes \alpha) = (q(F)s) \otimes \alpha - \frac{i}{2\pi}s \otimes L_{X_F}\alpha.}$$

Since every global section φ can be written locally in the form $\varphi = s\otimes\alpha$, this determines a unique polarized section $q(F)\varphi$ of $L \otimes S$.

Main Result

The main result of this section is the following

Theorem 15.21. *Let (L, ∇, H) be a prequantum line bundle on the symplectic manifold (M,ω) with reducible complex polarization P, and let S be a square root of $K_{-1}(P)$. The constructed representation $\mathbb{H}^S$ has as its quantum operators $q^S : \mathfrak{R}_P \to \mathcal{S}(\mathbb{H}^S)$ the linear maps*

$$q^S(F)(s \otimes \alpha) = q(F)s \otimes \alpha - \frac{i}{2\pi}s \otimes L_{X_F}\alpha\,,$$

[82] The partial Lie derivative on half-forms can be defined in a more natural manner by using the transformation property which we only have on the metalinear frame bundle $\widetilde{R}(P)$ linked with the square root bundle.

[83] A definition based on the local flow (ϱ_t^F) on $\Gamma(M, L \otimes S)$ induced by the local flow (Φ_t^F) of X_F (as in the last chapter, see Definition 14.8) can only be given using the metalinear structure linked to the square root. This will be explained in the next chapter.

for polarized sections $s \otimes \alpha \in \Gamma(M, L \otimes S)$*, where* $q(F)$ *is the prequantum operator.*

Moreover, half-form quantization is a full geometric quantization in the following sense:

1. $q^S(F)$ *is* $\mathbb{R}$*-linear and satisfies* (D1) *and* (D2)*, now for the new representation space* $\mathbb{H}^S = \mathbb{H}^S(M, L, P)$.
2. *If* X_F *is complete,* $F \in \mathfrak{R}_P$*, then* $q^S(F)$ *is self-adjoint.*

Example 15.22 (Harmonic Oscillator)**.** We want to show the effect of half-form quantization in the case of the harmonic oscillator, thereby continuing Example 10.23 and see that the quantized operator $q^S(H)$ will have the correct eigenvalues. Therefore, one speaks of "half-form correction".

The phase space is $M = T^*\mathbb{R}^n \cong \mathbb{C}^n$ with its standard symplectic form $\omega = dq^j \wedge dp_j$. We use complex coordinates $z_j = p_j + iq^j$, $j = 1, \dots, z_n$, so that $\omega = \frac{i}{2}\sum d\bar{z}_j \wedge dz_j$ with connection form and potential $\theta = \frac{i}{2}\sum_j \bar{z}_j dz_j$. The Hamiltonian H of the harmonic oscillator is the energy

$$H(z) = \frac{1}{2}\sum_{k=1}^{n} z_k \bar{z}_k.$$

The corresponding Hamiltonian vector field is

$$X_H = i\left(\sum z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\right).$$

The prequantum line bundle (L, ∇, H) is the trivial bundle $L = M$ with the connection ∇ given by θ and the natural Hermitian structure H on $L = M \times \mathbb{C}$. The polarization P is the holomorphic polarization P generated by the vector fields X_{z_j}. Observe that X_H preserves P, i.e. H is directly quantizable. The representation space $\mathbb{H}_P$ can be identified with the Bargmann space $\mathbb{F}$.

For the prequantum operator we know already, according to 10.23:

$$q(H) = \frac{1}{2\pi}\sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k}$$

on the space of holomorphic functions on $\mathbb{C}^n$.

In order to determine the additional term in the quantum operator

$$q^S(H)(s \otimes \alpha) = q(H) \otimes \alpha - \frac{i}{2\pi} s \otimes L_{X_H}\alpha\,,$$

we have to evaluate the partial Lie derivative $L_{X_H}\alpha$ for $\alpha \in \Gamma(M, K_{-1/2}(P))$.

The standard n-form $d\bar{z} := d\bar{z}_1 \wedge \dots \wedge d\bar{z}_n \in K(P)$ generates $K(P)$ and $K(P)$ is trivial. Moreover, $K_{-1}(P) = K^\vee \cong M \times \mathbb{C}$ has the square root $S = K_{-1/2}(P)$ generated by a $-1/2$-form α, $\alpha^{-2} = d\bar{z}$ which we denote by $\alpha =: d\bar{z}^{-1/2}$.

Cartan's formula yields $L_X d\bar{z} = i_X dd\bar{z} + d(i_X d\bar{z}) = d(i_X d\bar{z})$. From

$$i_{X_H} d\bar{z} = -i \sum_j \bar{z}_j (-1)^j d\bar{z}_1 \wedge \ldots \wedge \widehat{d\bar{z}_j} \wedge \ldots \wedge d\bar{z}_n$$

we deduce

$$L_{X_H} d\bar{z} = -ind\bar{z}\,,$$

and

$$L_{X_H} d\bar{z}^{-1/2} = \frac{1}{2} ind\bar{z}\,.$$

Therefore, the additional term is multiplication by

$$-\frac{i}{2\pi} i \frac{n}{2} = \frac{1}{2\pi}\frac{n}{2}\,.$$

As a consequence, the quantum operator for H is

$$\boxed{q^S(F) = \frac{1}{2\pi}\left(\sum_{k=1}^{n} z_k \frac{\partial}{\partial z_k} + \frac{n}{2}\right)}$$

acting on holomorphic functions $\phi \in \mathcal{O}(\mathbb{C}^n)$.

The eigenvalues will be determined by the equation

$$\frac{1}{2\pi}\left(\sum z_k \frac{\partial}{\partial z_k} + \frac{n}{2}\right)\phi = E\phi\,.$$

Using the result of Example 10.23 we conclude that the eigenvalues are the

$$E_M = \frac{1}{2\pi}\left(M + \frac{n}{2}\right)\,,\ M \in \mathbb{N}\,,$$

and the corresponding eigenspaces $\mathbb{V}_M$ are the spaces of M-homogenous polynomials in n complex variables. The $\mathbb{V}_M$ can be understood as ${\mathbb{V}_1}^{\odot M}$ and $\mathbb{F}$ as the Fock space of $\mathbb{V}_1$.

This is in accordance to the standard quantum model for the harmonic oscillator.

15.5 Square Root: Existence and Uniqueness

The objective of this section is to study the topological condition which is needed for the existence of a square root of a given complex line bundle, and to determine to which extent a possible square root is unique. The conditions are formulated in the context of Čech cohomology with values in the group $\mathbb{Z}_2$, cf. Section E.1.

Let K be a complex line bundle on the manifold M with transition functions h_{ij} with respect to an open cover $\mathfrak{U} = (U_j)$. We require that $\mathfrak{U}$ has the property that the $U_j, U_{ij} \ldots$ are all contractible.

From the characterization of vector bundles by transition functions we know that a square root S of K will have transition functions z_{ij} with $z_{ij}^2 = h_{ij}$, when h_{ij} are transition functions for K. As a consequence, we are looking for functions $z_{ij} \in \mathcal{E}(U_{ij}, \mathbb{C}^\times)$ satisfying

$$\begin{aligned} &1. \qquad z_{ij}^2 = h_{ij} \ \text{ on } \ U_{ij}\,, \\ &2. \qquad z_{ij}z_{jk}z_{ki} = 1 \ \text{ on } \ U_{ijk}. \end{aligned} \tag{65}$$

As a first try to find suitable z_{ij} we pick for each pair $(i,j) \in I^2$ a smooth square root $d_{ij} \in \mathcal{E}(U_{ij}, \mathbb{C}^\times)$ of h_{ij}, i.e. ${d_{ij}}^2 = h_{ij}$. This is possible since U_{ij} is contractible. Since h_{ij} satisfies the cocycle condition we have $d_{ij}^2 d_{jk}^2 d_{ki}^2 = 1$. But the condition 2. for the choice $z_{ij} = d_{ij}$, here $d_{ij}d_{jk}d_{ki} = 1$, will not be satisfied, in general. We define

$$a_{ijk} := d_{ij}d_{jk}d_{ki} \ \text{ on } \ U_{ijk}\,.$$

The collection $a := (a_{ijk})$ is a Čech cocycle in $C^2(\mathfrak{U}, \mathbb{Z}_2)$, since $a_{ijk} \in \mathbb{Z}_2 = \{1, -1\}$ [84] because of $a_{ijk}^2 = 1$. a induces a cohomology class $[a] \in \check{H}^2(\mathfrak{U}, \mathbb{Z}_2)$. This cohomology class is independent of the choice of the square roots d_{ij}, it only depends on the transitions functions h_{ij}. In fact, with another choice of $d'_{ij} \in \mathcal{E}(U_{ij}, \mathbb{C}^\times)$ with $(d'_{ij})^2 = h_{ij}$ we set $c_{ij} := d'_{ij}d_{ij}^{-1} \in \mathbb{Z}_2$ and obtain for the new cocycle $a' := (a'_{ijk})$

$$a'_{ijk} := d'_{ij}d'_{jk}d'_{ki} = d_{ij}d_{jk}d_{ki}c_{ij}c_{jk}c_{ki} = a_{ijk}c_{ij}c_{jk}c_{ki}\,.$$

Hence

$$\frac{a'_{ijk}}{a_{ijk}} = c_{ij}c_{jk}c_{ki}\,,$$

which means that $a'a^{-1}$ is the coboundary of $c = (c_{ij}) \in \check{C}^1(\mathfrak{U}, \mathbb{Z}_2)$. The two cocycles a, a' define the same cohomology class $[a] = [a'] \in \check{H}^2(\mathfrak{U}, \mathbb{Z}_2)$. Moreover, different transition functions for K lead to the same class, as well.

This cohomology class depending on the line bundle K will be denoted by $w(K) = [a] = [(h_{ij})]$ and it will be called the obstruction class. It is the obstruction for the existence of a square root, as we will see in the following.

Of course, $a = (a_{ijk})$ can be regarded as a cocycle in $\check{C}^2(\mathfrak{U}, \mathbb{C}^\times)$ with values in the group $\mathbb{C}^\times$, and by definition $a_{ijk} = d_{ij}d_{jk}d_{ki}$ it is a coboundary there. In order that a is a coboundary in $\check{C}^2(\mathfrak{U}, \mathbb{Z}_2)$, and hence trivial, it would require that a cocycle $b = (b_{ij}) \in \check{C}^1(\mathfrak{U}, \mathbb{Z}_2)$ exists which satisfies[85]

$$a_{ijk} = b_{ij}b_{jk}b_{ki}.$$

We have prepared the proof of the following proposition:

[84]We write the abelian group $\mathbb{Z}_2$ as the multiplicative group $\{1, -1\}$ instead of the additive group $\{[0], [1]\}$.

[85]When the condition is written additively, it is $a_{ijk} = b_{ij} + b_{jk} + b_{ki} = b_{jk} - b_{ik} + b_{ij} = \delta(b)_{ijk}$.

Proposition 15.23. *There exists a square root line bundle for the complex line bundle K over M if and only if the obstruction class $w(K) = [a][(h_{ij})]$ is trivial in $\check{H}^2(\mathfrak{U}, \mathbb{Z}_2)$, i.e. if it is the class* [1]. *One also says, $w(K)$ vanishes, when one emphasizes the additive notation $\mathbb{Z}_2 = \{[0], [1]\}$.*

Proof. We have stated earlier that the existence of a square root implies the existence of $z_{ij} \in \mathcal{E}(U_{ij}, \mathbb{C}^\times)$ satisfying $z_{ij}^2 = h_{ij}$ with $z_{ij}z_{jk}z_{ki} = 1$, see (65). Since $a_{ijk} = z_{ij}z_{jk}z_{ki}$, the obstruction class $[a] = [(a_{ijk})]$ is trivial.

Conversely, let the class $[a]$ given by $a_{ijk} := d_{ij}d_{jk}d_{ki}$ for a choice of d_{ij} and assume that it is trivial. Then $[a]$ is a boundary $\delta([b])$, i.e. there is a cocycle $b = (b_{ij}) \in \check{C}^1(\mathfrak{U}, \mathbb{Z}_2)$ with $a_{ijk} = \delta([b])_{ijk} = b_{ij}b_{jk}b_{ki}$. As a consequence, the functions

$$z_{ij} := \frac{d_{ij}}{b_{ij}} \in \mathcal{E}(U_{ij}, \mathbb{C}^\times)$$

satisfy 1. and 2. of (65), and thus define a square root S of K with z_{ij} as its transition functions. □

Concerning the uniqueness of the square root we prove:

Proposition 15.24. *Given a complex line bundle K the isomorphism classes of line bundles S with $S^2 \cong K$ are in one-to-one correspondence to the elements of the Čech cohomology group $\check{H}^1(\mathfrak{U}, \mathbb{Z}_2)$.*

Proof. Assume that $w(K) = [(h_{ij})]$ is trivial, so that there is a line bundle S with $S^2 \cong K$ with transition functions z_{ij} satisfying $z_{ij}^2 = h_{ij}$.

For another line bundle S' isomorphic to S its transition functions satisfy

$$z'_{ij} = \frac{h_i}{h_j} z_{ij}$$

with $h_j \in \mathcal{E}(U_j, \mathbb{C}^\times)$ according to (I) in Section 3.2. Because of $(z'_{ij})^2 = h_{ij} = z_{ij}^2$ it follows that $h_i \in \{1, -1\}$. Therefore,

$$\frac{z'_{ij}}{z_{ij}} \text{ is the coboundary of } \frac{h_i}{h_j},$$

and $(z_{ij}), (z'_{ij})$ determine the same cohomology class in $\check{H}^1(\mathfrak{U}, \mathbb{Z}_2)$.

Any cocycle $c = (c_{ij}) \in \check{C}^1(\mathfrak{U}, \mathbb{Z}_2)$ determines a $z'_{ij} := c_{ij}z_{ij}$ which defines another square root $S' = S([c])$ of K. It is not isomorphic to S when its cohomology class $[c] \in \check{H}^1(\mathfrak{U}, \mathbb{Z}_2)$ is not trivial. In this way, we obtain a bijection

$$S : \check{H}^1(\mathfrak{U}, \mathbb{Z}_2) \longrightarrow \{\text{isomorphism classes of square roots of } K\}\ ,\ [c] \mapsto S([c])\,.$$

□

Corollary 15.25. *For the program of half-form quantization it follows that* $K_{-1}(P)$ *has a square root if and only if the corresponding obstruction class* $w(K_{-1}(P))$ *of the bundle* $K_{-1}(P)$ *vanishes. This is equivalent to the vanishing of the obstruction* $w(K_P)$ *of the canonical bundle* $K_P = K_1(P))$ *of* P.

Moreover, whenever there exists a square root of $K_{-1}(P)$ *the Čech cohomology group* $\check{H}^1(\mathfrak{U}, \mathbb{Z}_2)$ *parametrizes the inequivalent square roots of* $K_{-1}(P)$.

Summary:

16 Metalinear Structure

16.1 Metalinear Frame Bundle

A naive approach to define half-forms on the frame bundle $R(P)$ of P (where P is a polarization or more generally a vector bundle) would be to require, that such a half-form corresponds to a function u on $R(P)$ with the following transformation property

$$u(bg) = (\det g)^{1/2}\, u(b) \tag{66}$$

for frames $b \in R(P)$ and $g \in \mathrm{GL}(n, \mathbb{C})$. However, the square root $(\det g)^{1/2}$ is not well-defined, in general. We have discussed this in Observation 15.6 in the preceding chapter.

To remove the ambiguity in the square root the general linear group $\mathrm{GL}(n, \mathbb{C})$ will be replaced be its connected double covering, the METALINEAR GROUP $\mathrm{ML}(n, \mathbb{C})$, and subsequently the frame bundle $R(P)$ by a metalinear frame bundle $\widetilde{R}(P)$.

The (complex) metalinear group $\mathrm{ML}(n, \mathbb{C})$ is the connected central extension of the Lie group $\mathrm{GL}(n, \mathbb{C})$ by $\mathbb{Z}_2$:

$$1 \to \mathbb{Z}_2 \to \mathrm{ML}(n, \mathbb{C}) \stackrel{\rho}{\to} \mathrm{GL}(n, \mathbb{C}) \to 1\,,$$

in particular, $\rho : \mathrm{ML}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ is a double covering.

The metalinear group can be defined as the quotient $(\mathbb{C} \times \mathrm{SL}(n, \mathbb{C}))/2\mathbb{Z}$ as we show in the following:

We start with the simply connected Lie group $\mathbb{C} \times \mathrm{SL}(n, \mathbb{C})$ with its group law $((u, s), (u', s')) \mapsto (u + u', ss')$, and consider the action

$$\mathbb{Z} \times (\mathbb{C} \times \mathrm{SL}(n, \mathbb{C})) \to \mathbb{C} \times \mathrm{SL}(n, \mathbb{C})$$

of $\mathbb{Z}$ on $\mathbb{C} \times \mathrm{SL}(n, \mathbb{C})$ given by

$$(k, (u, s)) \mapsto \left(u + \frac{2\pi i k}{n}, e^{-\frac{2\pi i k}{n}} s\right)\,.$$

Lemma 16.1. *The fibres of the homomorphism*

$$p : \mathbb{C} \times \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})\,,\ (u, s) \mapsto e^u s\,,$$

are the orbits of the above $\mathbb{Z}$-action. In other words,

$$\mathbb{C} \times \mathrm{SL}(n, \mathbb{C})/\mathbb{Z} = \mathbb{C} \times \mathrm{SL}(n, \mathbb{C})/\mathrm{Ker}\, p \cong \mathrm{GL}(n, \mathbb{C})\,.$$

Moreover, the injection

$$j : \mathbb{Z} \to \mathbb{C} \times \mathrm{SL}(n, \mathbb{C})\,,\ k \mapsto \left(\frac{2\pi i k}{n}, e^{-\frac{2\pi i k}{n}} I_n\right)\,,\ k \in \mathbb{Z}\,,$$

satisfies $\operatorname{Im} j = \operatorname{Ker} p$*, i.e. we obtain the following exact sequence*

$$1 \longrightarrow \mathbb{Z} \xrightarrow{j} \mathbb{C} \times \mathrm{SL}(n, \mathbb{C}) \xrightarrow{p} \mathrm{GL}(n, \mathbb{C}) \longrightarrow 1 .$$

Proof. It is easy to see that p is a homomorphism and surjective, so that $\mathbb{C} \times \mathrm{SL}(n, \mathbb{C}) / \operatorname{Ker} p \cong \mathrm{GL}(n, \mathbb{C})$. p is invariant under the action of $\mathbb{Z}$:

$$p\left(u + \frac{2\pi i k}{n}, e^{-\frac{2\pi i k}{n}} s\right) = e^{u + \frac{2\pi i k}{n}} e^{-\frac{2\pi i k}{n}} s = e^u s = p(u, s) .$$

In particular,

$$\operatorname{Ker} p = \left\{ \left(\frac{2\pi i k}{n}, e^{-\frac{2\pi i k}{n}} I_n \right) \mid k \in \mathbb{Z} \right\} = \operatorname{Im} j ,$$

and the orbits have the form

$$\mathbb{Z}(u, s) = (u, s) + \operatorname{Ker} p = p^{-1}(e^u s) ,$$

for $(u, s) \in \mathbb{C} \times \mathrm{SL}(n, \mathbb{C})$. □

We conclude that $p : \mathbb{C} \times \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ is a universal covering of $\mathrm{GL}(n, \mathbb{C})$ since $\mathrm{SL}(n, \mathbb{C})$ and hence $\mathbb{C} \times \mathrm{SL}(n, \mathbb{C})$ is simply connected. It follows, that $\pi_1(\mathrm{GL}(n, \mathbb{C})) \cong \mathbb{Z}$. This covering "contains" a 2-fold covering, which defines the metalinear group:

Definition 16.2. The quotient group $(\mathbb{C} \times \mathrm{SL}(n, \mathbb{C}))/2\mathbb{Z}$ with its Lie structure is called the (complex) metalinear group and will be denoted by $\mathrm{ML}(n, \mathbb{C})$.

Two elements $(u, s), (u', s') \in \mathbb{C} \times \mathrm{SL}(n, \mathbb{C})$ are equivalent with respect to $2\mathbb{Z}$ if and only if there is $k \in \mathbb{Z}$ such that $u = u' + \frac{4\pi i k}{n}$ and $s = \exp(-\frac{4\pi i k}{n}) s'$. As a consequence, the coset $[u, s]$ of (u, s) is

$$\pi(u, s) := [u, s] = \left\{ \left(u + \frac{4\pi i k}{n}, e^{-\frac{4\pi i k}{n}} s \right) \mid k \in \mathbb{Z} \right\} , \; (u, s) \in \mathbb{C} \times \mathrm{SL}(n, \mathbb{C}) .$$

The definition of $\mathrm{ML}(n, \mathbb{C})$ comprises the quotient homomorphism

$$\rho : \mathrm{ML}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C}) , \; [u, s] \mapsto e^u s ,$$

which is a twofold covering such that the following diagram is commutative

$$\begin{array}{ccc} \mathbb{C} \times \mathrm{SL}(n, \mathbb{C}) & & \\ {\scriptstyle p'} \downarrow & \searrow^{p} & \\ \mathrm{ML}(n, \mathbb{C}) & \xrightarrow{\rho} & \mathrm{GL}(n, \mathbb{C}) \end{array}$$

where p' denotes the projection $\mathbb{C} \times \mathrm{SL}(n, \mathbb{C}) \to (\mathbb{C} \times \mathrm{SL}(n, \mathbb{C}))/2\mathbb{Z} = \mathrm{ML}(n, \mathbb{C})$.

The kernel of ρ consists of the elements $[u, s]$ satisfying $e^u s = I_n$[86] which implies $s = e^{-u} I_n$, hence $\det s = e^{-nu} = 1$, where $u = \frac{2\pi i h}{n}$ for suitable $h \in \mathbb{Z}$. As a consequence,

$$\operatorname{Ker} \rho = \left\{ \left(\frac{2\pi i k}{n} + \frac{4\pi i h}{n}, e^{-\frac{2\pi i k}{n}} e^{-\frac{4\pi i h}{n}} I_n \right) \mid k, h \in \mathbb{Z} \right\} = \left\{ [0, I_n], \left[\frac{2\pi i}{n}, e^{-\frac{2\pi i}{n}} I_n \right] \right\} \cong \mathbb{Z}_2 \,.$$

We obtain the above mentioned exact sequence

$$1 \to \mathbb{Z}_2 \xrightarrow{\iota} \mathrm{ML}(n, \mathbb{C}) \xrightarrow{\rho} \mathrm{GL}(n, \mathbb{C}) \to 1 \,,$$

where (we write again $\mathbb{Z}_2$ multiplicatively: $\mathbb{Z}_2 = \{1, -1\}$)

$$\iota(1) := [0, I_n], \iota(-1) := \left[\frac{2\pi i}{n}, e^{-\frac{2\pi i}{n}} I_n \right] .$$

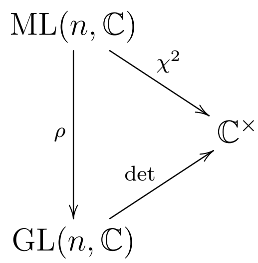
Observe, that this exact sequence characterizes $\mathrm{ML}(n, \mathbb{C})$ as a connected double covering of $\mathrm{GL}(n, \mathbb{C})$.

We also want to note that the fibre of ρ over an element $g \in \mathrm{GL}(n, \mathbb{C})$ is

$$\rho^{-1}(g) = \left\{ \left[u, e^{-u} g \right], \left[u + \frac{2\pi i}{n}, e^{-(u + \frac{2\pi i}{n})} g \right] \right\} ,$$

where $u \in \mathbb{C}$ is a complex number with $\det g = e^{nu}$.

The Lie group homomorphism $\chi : \mathrm{ML}(n, \mathbb{C}) \to \mathbb{C}^\times$ defined by $\chi([u, g]) := e^{\frac{n}{2} u}$ satisfies $\chi^2 = \det \circ \rho$, i.e. the following diagram is commutative



In this sense, χ can be regarded as to be the square root of the determinant.

In order to use this square root not only for a single point of M but globally over the manifold M we consider an equivariant double (i.e. 2-to-1) covering $\tilde{\rho} : \widetilde{R}(P) \to R(P)$ of the frame bundle $R(P)$, where P is a polarization. Over a point $a \in M$ where the fibre $R_a(P)$ is essentially $\mathrm{GL}(n, \mathbb{C})$ the covering $\tilde{\rho}_a : \widetilde{R}_a(P) \to R_a(P)$ should represent the covering $\rho : \mathrm{ML}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ just described. More precisely:

Definition 16.3. A Metalinear Frame Bundle for a polarization $P \subset TM^{\mathbb{C}}$ is a principal fibre bundle $\tilde{\pi} : \widetilde{R}(P) \to M$ with structure group $\mathrm{ML}(n, \mathbb{C})$ together with a 2-to-1 covering $\tilde{\rho} : \widetilde{R}(P) \to R(P)$, such that

[86] Here I_n denotes, as in other occasions, the unit $n \times n$- matrix.

1. $\tilde{\rho}$ is compatible with the projections, i.e. the diagram

$$\begin{array}{ccc} \widetilde{R}(P) & \xrightarrow{\tilde{\rho}} & R(P) \\ & {\scriptstyle\tilde{\pi}}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & M & \end{array}$$

is commutative: $\tilde{\pi} = \tilde{\rho} \circ \pi$, and

2. $\tilde{\rho}$ is equivariant, i.e. the following diagram is commutative

$$\begin{array}{ccc} \widetilde{R}(P) \times \mathrm{ML}(n,\mathbb{C}) & \longrightarrow & \widetilde{R}(P) \\ \downarrow{\scriptstyle(\tilde{\rho},\rho)} & & \downarrow{\scriptstyle\tilde{\rho}} \\ R(P) \times \mathrm{GL}(n,\mathbb{C}) & \longrightarrow & R(P)\,, \end{array}$$

where the horizontal arrows are the right actions of the respective groups: In other words, $\tilde{\rho}(\tilde{b}\tilde{g}) = \tilde{\rho}(\tilde{b})\rho(\tilde{g}))$ for $(\tilde{b}, \tilde{g}) \in \widetilde{R}(P) \times \mathrm{ML}(n, \mathbb{C})$.

Two such metalinear frame bundles $\widetilde{R}(P)$ and $\tilde{\rho}' : \widetilde{R}'(P) \to R(P)$ for P are equivalent, if there exists an equivariant diffeomorphism $\phi : \widetilde{R}(P) \to \widetilde{R}'(P)$, i.e. a diffeomorphism satisfying $\phi(\tilde{b}\tilde{g}) = \phi(\tilde{b})\tilde{g}$ and $\phi \circ \tilde{\pi}' = \tilde{\pi}$. Finally, a Metalinear Structure on P is an equivalence class of metalinear frame bundles for P.

Given a metalinear frame bundle $\widetilde{R}(P)$ and a frame $b \in R_a(P)$ each of the two objects in $\tilde{\rho}^{-1}(b) \subset \widetilde{R}(P)$ is called metaframe. A metaframe is sometimes denoted by $\tilde{b}$ although this notation is slightly ambiguous.

In general, a metalinear frame bundle does not exist. It exists whenever a certain obstruction class in $\check{H}^2(M, \mathbb{Z}_2)$, which is induced by P resp. $R(P)$, is trivial. In case of existence the metalinear structures on P are parametrized by the Čech cohomology group $\check{H}^1(M, \mathbb{Z}_2)$. This is similar to the existence and uniqueness of square roots of the canonical bundle $K(P)$, see Section 15.5 in the preceding chapter, and will be explained in Section 16.3 below in detail.

Let $\tilde{\rho} : \widetilde{R}(P) \to R(P)$ a metalinear frame bundle for P and let $\tilde{g}_{ij}$ be transition functions for this metalinear frame bundle with respect to some open cover $\mathfrak{U} = (U_j)$, again with the property that $U_i, U_{ij}, U_{ijk}, ...$ are contractible. Then there exist functions $u_{ij} : U_{ij} \to \mathbb{C}$ and $s_{ij} : U_{ij} \to \mathrm{SL}(n, \mathbb{C})$ such that $\tilde{g}_{ij} = [u_{ij}, s_{ij}]$. It is easy to show that $g_{ij} := \rho \circ \tilde{g}_{ij} = e^{u_{ij}} s_{ij}$ are transition functions for the frame bundle $R(P)$ (see the proof of Proposition 16.8 below). Moreover, $z_{ij} := \chi(\tilde{g}_{ij}) = \exp \frac{n}{2} u_{ij}$ is a cocycle. Therefore (z_{ij}) defines a complex line bundle $S \to M$, see Proposition 3.9.

Now the square of S, the bundle $S^2 := S \otimes S$, has transition functions z_{ij}^2 satisfying $z_{ij}^2 = \chi^2(\tilde{g}_{ij}) = \det g_{ij}$ by definition of the character χ. We know by Observation 15.4 that $\det g_{ij}$ are the transition functions of $K_{-1}(P)$ as well. As a result, $S \otimes S \cong K_{-1}(P)$ and the dual $S^\vee$ is a square root of the canonical bundle $K(P) = K_1(P)$.

We conclude

Proposition 16.4. *For a polarization P the equivalence classes of square roots of the bundle $K_{-1}(P)$ are in one-to-one correspondence to the metalinear structures on P. More explicitly, we can describe this bijection with the aid of transition functions: Let $[(\tilde{g}_{ij})]$ denote the equivalence class of the metalinear frame bundle $\widetilde{R}(P)$ determined by transition functions $\tilde{g}_{ij} : U_{ij} \to \mathrm{ML}(n,\mathbb{C})$, i.e. $[(\tilde{g}_{ij})]$ is a metalinear structure on P. Then the bijection is given by*

$$[(\tilde{g}_{ij})] \mapsto [(\chi(\tilde{g}_{ij})] .$$

Proof. We have just seen that a metalinear structure on P with transition functions $\tilde{g}_{ij}$ induces a square root of $K_{-1}(P)$ determined by transition functions $\chi(\tilde{g}_{ij})$.

Conversely, any square root S of $K_{-1}(P)$ is given by transition functions z_{ij} satisfying $z_{ij}^2 = \det g_{ij}$. We can find u_{ij} with

$$z_{ij} = \exp \frac{n}{2} u_{ij} .$$

Since (z_{ij}) is a cocycle, $\exp \frac{n}{2}(u_{ij} + u_{jk} + u_{ki}) = 1$, i.e. $\frac{n}{2}(u_{ij} + u_{jk} + u_{ki}) = 2\pi i m$, thus $u_{ij} + u_{jk} + u_{ki} = \frac{4\pi i m}{n}$, where m is a suitable integer $m \in \mathbb{Z}$. We define $\tilde{g}_{ij} := [u_{ij}, \exp(-u_{ij} g_{ij})] : U_{ij} \to \mathrm{ML}(n,\mathbb{C})$. We show that $(\tilde{g}_{ij})$ is a cocycle:

$$\tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} = [u_{ij} + u_{jk} + u_{ki}, e^{-u_{ij}} e^{-u_{jk}} e^{-u_{ki}} g_{ij} g_{jk} g_{kj}] = \left[\frac{4\pi i m}{n}, e^{-\frac{4\pi i m}{n}} I_n\right] = [0, I_n] = 1$$

by the definition of the metalinear group as $(\mathbb{C} \times \mathrm{SL}(n,\mathbb{C}))/2\mathbb{Z}$. This cocycle $(\tilde{g}_{ij})$ defines a metalinear frame bundle $\widetilde{R}(P)$ determined by the square root of $K_{-1}(P)$ given by the transition function z_{ij}. Moreover, $z_{ij} = \chi(\tilde{g}_{ij})$.

As a result, the assignment $[(\tilde{g}_{ij})] \mapsto [(\chi(\tilde{g}_{ij})]$ is bijective. □

Note, that the set of equivalence classes of square roots of K_{-1} is also in bijection with $\check{H}^1(\mathfrak{U}, \mathbb{Z}_2) \cong \check{H}^1(M, \mathbb{Z}_2)$, see Proposition 15.24, and thus the set of equivalence clases of metalinear structure on P are in bijection with $\check{H}^1(M, \mathbb{Z}_2)$.

A slightly different look at the line bundle S induced by the metalinear frame bundle is the following: With respect to the charecter $\chi : \mathrm{ML}(n,\mathbb{C}) \to \mathbb{C}^\times$, $(z, g) \mapsto z$, we obtain the associated complex line bundle to the principal fibre bundle $\widetilde{R}(P)$

$$S' := \tilde{R}(P) \times_{\mathrm{ML}(n,\mathbb{C})} \mathbb{C} = \tilde{R}(P) \times_\chi \mathbb{C} .$$

The transition functions for S' turn out to be $\chi(\tilde{g}_{ij}) = z_{ij}$ according to Proposition D.9. It follows that S and S' are isomorphic.

In the same way one can prove:

Lemma 16.5. *In case a square bundle of $K(P)$ exists, for each half integer r the bundle $K_r(P)$ of r-forms is isomorphic to the associated complex line bundle*

$$\tilde{R}(P) \times_{\mathrm{ML}(n,\mathbb{C})} \mathbb{C} = \tilde{R}(P) \times_{\chi^{-2r}} \mathbb{C} .$$

With this result we can formulate the correct form of the transformation property adjusting the naive ansatz (66):

Corollary 16.6. *Let $\tilde{\rho} : \widetilde{R}(P) \to R(P)$ be a metalinear frame bundle, where P is a polarization of the symplectic manifold (M, ω). For each half integer r the sections $\alpha \in \Gamma(M, K_r(P))$ are in one-to-one correspondence with the functions $\tilde{\alpha}^\sharp = u : \widetilde{R}(P) \to \mathbb{C}$ satisfying*

$$u(\tilde{b}\tilde{g}) = \chi(\tilde{g})^{2r} u(\tilde{b}) \,,$$

where $\tilde{b} \in \widetilde{R}(P)$ and $\tilde{g} \in ML(n, \mathbb{C})$. In particular, with $r = \frac{1}{2}$

$$u(\tilde{b}\tilde{g}) = \chi(\tilde{g}) u(\tilde{b}) \,,$$

Moreover, for $r \in \mathbb{Z}$

$$u(\tilde{b}\tilde{g}) = (\det \rho(\tilde{g}))^r u(\tilde{b}) = (\det g)^r u(\tilde{b}) \,,$$

when $\rho(\tilde{g}) = g$.

Proof. The transformation rule follows immediately from the construction of the associated bundle $\tilde{R}(P) \times_{\chi^{-2r}} \mathbb{C}$, cf. Corollary D.8. □

Note, that in the case of an integer $r \in \mathbb{Z}$ a section $\alpha \in \Gamma(M, K_r(P))$ induces $\alpha^\sharp : R(P) \to \mathbb{C}$ with

$$\alpha^\sharp(bg) = (\det g)^r \alpha^\sharp(b)$$

and $\tilde{\alpha}^\sharp : \widetilde{R}(P) \to \mathbb{C}$ with

$$\tilde{\alpha}^\sharp(\tilde{b}\tilde{g}) = (\det g)^r \tilde{\alpha}^\sharp(\tilde{b}) \,,\; g = \tilde{\rho}(\tilde{g}) \,.$$

The two functions are simply related by $\tilde{\alpha}^\sharp = \alpha^\sharp \circ \tilde{\rho}$ when $r \in \mathbb{Z}$.

16.2 Half-Form Quantization Based on the Metalinear Group

In this section a fixed metalinear structure is given by a metalinear frame bundle $\tilde{R}(P)$ where P is a reducible complex polarization on the quantizable symplectic manifold (M, ω) with a prequantum bundle (L, H, ∇). The metalinear frame bundle $\tilde{R}(P)$ induces a square root $S = K_{-1/2}(P)$ of the dual $K_{-1}(P)$ of the canonical bundle $K(P)$, as we have seen in the preceding section.

With the aid of this square root line bundle $S = K_{-1/2}(P)$ the programme of half-form quantization can be carried through in the same way as is done in Section 15.4.

We mention four occasions where, in comparison to the Section 15.4 and before, the transformation property of half-forms can be used directly in order to make the arguments more transparent. Recall, that the transformation property in question is (cf. Corollary 16.6):

$$\tilde{\alpha}^\sharp(\tilde{b}\tilde{g}) = \chi(\tilde{g})^{-1} \tilde{\alpha}^\sharp(\tilde{b})$$

for sections $\alpha \in \Gamma(M, K_{-1/2}(P))$, where $(\tilde{b}, \tilde{g}) \in \widetilde{R}(P) \times \mathrm{ML}(n, \mathbb{C})$.

1. Proof of the statement of Lemma 15.9: "Any two sections $\alpha, \beta \in \Gamma(M, K_{-1/2}(P))$ determine a -1-density $\bar{\alpha}\beta \in \Gamma(M, \delta_{-1}(P))$."

   Here, $\bar{\alpha}\beta =: \mu$ is given by
   $$\mu^\sharp(b) := \overline{\tilde{\alpha}^\sharp}(\tilde{b})\tilde{\beta}^\sharp(\tilde{b})$$
   for $\tilde{\rho}(\tilde{b}) = b$. To prove this statement, we first observe that
   $$\overline{\chi(\tilde{g})}\chi(\tilde{g}) = |\chi(\tilde{g})|^2 = |\det \rho(\tilde{g})||\det g|$$
   when $\rho(\tilde{g}) = g$. The new transformation property
   $$\tilde{\alpha}^\sharp(\tilde{b}\tilde{g}) = \chi(\tilde{g})^{-1}\tilde{\alpha}^\sharp(\tilde{b})$$
   (see Corollary 16.6) yields:
   $$\mu^\sharp(bg) = \overline{\tilde{\alpha}^\sharp(\tilde{b}\tilde{g})}\tilde{\beta}^\sharp(\tilde{b}\tilde{g}) = \overline{\chi(\tilde{g})^{-1}\tilde{\alpha}^\sharp(\tilde{b})}\chi(\tilde{g})^{-1}\tilde{\beta}^\sharp(\tilde{b}) = (\overline{\chi(\tilde{g})}\chi(\tilde{g}))^{-1}\mu^\sharp(b) = |\det g|^{-1}\mu^\sharp(b)\,.$$
   Hence, $\mu^\sharp$ is well-defined and $\mu = \bar{\alpha}\beta \in \Gamma(M, \delta_{-1}(P))$ because of $\mu^\sharp(bg) = |\det g|^{-1}\mu^\sharp(b)$.

2. Definition of partial connection $\nabla_X\sigma$ for $\sigma \in \Gamma(U, K_\ell), 2\ell \in \mathbb{Z}$, and $X \in \Gamma(U, P)$: Locally, there exists a metaframe field $\tilde{\xi} : U \to \widetilde{R}(P)$ such that the corresponding frame field $\xi = \tilde{\rho} \circ \tilde{\xi}$ is Hamiltonian. We obtain as in the case of $R(P)$ a section $\sigma_{\tilde{\xi}} \in \Gamma(U, K_\ell)$ such that for the induced $\tilde{\sigma}^\sharp_{\tilde{\xi}} : \widetilde{R}(P)|_U \to \mathbb{C}$ the following holds $\tilde{\sigma}^\sharp_{\tilde{\xi}}(\tilde{\xi}) = 1$. A general section $\sigma \in \Gamma(U, K_\ell)$ satisfies $\sigma = \tilde{\sigma}^\sharp(\tilde{\xi})\sigma_{\tilde{\xi}}$ and the definition is
   $$\nabla_X\sigma := L_X(\tilde{\sigma}^\sharp(\tilde{\xi}))\sigma_{\tilde{\xi}}\,.$$
   This direct definition avoids the two step definition using the square σ^2, cf. Corollary 15.15.

3. Definition of partial Lie derivative: Let $X \in \Gamma(M, P)$ preserve P and denote the flow of X by $\Phi_t : M_t \to M_{-t}$. Let $\ell \in \{\frac{1}{2}, -\frac{1}{2}\}$. Then for $\alpha \in \Gamma(M, K_\ell(P))$ the partial Lie derivative $L_X\alpha$ given by
   $$(\widetilde{L_X\alpha})^\sharp(\tilde{\xi}) := \frac{d}{dt}\tilde{\alpha}^\sharp(\widetilde{T\Phi_t}(\tilde{\xi}))\Big|_{t=0}\,,$$
   $\tilde{\xi} \in \widetilde{R}(P)$, is well-defined since $T\Phi_t$ can be transferred to $\widetilde{R}(P)$ as $\widetilde{T}\Phi_t : \widetilde{R}(P) \to \widetilde{R}(P)$ such that $T\Phi_t \circ \tilde{\rho} = \tilde{\rho} \circ \widetilde{T\Phi_t}$.

   Thus we avoid the 2 step definition 15.19.

4. Definition of the quantum operator for classical variables F using the local flow of F in dynamic form, see Proposition 7.13 and Definition 14.8.

16.3 Metalinear Structure: Existence and Uniqueness

This section discusses under which topological conditions on the manifold M and the polarization P there exists a metalinear frame bundle for P and to which extent it will be unique up to isomorphism. These questions will be investigated in the case of a general principal fibre bundle $\pi : B \to M$ with structure group $\mathrm{GL}(n, \mathbb{C})$ instead of $R(P)$. A comparison with spin structures on a manifold will be described in the next chapter.

Definition 16.7. For a principal fibre bundle $\pi : B \to M$ with structure group $\mathrm{GL}(n, \mathbb{C})$ a metalinear bundle over B is a principal fibre bundle $\tilde{\pi} : \widetilde{B} \to M$ with structure group $\mathrm{ML}(n, \mathbb{C})$ together with a 2-to-1 covering $\tilde{\rho} : \widetilde{B} \to B$, such that

1. $\tilde{\rho}$ is compatible with the projections, i.e. the diagram

$$\begin{array}{ccc} \widetilde{B} & \xrightarrow{\tilde{\rho}} & B \\ & {\scriptstyle \tilde{\pi}} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & M & \end{array}$$

is commutative: $\tilde{\pi} = \pi \circ \tilde{\rho}$, and

2. $\tilde{\rho}$ is equivariant, i.e. the following diagram is commutative

$$\begin{array}{ccc} \widetilde{B} \times \mathrm{ML}(n, \mathbb{C}) & \longrightarrow & \widetilde{B} \\ \Big\downarrow {\scriptstyle (\tilde{\rho}, \rho)} & & \Big\downarrow {\scriptstyle \tilde{\rho}} \\ B \times \mathrm{GL}(n, \mathbb{C}) & \longrightarrow & B\,, \end{array} \tag{67}$$

where the horizontal arrows are the right actions of the respective groups: We have $\tilde{\rho}(\tilde{b}\tilde{g}) = \tilde{\rho}(\tilde{b})\rho(\tilde{g})$ for $(\tilde{b}, \tilde{g}) \in \widetilde{R}(P) \times \mathrm{ML}(n, \mathbb{C})$.

Two such metalinear bundles $\tilde{\rho} : \widetilde{B} \to B$ and $\tilde{\rho}' : \widetilde{B}' \to B$ over B are called to be equivalent if there exists an equivariant map $\phi : \widetilde{B} \to \widetilde{B}'$, i.e. $\tilde{\rho} = \tilde{\rho}' \circ \phi$ and $\phi(\tilde{b}\tilde{g}) = \phi(\tilde{b})\tilde{g}$ for $(\tilde{b}, \tilde{g}) \in \widetilde{B} \times \mathrm{ML}(n, \mathbb{C})$. An equivalence class of metalinear bundles over B is called a Metalinear Structure.

Evidently, when P is a polarization on a symplectic manifold (M, ω), then a metalinear structure on $B = R(P)$ is represented by a metalinear frame bundle for P as defined in Section 16.1.

Let $\tilde{\rho} : \tilde{B} \to B$ be such a metalinear bundle over B and let $\tilde{g}_{ij} : U_{ij} \to \mathrm{ML}(n, \mathbb{C})$ be transition functions for $\tilde{B}$ with respect to an open cover $\mathfrak{U} = (U_j)$, where the $U_j, U_{ij} \ldots$ are contractible. Then $\tilde{g}_{ij}$ has the form $\tilde{g}_{ij} = [u_{ij}, s_{ij}]$ with suitable $u_{ij} : U_{ij} \to \mathbb{C}, s_{ij} : U_{ij} \to \mathrm{SL}(n, \mathbb{C})$, as before. Moreover, $z_{ij} := \chi(\tilde{g}_{ij}) = e^{\frac{n}{2}u_{ij}} : U_{ij} \to \mathbb{C}^\times$, $g_{ij} := \rho(\tilde{g}_{ij}) = e^{u_{ij}} s_{ij} : U_{ij} \to \mathrm{GL}(n, \mathbb{C})$ satisfy

$$\begin{array}{ll} 1. & z_{ij}^2 = \det g_{ij} \quad \text{on} \quad U_{ij}\,, \\ 2. & z_{ij} z_{jk} z_{ki} = 1 \quad \text{on} \quad U_{ijk}. \end{array} \tag{68}$$

Proposition 16.8. *The g_{ij} are transition functions for the bundle B. Conversely, when g_{ij} are transition functions for B and $u_{ij} : U_{ij} \to \mathbb{C}$ can be found such that with $z_{ij} := \exp \frac{n}{2} u_{ij}$ the above conditions 1. and 2. are satisfied, then $\tilde{g}_{ij} := [u_{ij}, e^{-u_{ij}} g_{ij}] : U_{ij} \to \mathrm{ML}(n, \mathbb{C})$ are transition functions of a metalinear bundle $\tilde{\pi} : \widetilde{B} \to B$ over B.*

Proof. To show the lemma is easy, only the various conditions have to be checked, and this is not very interesting. Nevertheless, in order to get acquainted with the new structure, which is important also in the next chapter, we present a detailed proof.

So let $\tilde{\rho} : \tilde{B} \to B$ a metalinear structure over B. The compatibility conditions on $\tilde{\rho}, \rho$ and the group actions (see the commutative diagram (67) in the definition) imply that g_{ij} are transition functions for B as we see by investigating the local situation: Let $\tilde{\psi}_j : \widetilde{B}_{U_j} \to U_j \times \mathrm{ML}(n, \mathbb{C})$ be the local trivializations with respect to the cover (U_j) which define the transition functions $\tilde{g}_{ij}$ by

$$\tilde{\psi}_i \circ \tilde{\psi}_j^{-1}(a, \tilde{h}) = (a, \tilde{g}_{ij}(a).\tilde{h}) \, , \; (a, \tilde{h}) \in U_{ij} \times \mathrm{ML}(n, \mathbb{C}) \, .$$

The double covering $\tilde{\rho}$ induces local trivializations $\psi_j : B_{U_j} \to U_j \times \mathrm{GL}(n, \mathbb{C})$ for the principal bundle B: Each $b \in B$ with $\pi(b) \in U_j$ has two inverse images $\tilde{b}_\pm \in \tilde{\rho}^{-1}(b)$, and $\psi_j(b) := (\mathrm{id} \times \rho)(\tilde{\psi}_j(\tilde{b}_\pm))$ is well-defined: $\tilde{\psi}_j(\tilde{b}_\pm) = (a, \tilde{h}_\pm)$ with two $h_\pm \in \mathrm{SL}(n, \mathbb{C})$ with a common image $\rho(h_\pm) = h$, hence $(\mathrm{id} \times \rho)(\tilde{\psi}_j(\tilde{b}_+)) = (a, h) = (\mathrm{id} \times \rho)(\tilde{\psi}_j(\tilde{b}_-))$. Since $\tilde{\rho}$ is a local diffeomorphism, ψ_j is smooth, and, moreover, a diffeomorphism which respects the action of $\mathrm{GL}(n, \mathbb{C})$. Hence, the ψ_j are local trivializations of the principal fibre bundle B satisfying $\psi_j \circ \tilde{\rho} = (\mathrm{id} \times \rho) \circ \tilde{\psi}_j$, i.e. the following diagram is commutative

$$\begin{array}{ccc} \widetilde{B}_{U_j} & \xrightarrow{\;\tilde{\psi}_j\;} & U_j \times \mathrm{ML}(n, \mathbb{C}) \\ {\scriptstyle \tilde{\rho}} \downarrow & & \downarrow {\scriptstyle \mathrm{id} \times \rho} \\ B_{U_j} & \xrightarrow[\;\psi_j\;]{} & U_j \times \mathrm{GL}(n, \mathbb{C}) \, . \end{array}$$

Finally, the diagram helpsto varify

$$\begin{aligned} \psi_i \circ \psi_j^{-1}(a, h) &= \psi_i \circ \tilde{\rho} \circ \tilde{\psi}_j^{-1}(a, \tilde{h}_\pm) \\ &= (\mathrm{id} \times \rho) \circ \tilde{\psi}_i \circ \psi_j^{-1}(a, \tilde{h}_\pm) \\ &= (\mathrm{id} \times \rho)(a, \tilde{g}_{ij}(a).\tilde{h}_\pm) \\ &= (a, \rho(\tilde{g}_{ij}(a).\tilde{h}_\pm)) \\ &= (a, (\rho(\tilde{g}_{ij}(a)).(\rho(\tilde{h}_\pm)))) \\ &= (a, g_{ij}(a).h) \, , \end{aligned}$$

As a result, $g_{ij} = \rho(\tilde{g}_{ij})$ are transition functions for B.

Conversely, 1. and 2. imply that $\tilde{g}_{ij} : U_{ij} \to \mathrm{ML}(n, \mathbb{C})$, given by $\tilde{g}_{ij} := [u_{ij}, e^{-u_{ij}} g_{ij}]$, are well-defined and satisfy the cocycle condition (C) which we have shown already in the special case of $B = R(P)$ in Proposition 16.4: In fact, $(z_{ij}) = (\exp \frac{n}{2} u_{ij})$ is a cocycle, i.e. $\exp \frac{n}{2}(u_{ij} + u_{jk} + u_{ki}) = 1$, such that $\frac{n}{2}(u_{ij} + u_{jk} + u_{ki}) = 2\pi i m$ for a suitable $m \in \mathbb{Z}$. Hence,

$$\tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} = [u_{ij} + u_{jk} + u_{ki}, e^{-u_{ij}}e^{-u_{jk}}e^{-u_{ki}}g_{ij}g_{jk}g_{kj}] = \left[\frac{4\pi i m}{n}, e^{-\frac{4\pi i m}{n}} I_n\right] = [0, I_n] = 1\,.$$

The corresponding principal fibre bundle $\tilde{\pi} : \widetilde{B} \to M$ having $\tilde{g}_{ij}$ as its transition functions can be constructed as in the case of line bundles, see Proposition 3.9. The fact $g_{ij} = \rho \circ \tilde{g}_{ij}$ gives immidiately the 2-1 covering map $\tilde{\rho} : \widetilde{B} \to B$ with all its compatibilities establishing that in this way we obtain a metalinear structure over B.

□

We now come to the essential part of this section where we determine when a metalinear structure exists and to which extent it is unique. From the preceding proposition we know that a metalinear structure on a principal $\mathrm{GL}(n, \mathbb{C})$-bundle B with transition functions g_{ij} exists if and only if one can find $u_{ij} : U_{ij} \to \mathbb{C}$ such that for $z_{ij} := e^{\frac{n}{2}u_{ij}}$ and g_{ij} the conditions 1. and 2. are satisfied.

In the following, we require again that the open cover $\mathfrak{U} = (U_j)$ has the property that the $U_j, U_{ij} \ldots$ are all contractible.

We pick for each $i, j \in I$ a smooth square root $d_{ij} \in \mathcal{E}(U_{ij}, \mathbb{C}^\times)$ of $\det g_{ij}$, i.e. $d_{ij}^2 = \det g_{ij}$. Since g_{ij} satisfies the cocycle condition this holds for $\det g_{ij}$ as well and thus $d_{ij}^2 d_{jk}^2 d_{ki}^2 = 1$. In order to discuss condition 2., here $d_{ij}d_{jk}d_{ki} = 1$, we define

$$a_{ijk} := d_{ij}d_{jk}d_{ki} \;\text{ on }\; U_{ijk}\,.$$

The collection $a := (a_{ijk})$ is a cocycle with $a_{ijk} \in \mathbb{Z}_2 = \{1, -1\}$ since $a_{ijk}^2 = 1$. It induces a cohomology class $[a] \in \check{H}^2(\mathfrak{U}, \mathbb{Z}_2) \cong \check{H}^2(M, \mathbb{Z}_2)$. This cohomology class is independent of the choice of the square roots d_{ij}, it only depends on the transitions functions g_{ij}. For different transition functions with respect to possibly other covers $\mathfrak{U}'$ we obtain the same class depending only on B. This cohomology class $w(B) := [a]$ will be called the obstruction class.

$a = (a_{ijk})$ can be regarded as a cocycle in $\check{C}^2(\mathfrak{U}, \mathbb{C})$, and by definition it is a coboundary there. In order that a is a coboundary in $\check{C}^2(\mathfrak{U}, \mathbb{Z}_2)$, there should exist a cocycle $b = (b_{ij}) \in \check{C}^1(\mathfrak{U}, \mathbb{Z}_2)$ which satisfies

$$a_{ijk} = b_{ij}b_{jk}b_{ki}.$$

We have prepared the proof of the following proposition:

Proposition 16.9. *There exists a metalinear structure over B if and only if the obstruction class $w(B) = [a]$ is trivial, i.e. if it is the class* [1]. *One also says, it vanishes, when one emphasizes the notation $\mathbb{Z}_2 = \{[0], [1]\}$.*

Proof. We just have deduced that the existence of a metalinear structure over B implies the existence of $u_{ij} \in \mathcal{E}(U_{ij}, \mathbb{C}$ such that $z_{ij} := \exp \frac{n}{2} u_{ij}$ satisfy $z_{ij}^2 = \det g_{ij}$ and $z_{ij} z_{jk} z_{ki} = 1$, see (68). Since $a = z_{ij} z_{jk} z_{ki} = 1$, the obstruction class $[a]$ is trivial.

Conversely, let a be given by $a_{ijk} := d_{ij} d_{jk} d_{ki}$ for a choice of d_{ij} with $d_{ij}^2 = \det g_{ij}$. When $[a]$ is trivial in $\check{H}^2(\mathfrak{U}, \mathbb{Z}_2)$, then there is $b = (b_{ij}) \in \check{C}^1(\mathfrak{U}, \mathbb{Z}_2)$ with $a_{ijk} = b_{ij} b_{jk} b_{ki}$. As a consequence,

$$z_{ij} := \frac{d_{ij}}{b_{ij}}$$

satisfy 1. and 2. of (68).

□

Corollary 16.10. *For the program of geometric quantization it follows for a complex polarization P that a metalinear frame bundle $\widetilde{R}(P) \to R(P)$ exists if and only if the obstruction class $w(R(P))$ is trivial.*

If $w(R(P))$ is trivial and $\widetilde{R}(P)$ is a metalinear frame bundle we have a natural bijection between $\check{H}^1(M, \mathbb{Z}_2)$ and the set of metalinear structures in the same way as for the case of square roots, see the end of the preceding chapter.

Examples 16.11. 1. Momentum Space (T^*Q, ω).

2. $\mathbb{P}^{2k+1}(\mathbb{C})$, but not $\mathbb{P}^{2k}(\mathbb{C})$, $k \in \mathbb{Z}$.

3. Graßmannian $Gr(2,4)$.

Summary:

17 Metaplectic Structure

In case of a quantizable symplectic manifold (M, ω) with a prequantum line bundle and with two different complex polarizations admitting a square root of the canonical bundle we want to compare the corresponding representations induced by the half-form quantization (see Section 15). In order to construct a pairing between the two representation spaces the square roots have to fit together. In other words, the metalinear structures related to the square roots (see 16) have to be compatible. In which way? It turns out that the different metalinear structures on the polarizations should be induced by a joint metaplectic structure given on the symplectic manifold (M, ω).

As the main result of this section we show, that a metaplectic structure on (M, ω) determines on each positive complex polarization P of (M, ω) a unique metalinear structure thereby yielding a square root of the canonical bundle K_P for each P.

This result is applied to try to construct a pairing between representations determined by different positive presentations.

Moreover, this chapter ends with a description of the metaplectic representation and its relation to the CCR representation.

17.1 Metaplectic Frame Bundle

Let (M, ω) be a symplectic manifold. A SYMPLECTIC FRAME at $a \in M$ is an ordered basis

$$(u; v) = (u_1, \dots, u_n, v_1, \dots, v_n) \text{ of } T_aM$$

such that $\omega(u_i, v_j) = \delta_{ij}$ and $\omega(u_i, u_j) = \omega(v_k, v_l) = 0, i, j, k, l \leq n$. The collection $R_a(M, \omega)$ of symplectic frames at $a \in M$ is in one-to-one correspondence with the symplectic group $\mathrm{Sp}(n, \mathbb{R})$, the group of linear canonical transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$, with respect to the standard symplectic form.

The symplectic group $\mathrm{Sp}(n)$ can be defined (see Examples C.6) as the concrete matrix Lie group of $2n \times 2n$ real block matrices

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C, D are real $n \times n$-matrices,satisfying

$$S^T J S = J, \text{ with } J = J_n := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ i.e.}$$

$$A^T D - C^T B = 1, \quad A^T C = C^T A, \quad D^T B = B^T D. \tag{69}$$

There is a natural right action of $\mathrm{Sp}(n)$ on $R_a(M, \omega)$:

$$(u; v) \times \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (uA + vC; uB + vD),$$

which provides a bijection $\mathrm{Sp}(n) \to R_a(M,\omega)$ by fixing a frame $(u^0; v^0) \in R_a(M,\omega)$: For each $(u;v) \in R_a(M,\omega)$ there exists exactly one $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n)$ with $(u^0; v^0)S = (u;v)$.

Lemma 17.1. *The subgroup U_n of $\mathrm{Sp}(n)$ consisting of all block matrices S with $A = D$ and $C = -B$ is isomorphic to $\mathrm{U}(n)$. The $S \in \mathrm{Sp}(n)$ which are in U_n can be characterized by $JS = SJ$ or $S^{-1} = S^T$. U_n is a maximal compact subgroup of $\mathrm{Sp}(n,\mathbb{R})$.*

Proof. In fact,

$$\begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto A + iB =: \gamma \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

$A^T A + B^T B = 1$, $\quad A^T B = B^T A$ (cf. (69)), is an isomorphism $\gamma : U_n \to U(n)$: For $U := A + iB$ we see $\overline{U}^T U = 1$ by

$$\overline{(A+iB)}^T (A+iB) = A^T A + B^T B + i(A^T B - B^T A) = 1 + i0 = 1\,.$$

Therefore, γ is a well-defined homomorphism and it is easy to see that it is bijective.

Given

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(n),$$

the condition $JS = SJ$ is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} C & D \\ -A & -B \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -B & A \\ -D & C \end{pmatrix}$$

which is equivalent to $A = D$ and $B = -C$.

$S^{-1} = S^T$ implies $S^{-1} J S = J$, hence $JS = SJ$ for $S \in \mathrm{Sp}(n)$ □

Note, that $\mathrm{U}(n)$ occurs also as a maximal compact subgroup of the general linear group $\mathrm{GL}(n,\mathbb{C})$.

The symplectic group is homeomorphic to the product of the unitary group $U(n)$ and a contractible space: Let $S_+(n)$ denote the symmetric and positive symplectic matrices, then

$$S_+(n) \times U_n \to \mathrm{Sp}(n)\,,\ (S,U) \mapsto S \circ U\,,$$

is a homeomorphism.

Therefore, since $S_+(n)$ is contractible, the fundamental group of the symplectic group is $\pi_1(\mathrm{Sp}(n)) = \pi_1(\mathrm{U}(n)) = \mathbb{Z}$ [87]. As a consequence, there exists a double covering of $\mathrm{Sp}(n)$, denoted by $\mathrm{Mp}(n)$, which is again a Lie group, and we have the central extension

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Mp}(n) \xrightarrow{\rho} \mathrm{Sp}(n) \longrightarrow 1\,.$$

[87] Here, we use the result $\pi_1(\mathrm{U}(n)) = \mathbb{Z}$.

$\mathrm{Mp}(n,\mathbb{R})$ is called the METAPLECTIC GROUP.

The collection of all the symplectic frames over a symplectic manifold (M,ω) defines the symplectic frame bundle on (M,ω)

$$R(M,\omega) := \bigcup_{a\in M} R_a(M,\omega) \xrightarrow{\pi} M\,,\ R_a(M,\omega) \ni (u;v) \mapsto a \in M\,.$$

$R(M,\omega)$ is a right principal bundle over M with structure group $\mathrm{Sp}(n)$. This fibre bundle comes automatically with the structure of a symplectic manifold and is constructed in an analogous way as the frame bundle $R(E)$ of a complex vector bundle $E \to M$ (see Construction D.4). In particular, the symplectic frame bundle $R(M,\omega)$ is a subbundle of the (tangent) frame bundle $R(M) = R(TM)$ of all bases of T_aM, $a \in M$, with structure group $\mathrm{GL}(2n,\mathbb{R})$.

Definition 17.2. A METAPLECTIC FRAME BUNDLE over a symplectic manifold (M,ω) is a right principal bundle $\tilde\pi : \widetilde{R}(M,\omega) \to M$ with structure group $\mathrm{Mp}(n,\mathbb{R})$ together with a double covering

$$\tilde\rho : \widetilde{R}(M,\omega) \to R(M,\omega)\,,$$

such that

1. $\tilde\rho$ is compatible with the projections, i.e. the diagram

$$\begin{array}{ccc} \widetilde{R}(M,\omega) & \xrightarrow{\tilde\rho} & R(M,\omega) \\ & {\scriptstyle\tilde\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & M & \end{array}$$

is commutative: $\tilde\pi = \tilde\rho \circ \pi$, and

2. $\tilde\rho$ is equivariant, i.e. the following diagram is commutative

$$\begin{array}{ccc} \widetilde{R}(M,\omega) \times \mathrm{Mp}(n) & \longrightarrow & \widetilde{R}(M,\omega) \\ \Big\downarrow{\scriptstyle(\tilde\rho,\rho)} & & \Big\downarrow{\scriptstyle\tilde\rho} \\ R(M,\omega) \times \mathrm{Sp}(n) & \longrightarrow & R(M,\omega), \end{array}$$

where the horizontal arrows are the right actions of the respective groups: In other words, the condition 2. is

$$\tilde\rho(\tilde b\tilde g) = \tilde\rho(\tilde b)\rho(\tilde g))$$

for $(\tilde b,\tilde g) \in \widetilde{R}(M,\omega) \times \mathrm{Mp}(n)$.

Two such metaplectic frame bundles $\widetilde{R}(M,\omega)$ and $\tilde\rho' : \widetilde{R}'(M,\omega) \to R(M,\omega)$ on a symplectic manifold (M,ω) are equivalent, if there exists an equivariant diffeomorphism $\tilde\phi : \widetilde{R}(M,\omega) \to \widetilde{R}'(M,\omega)$, i.e. a diffeomorphism satisfying $\tilde\rho' \circ \tilde\phi = \tilde\rho$ and $\tilde\phi(\tilde b\tilde g) = \tilde\phi(\tilde b)\tilde g$ and $\phi \circ \pi' = \tilde\pi$. Finally, a METAPLECTIC STRUCTURE on P is an equivalence class of metaplectic frame bundles for (M,ω).

Remark 17.3. This definition is completely analogous to the definition of a metalinear structure (see Definitions 17.2 or 16.7), and can be generalized to further geometric situations as we explain in the next section. Moreover, the similarity extends to the questions of existence and uniqueness.

A symplectic manifold (M, ω) admits a metaplectic structure if and only if the second Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}_2)$ of M is trivial. Here, the second Stiefel-Whitney class $w_2(M)$ is the modulo 2 reduction of the first Chern class $c_1(M) \in H^2(M, \mathbb{Z})$. Hence, (M, ω) admits metaplectic structures if and only if $c_1(M)$ is even, i.e. $w_2(M)$ is zero. In case of $w_2(M) = 0$ the equivalence clases of metaplectic structures are in 1-to-1 correspondece to $H^1(M, \mathbb{Z}_2)$.

17.2 General Metastructures

The notion of a metaplectic structure of a symplectic manifold is not only analogous to the notion of a metalinear structure of a complex polarization but also to the notion of a spin structure of an oriented Riemannian manifold (M, g) and other geometric structures like the metaunitary structure of a Hermitian vector bundle (E, H). For the general definition recall the notion of central extension of a Lie group G:

Let

$$1 \longrightarrow Z \xrightarrow{\iota} \widetilde{G} \xrightarrow{\rho} G \longrightarrow 1\,,$$

an exact sequence of Lie groups where Z is abelian with $\iota(Z)$ in the center of $\widetilde{G}$ and where we write the trivial group as 1. In this situation $\widetilde{G}$ is called a central extension of G by Z. Sometimes, the exact sequence is called central extension.

Definition 17.4. Consider the central extension above, and let $B \to X$ be a G-bundle, i.e. a principal fibre bundle over X with structure group G. A ρ-lifting is a $\widetilde{G}$-bundle $\widetilde{B} \to X$ together with a ρ-equivariant principal bundle morphism $\tilde{\rho} : \widetilde{B} \to B$. That is, the following diagrams are commutative:

$$\begin{array}{ccc} \widetilde{B} & \xrightarrow{\tilde{\rho}} & B \\ & {\scriptstyle\tilde{\pi}}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & X & \end{array}$$

$$\begin{array}{ccc} \widetilde{B} \times G & \longrightarrow & \widetilde{B} \\ \downarrow{\scriptstyle(\tilde{\rho},\rho)} & & \downarrow{\scriptstyle\tilde{\rho}} \\ B \times G & \longrightarrow & B, \end{array}$$

Two ρ-liftings $\tilde{\rho} : \widetilde{B} \to B$ and $\tilde{\rho}' : \widetilde{B}' \to B$ are equivalent if there exists an equivariant principal bundle map $\phi : \widetilde{B} \to \widetilde{B}'$ with $\tilde{\rho} = \tilde{\rho}' \circ \phi$. In particular the following diagram is commutative:

$$\begin{array}{ccc} \widetilde{B} & \xrightarrow{\phi} & \widetilde{B}' \\ & {\scriptstyle\tilde{\rho}}\searrow \quad \swarrow{\scriptstyle\tilde{\rho}'} & \\ & B & \end{array}$$

In case of a central extension of the Lie group G by $\mathbb{Z}_2$,

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\iota} \widetilde{G} \xrightarrow{\rho} G \longrightarrow 1\,,$$

a ρ-lifting is a double coverimg $\tilde{\rho} : \widetilde{B} \to B$ with the usual equivariance properties. An equivalence class of ρ-liftings could be called a meta-G-structure in this case.

Metalinear and Metaplectic Structure

It is easy to see that the notions of metalinear structure and metaplectic structure fit into the new definition of a ρ-lifting with respect to the corresponding central extensions

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\iota} \mathrm{ML}(n, \mathbb{C}) \xrightarrow{\rho} \mathrm{GL}(n\mathbb{C}) \longrightarrow 1\,,$$

of $\mathrm{GL}(n, \mathbb{C})$ as the structure group of the frame bundle $R(E)$ of a complex vector bundle E resp.

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\iota} \mathrm{Mp}(n) \xrightarrow{\rho} \mathrm{Sp}(n) \longrightarrow 1\,,$$

of $\mathrm{Sp}(n)$ as the structure group of the symplectic frame bundle by $\mathbb{Z}_2$.

Let us mention another "metastructure" which we need in the following, namely the metaunitary structure, before we discuss the spin structure as another example.

Metaunitary Structure

Here, the group G is the unitary group $G = \mathrm{U}(n) = \mathrm{U}(n, \mathbb{C})$. The unitary group is the structure group of the frame bundle $R(E, H)$ of orthonormal frames of a hermitian vector bundle (E, H).

The metaunitary group $\mathrm{MU}(n)$ is the (connected) double covering of $U(n)$ determined by the following central extension:

$$1 \longrightarrow \mathbb{Z}_2 \xrightarrow{\iota} \mathrm{MU}(n) \xrightarrow{\rho} U(n) \longrightarrow 1\,.$$

For the unitary frame bundle $R(E, H)$ corresponding to a Hermitien vector bundle (E, H) a ρ-lifting $\tilde{\rho} : \widetilde{R}(E, H) \to R(E, H)$ is a metaunitary frame bundle and leads to the notion of a metaunitary structure.

Spin Structure

Let X be an oriented manifold of real dimension k with Riemannian metric g and let $\pi : R(X, g) \to X$ be the principal fibre bundle of oriented orthonormal frames with structure group $\mathrm{SO}(k)$ called orthonormal frame bundle. $R(X, g)$ can be constructed in a way similar to the construction of the frame bundle $R(X)$ or other frame bundles we have encountered. In particular, the fibre $\pi^{-1}(x) = R_x(X,g)$ is the set of ordered bases b of T_xX, which are oriented and orthonormal with respect to $g(x)$. $R_x(X,g)$ can be parametrized by $\mathrm{SO}(k)$ by the action $(b,g) \mapsto bg$ of $\mathrm{SO}(k)$: In particular, for a fixed oriented, orthonormal and ordered basis $b \in R_x(X,g)$, $x \in X$ the map $\mathrm{SO}(k) \to B_x$, $g \mapsto bg$ is bijective.

Recall, that the spin group $\mathrm{Spin}(k)$ is the 2-1 covering group of the special orthogonal group $\mathrm{SO}(k)$, or in other words, $\mathrm{Spin}(k)$ is the central extension of $\mathrm{SO}(k)$ by $\mathbb{Z}_2$, with respect to the exact sequence of Lie groups:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(k) \xrightarrow{\rho} \mathrm{SO}(k) \longrightarrow 1 .$$

ρ denotes the covering map $\rho : \mathrm{Spin}(k) \to \mathrm{SO}(k)$.

Definition 17.5. A Spin Frame Bundle for the orthonormal frame bundle $\pi : R(X,g) \to M$ over a k-dimensional oriented Riemannian manifold (X,g) is ρ-lifting

$$\tilde{\rho} : \widetilde{R}(X,g) \to R(X,g) .$$

A spin structure is an equivalence class of spin frame bundles.

In the following we describe the cohomological condition for a principal G-bundle $B \to X$ to admit a ρ-lifting. For a Lie group G let us denote by $H^1(X,G)$ the set of equivalence classes of principal G-bundles. In case of an abelian group G and a suitable cover $\mathfrak{U}$ we have $H^1(X;G) = \check{H}^1(\mathfrak{U},G)$ and this is is essentially $\check{H}^1(X,G)$.

Remark 17.6. The central extension induces a long exact sequence

$$\longrightarrow H^1(X,Z) \longrightarrow H^1(X,\widetilde{G}) \longrightarrow H^1(X,G) \xrightarrow{\delta} H^2(X,Z)$$

of homomorphisms.

We give an explicit description of the so-called connecting homomorphism $\delta : H^1(X,G) \longrightarrow H^2(X,Z)$: A class $h \in H^1(X,G)$ will be presented by transition functions h_{ij} of a G-bundle B. Since ρ is surjective, there are $\tilde{h}_{ij} \in \widetilde{G}$ with $\rho(\tilde{h}_{ij}) = h_{ij}$. In general, $\tilde{h}_{ij}$ does not satisfy the cocycle condition. But $c_{ijk} := \tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki} \in \widetilde{G}$ is mapped to 1 by ρ: $\rho(c_{ijk}) = h_{ij}h_{jk}h_{ki} = 1$, i.e. c_{ijk} is in the kernel of ρ. By the exactness of the sequence there are $z_{ijk} \in Z$ with $\iota(z_{ijk}) = c_{ijk}$. Now, $\delta(h) := [(z_{ijk})] \in H^2(X,Z)$ is well-defined and yields a homomorphisem $\delta : H^1(X,G) \longrightarrow H^2(X,Z)$.

Let us call $w_\rho(B) := \delta([B])$ the obstruction class of of the G-bundle B with respect to ρ.

Proposition 17.7. *A ρ-lifting of B exists if and only if $w_\rho(B)$ is trivial. Moreover, when a ρ-lifting exists, the equivalence classes of ρ-liftings of B are in a 1-to-1 correspndence to the cohomology classes of $H^1(X,Z)$.*

Proof. Let $\widetilde{B} \to B$ a ρ-lifting with transition functions $\tilde{g}_{ij}$ of $\widetilde{B}$. Then $g_{ij} := \rho(\tilde{g}_{ij})$ are transition functions of B and represent $[B] = [g_{ij}] \in H^1(M, G)$. Since $\tilde{g}_{ij}$ satisfy the cocycle condition the z_{ijk} with $\iota(z_{ijk}) = \tilde{g}_{ij}\tilde{g}_{jk}\tilde{g}_{ki} = 1$ define the trivial class $\delta([B]) = \delta([z_{ijk}]) \in H^2(M, Z)$.

Conversely, let g_{ij} be transition functions of a G-bundle B with trivial $w_\rho(B)$. There are $\hat{g}_{ij}$ with $\rho(\hat{g}_{ij}) = g_{ij}$ and z_{ijk} such that $\iota(z_{ijk}) = \hat{g}_{ij}\hat{g}_{jk}\hat{g}_{ki}$ as well as $[z_{ijk}] = \delta([B]) = 1$. Hence, there are $z_{ij} \in Z$ with $z_{ijk} = z_{ij}z_{jk}z_{ki}$. Now, the functions

$$\tilde{g}_{ij} := \frac{\hat{g}_{ij}}{z_{ij}}$$

satisfy the cocycle condition, since all z_{ij} are central elements, and therefore determine a ρ-lifting.

Moreover, when $\tilde{\rho} : \widetilde{B} \to B$ is a ρ-lifting with transition functions $\tilde{g}_{ij}$ and if we choose $[z_{ij}] \in H^1(M, Z)$, then $g'_{ij} := \tilde{g}_{ij}z_{ij}$ defines a cocycle and, hence, another ρ-lifting. This ρ-lifting is equivalent to the $\widetilde{B}$ if and only if $[z_{ij}]$ is trivial in $H^1(M, Z)$. □

We recognize our results for the existence of a metalinear structure of a $\mathrm{GL}(n, \mathbb{C})$-bundle B proved in the previous chapter.

Completely in the same way one can prove a corresponding result for a $\mathrm{U}(n)$-bundle $B \to M$ and its possible metaunitary structures $\tilde{\rho} : \widetilde{B} \to B$ with respect to the exact sequence $1 \to \mathbb{Z}_2 \to \mathrm{MU}(n) \to \mathrm{U}(n) \to 1$.

And for the spin case we obtain: A spin structure for the orthonormal frame bundle $R(X, g)$ exists if and only if $w_\rho(R(X, g) \in H^2(X, \mathbb{Z}_2))$ is trivial [88] When this is the case, the equivalence classes of spin structures are parametrized by $H^1(X, \mathbb{Z}_2)$.

Enlarged Extension

There are interesting manifolds for which a metaplectic structure does not exist. For instance, for the complex projective space $M = \mathbb{P}^n(\mathbb{C})$ with n even the obstruction class $w_\rho(Sp(M, \omega))$ is not trivial[89]. In order to include such symplectic manifolds in the program of Geometric Quantization one can modify the exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \stackrel{\iota}{\longrightarrow} \widetilde{G} \stackrel{\rho}{\longrightarrow} G \longrightarrow 1\,,$$

on which the ρ-lifting is based by a suitable Lie group H which contains $\mathbb{Z}_2$ in its center (cf. [FH79]):

$$\widetilde{G}^H := \widetilde{G} \times_{\mathbb{Z}_2} H\,.$$

We obtain the exact sequence

$$1 \longrightarrow H \stackrel{\iota^H}{\longrightarrow} \widetilde{G}^H \stackrel{\rho^H}{\longrightarrow} G \longrightarrow 1\,.$$

[88] This obstruction class is the second Stiefel-Whitney class of the real vector bundle TX.

[89] It is the second Stiefel-Whitney class of $M = \mathbb{P}^n(\mathbb{C})$, the nontrivial element of $H^2(\mathbb{P}^n(\mathbb{C}), \mathbb{Z}_2) = \mathbb{Z}_2$.

In particular, with $H = \mathrm{U}(1), G = \mathrm{Sp}(n)$, and the abbreviation $\mathrm{Mp}^c(n) := \mathrm{Mp}(n)^{\mathrm{U}(1)}$, $\rho^c := \rho^{\mathrm{U}(1)}$, ..., we obtain the exact sequence

$$1 \longrightarrow \mathrm{U}(1) \xrightarrow{\iota^c} \mathrm{Mp}^c(n) \xrightarrow{\rho^c} Sp(n) \longrightarrow 1\,,$$

where ι^c and ρ^c are the obvious homomorphisms.

The group $\mathrm{Mp}^c(n)$ is called generalized metaplectic group. It will be appear again in the context of metaplectic representation in Section 18. we obtain the following diagram where rows and columns are exact:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathbb{Z}_2 & \xrightarrow{\iota} & \mathrm{Mp}(n) & \xrightarrow{\rho} & \mathrm{Sp}(n) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \zeta & & \downarrow \mathrm{id} & & \\
1 & \longrightarrow & \mathrm{U}(1) & \xrightarrow{\iota^c} & \mathrm{Mp}^c(n) & \xrightarrow{\rho^c} & \mathrm{Sp}(n) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \eta & & & & \\
 & & \mathrm{U}(1)/\mathbb{Z}_2 \cong \mathrm{U}(1) & \xrightarrow{\mathrm{id}} & \mathrm{U}(1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Here, ζ is the inclusion and $\eta : \mathrm{Mp}^c(n) \to \mathrm{U}(1)$ is the character whose restriction to $\mathrm{U}(1) \subset \mathrm{Mp}^c(n)$ is $\lambda \to \lambda^2$.

As a result, in same cases it is reasonable to study ρ^c-liftings to obtain so-called generalized metalinear frames, i.e. frames in the corresponding $\mathrm{Mp}^c(n)$-bundle over the symplectic frame bundle $R(M,\omega)$. In particular, since an $\mathrm{Mp}^c(n)$-structure exists for every symplectic manifold, see below.

17.3 Square Roots, Metalinear und Metaplectice Structures

After these preparations on metastructures we show in this section how a given metaplectic structure on the symplectic manifold (M,ω) induces on every positive polarization $P \subset TM^{\mathbb{C}}$ on (M,ω) a metalinear structure and thus a square root of the canonical bundle K_P. This section is developed along the lines of a paper of Rawnsley [Raw78].

We begin with the reduction of the symplectic frame bundle $R(M,\omega)$ on the symplectic manifold to a $\mathrm{U}(n)$-bundle. We pick a compatible almost complex structure J which is positive[90]:

[90] Such an almost complex structure exists on any symplectic manifold (M,ω): One equips M with a Riemannian metric g and uses the condition 2. – $g(X,Y) = \omega(X,JY)$ – to determine J.

1. J is a section $M \to \mathrm{End}\,(TM, TM)$ satisfying $J \circ J = -1 = -\mathrm{id}_{TM}$.
2. $g^J_a(X, Y) := \omega_a(X, J_a Y)$ defines a symmetric, positive definite bilinear form g^J on $T_a M$ for each $a \in M$. Hence, (M, g^J) is a Riemannian manifold.
3. $\omega(X, Y) = \omega(JX, JY)$ for all $X, Y \in T_a M$, $a \in M$[91].

Proposition 17.8. *Let*

$$R(M, \omega, J) := \{(u; v) \in R(M, \omega) \mid v = (Ju_1, \dots, Ju_n)\}\,.$$

Then $R(M, \omega, J)$ is a reduction of $R(M, \omega)$ to a U(n)*-bundle and every* U(n)*-reduction arises this way. Moreover, the* Sp(n)*-bundle $R := R(M, \omega, J) \times_{U(n)} \mathrm{Sp}(n)$ is a symplectic frame bundle isomorphic to the symplectic frame bundle $R(M, \omega)$.*

Proof. The condition $(Ju_1, \dots, Ju_n) := Ju = v$ for a symplectic frame $(u; v)$ remains true for those $g \in \mathrm{Sp}(n)$, which satisfy $Ju' = v'$ when $(u'; v') := (u; v)g$, i.e. $Jg = gJ$. These group elements g form exactly the subgroup U_n of Sp(n) isomorphic to U(n) according to Lemma 17.1.

The isomorphism $R \cong R(M, \omega)$ is given by

$$[(u; v), g] \mapsto (u; v)g\,,\ ((u; v), g) \in R(M, \omega, J) \times \mathrm{Sp}(n)\,.$$

□

With this almost complex structure J the real vector bundle TM becomes a complex vector bundle TM^J of dimension n on which J and ω induce the hermitian metric H^J:

$$H^J_a(X, Y) := g^J_a(X, Y) - i\omega_a(X, Y)\,,\ X, Y \in T_a M\,,\ a \in M\,.$$

The U(n)-bundle $R(TM^J, H^J)$ of unitary frames of (TM^J, H^J) is isomorphic to the reduced bundle $R(M, \omega, J)$ by the map

$$R(M, \omega, J) \ni (u; v) \mapsto u \in R(TM^J, H^J)\,.$$

In fact, for $(u; v) \in R(M, \omega, J)$ with $u = (u_1, \dots, u_n)$ the condition

$$H^J(u_i, u_k) = (u_i, v_k) = \omega(u_i, Ju_k) + i\omega(u_i, u_k) = \omega(u_i, v_k) = \delta_{ik}$$

is satisfied, so u is an orthonormal frame, and every orthonormal frame u can be expanded to become a symplectic frame $(u; v)$.

Now, let P be a positive polarization. We are interested in a natural bijection between the metalinear structures on P and the metaplectic structures on (M, ω).

[91] 3. follows from 2.

We have learned in the previous chapter that the metalinear structures on P are parametrized by $\check{H}^1(\mathfrak{U}, \mathbb{Z}_2)$.

The same is true for the metaunitary structures on (TM^J, H^J), cf. Remark 17.7. Regarding the isomorphisms $R(M,\omega) \cong R(TM^J, H^J)$ we conclude that the equivalence classes of metaplectic frame bundles of (M,ω) is parametrized by $\check{H}^1(\mathfrak{U}, \mathbb{Z}_2)$ as well. As a result, there is a bijection between the set of metalinear structures on P and the set of metaplectic structures on M given by any bijection of $\check{H}^1(\mathfrak{U}, \mathbb{Z}_2)$, in particular by the identity id.

However, we are interested to describe and construct a natural bijection using geometric insight. So we have to go into some details.

As a first step of our way from a metaplectic frame bundle $\widetilde{R}(M,\omega)$ to the corresponding metalinear frame bundle $\widetilde{R}(P)$ on a given positive polarization P we just have started with the symplectic frame bundle $R(M,\omega)$ and arrived – with the help of Proposition 17.8 – at the $\mathrm{U}(n)$-bundle $R(TM^J, H^J) \cong R(M,\omega,J)$ of hermitian frames of the tangent bundle of M as a reduction of $R(M,\omega)$. The strategy is to identify this bundle with the $\mathrm{U}(n)$-bundle $R(P, H^J)$ of unitary frames of P where P is the positive polarization in question, which will induce a metalinear structure on P.

Conversely, a metalinear structure $\widetilde{R}(P)$ on P gives a metaunitary structure $\widetilde{R}(P, H^J)$ by reduction. Using the close relations $R(M,\omega) \to R(M,\omega,J) \cong R(P^J, J)$ this metastructure is transferred to yield a metaplectic structure on M.

We introduce the notion of Lagrangian subbundle which generalizes the notion of polarization:

Definition 17.9. A Lagrangian subbundle $P \subset TM^{\mathbb{C}}$ is an n-dimensional complex vector bundle in the complexification $TM^{\mathbb{C}}$ of the tangent bundle TM such that each $P_a, a \in M$ is a Lagrangian subspace of $(T_aM^{\mathbb{C}}, \omega_a)$, i.e. $\omega|_{P\times P} = 0$.

P is called positive, if $-i\omega_a(\overline{X}, X) \geq 0$ for all $a \in M$, $X \in P_a$.

A polarization is an involutive Lagrangian subbundle as introduced in Chapter 9. The results we present in the following hold for the more general case of Lagrangian subbundles.

Proposition 17.10. *Let P be a positive Lagrangian subbundle of $TM^{\mathbb{C}}$ and let J be a positive almost complex structure on (M,ω). $TM^{(1,0)}$ denotes the i-eigenspace of J : $TM^{\mathbb{C}} \to TM^{\mathbb{C}}$ and TM^J the tangential bundle equipped with the complex multiplication by J. Then the complex vector bundles P, TM^J and $TM^{(1,0)}$ are isomorphic in a natural way.*

As a consequence, when the hermitian form H^J is transformed to P by the isomorphim $TM^J \to P$ we have a natural isomorphism $R(TM^J, H^J) \cong R(P, H^J)$.

Moreover the Chern classes of the three vector bundles agree with $c_i(M,\omega)$, the Chern class of M.

Proof. The i-eigenspace of J is $\operatorname{Im}\pi$ for the projection

$$\pi : TM^{\mathbb{C}} \to TM^{\mathbb{C}}\,,\ \pi(X) := \frac{1}{2}(X - iJX)\,.$$

Moreover, $TM^{(1,0)} = \pi(TM)$ and the restriction of π to the complex tangent bundle TM^J is an isomorphism, i.e. $\mathbb{R}$-linear and

$$\pi(JX) = \frac{1}{2}(JX - iJJX) = \frac{1}{2}(JX + iX) = i(\pi(X))\,,\ X \in TM\,.$$

$TM^{(1,0)}$ is a positive Lagrangian and $-i\omega(\overline{X}, X) > 0$ for $X \in\ TM^{(1,0)},$, $X \neq 0$. Hence, the orthogonal complement $\overline{TM^{(1,0)}} = \operatorname{Im}(1-\pi)$ satisfies $-i\omega(\overline{X}, X) < 0$ for $X \in \overline{TM^{(1,0)}}$, $X \neq 0$. We conclude $P \cap \overline{TM^{(1,0)}} = \emptyset$. As a result, the restriction of π to P is an isomorphism, since $\pi(P) \subset TM^{(1,0)}$ and $\pi|_P$ is injective: For $X, Y \in P$ the equality $\pi(X) = \pi(Y)$ implies $X - Y \in \operatorname{Ker}\pi = \overline{TM^{(1,0)}}$, hence $X - Y \in P \cap \overline{TM^{(1,0)}} = \emptyset$, i.e. $X = Y$. □

Construction

After these preparations we now concentrate on a positive Lagrangian P on (M, ω) and assume that it is equipped with a metalinear structure represented by a metalinear frame bundle $\tilde{\rho} : \widetilde{R}(P) \to R(P)$. We use this structure to construct an associated metaplectic frame bundle $\widetilde{R}(M, \omega) \to R(M, \omega)$ and hence a metaplectic structure on (M, ω).

As before, pick a positive compatible almost complex structure J on M and transform the Hermitian form H^J on TM^J to P by the isomorphism $TM^J \cong P$ (cf. Proposition 17.10). The Lagrangian P thus becames a Hermitian vector bundle (P, H^J). The corresponding unitary frame bundle $R(P, H^J)$ is a subbundle of the $\mathrm{GL}(n, \mathbb{C})$-frame bundle $R(P)$. The subbundle

$$\widetilde{B} := (\tilde{\rho})^{-1}(R(P, H^J)) \subset \widetilde{R}(P)$$

is a $MU(n)$-bundle and turns out to be a metaunitary frame bundle

$$\tilde{\rho} : \widetilde{R}(P, H^J) := \widetilde{B} \to R(P, H^J)\,.$$

Again using the isomorphism $TM^J \cong P$ this construction yields a metaunitary frame bundle

$$\widetilde{R}(TM^J, H^J) \to R(TM^J, H^J)\,.$$

Considering the isomorphism $R(M, \omega) \cong R(M, \omega, J) \times_{\mathrm{U}(n)} \mathrm{Sp}(n)$ of Proposition 17.10 we define the $\mathrm{Mp}(n)$-bundle

$$\widetilde{R}^J := \widetilde{R}(TM^J, H^J) \times_{\mathrm{MU}(n)} \mathrm{Sp}(n) \longrightarrow R(M, \omega)$$

with the 2-to-1 mapping

$$\sigma : \widetilde{R}^J \to R(TM^J, H^J)\times_{\mathrm{U}(n)}\mathrm{Sp}(n)\,,\ [(b,S)] \mapsto [(\tilde\rho(b),S)]\,, (b,S) \in \widetilde{R}(TM^J,H^J)\times \mathrm{Sp}(n)\,.$$

Then it is not difficult to check that

$$\sigma : \widetilde{R}(TM^J, H^J) \times_{\mathrm{MU}(n)} \mathrm{Sp}(n) \to R(TM^J, H^J) \times_{\mathrm{U}(n)} \mathrm{Sp}(n)$$

is a metaplectic frame bundle determining a metaplectic structure on (M,ω).

Proposition 17.11. *For each positive Lagrangian P the above construction exhibits a natural bijection between the metalinear structures on P and the metaplectic structures on M.*

Proof. We have constructed a map $[\widetilde{R}(P)] \mapsto [\widetilde{R}(TM^J,H^J) \times_{\mathrm{MU}(n)} \mathrm{Sp}(n)]$ from the metalinear structures on P to the metaplectic structures on M. Since both sets are paramterized bx $\check{H}^1(M;\mathbb{Z}_2)$ it is enough to check that the assignment is injective.

The inverse map can be determined by reversing each step: Given a metaplectic frame bundle $\tilde\rho : \widetilde{R}(M,\omega) \to R(M,\omega)$ the reduction to $\mathrm{U}(n)$, $\widetilde{B}^J := \tilde\rho^{-1}(R(M,\omega,J))\,,\ \widetilde{B}^J \to R(M,\omega,J) \cong R(P,H^J)$ yields a metaunitary frame bundle $\widetilde{B}^J \to R(P,H^J)$ of (P,H^J). $\widetilde{R}(P) := \widetilde{B}^J \times_{\mathrm{MU}(n)} \mathrm{ML}(n,\mathbb{C}) \to R(P)$ is the metalinear frame bundle of (P) yielding the inverse map $[\widetilde{R}(M,\omega)] \mapsto [\widetilde{B}^J \times_{\mathrm{MU}(n)} \mathrm{ML}(n,\mathbb{C})]$.

□

Remark 17.12. Blattner [Bla77] (see also Sniaticky [Sni80] and Tuynman [Tuy16]) presents an alternative proof of Proposition 17.11 using as an intermediate structure the bundle of positive Lagrangian frames $F_+(M,\omega)$ on (M,ω). A positive Lagrangian frame at $a \in M$ is an n-tuple $(u_1,\ldots,u_n)$ of vectors $u_j \in T_aM^{\mathbb{C}}$ such that $\omega_a(u_i,u_k) = 0$ and $-i\omega_a(\bar u_i, u_k)$ is positive semi definite. Note, when P is a positive Lagrangian then each element $b \in R(P)$ of the frame bundle $R(P)$ is a positive Lagrangian frame. $F_+(M,\omega)$ consists of the collection of all positive Lagrangian frames at all points of M,

$$F_+(M,\omega) = \{b \in P \mid P \subset TM^{\mathbb{C}} \text{ positive Lagrangian subbundle}\}\,.$$

In particular, every positive polarization P is a subbundle of $F_+(M,\omega)$. On $F_+(M,\omega)$ there are natural actions of $\mathrm{Sp}(n)$ (from the left) and of $\mathrm{GL}(n,\mathbb{C})$ from the right. Given a metaplectic frame bundle $\widetilde{R}(M,\omega) \to R(M,\omega)$ by direct geometric considerations about $F_+(M,\omega)$ a "metaplectic" bundle of positive Lagrangian frames

$$\tilde\rho : \widetilde{F}_+(M,\omega) \to F_+(M,\omega)$$

can be constructed which induces on each positive Lagrangian $R(P) \subset F_+(M,\omega)$ a metalinear frame bundle $\widetilde{R}(P) := (\tilde\rho)^{-1}(R(P))$.

Remark 17.13. A different way to introduce and apply the concept of metaplectic structure has been suggested by Woodhouse [Woo91], p.232. In order to explain this approach we need the space $L_+(M,\omega)$ of positive Lagrangian subspaces over M:

$$L_+(M,\omega) := \bigcup_{a\in M} \{P \subset T_aM^{\mathbb{C}} \mid P \text{ positive Lagrangian subspace of } (T_M,\omega_a)\}\,.$$

This is a topological fibre bundle with fibres

$$\{P \subset T_aM^{\mathbb{C}} \mid P \text{ positive Lagrangian subspace of } (T_M,\omega_a)\}.$$

A metaplectic structure on (M,ω) is defined to be a square root of the canonical bundle $K = K_{L+M}$ of $L_+(M,\omega)$. A positive Lagrangian P on (M,ω) is essentially the same as a section $s : M \to Ł_+(M,\omega)\,,\ a \mapsto P_a$: $P \cong s(M)$. P obtains a square root of the canonical bundle K_P by pullback: Since $K_P = s^*(K)$, the given metaplectic structure induces $S_P = s^*(S)$ directly. In this way, a given metaplectic structure induces directly a square root on every positive polarization of (M,ω).

17.4 Half-Form Pairing

(incomplete)

Let (M,ω) be a quantizable symplectic manifold with a prequantum line bundle (L,∇,H). Let P be a polarization of (M,ω) and assume that $K_{-1}(P)$ has a square root $S = K_{-1/2}(P)$. Then we can build the corresponding half-form quantization with its representation space $\mathbb{H}^S_P = \mathbb{H}^S(M,L,P)$ as in Section 15.4.

For another polarization P' we know by the considerations of this chapter that there exists a square root S' of $K_{-1}(P')$. We want to compare the representation spaces $\mathbb{H}^{S'}_{P'}$ and $\mathbb{H}^{S'}_{P'}$ and hope that this can be achieved by a pairing

$$\mathbb{H}^S_P \times \mathbb{H}^{S'}_{P'} \longrightarrow \mathbb{C}\,.$$

As a model we can use the attempts to find a pairing of half-density quantizations as presented in Section 14.5.

For the rest of this section, P and P' are compatible in the sense of Definition 14.16, that is the intersection $D := P \cap \overline{P'} \cap TM$ is an integrable distribution, the quotient M/D exists as a manifold such that the projection $\pi : M \to M/D$ is a submersion, and the distribution $E := (P + \overline{P'}) \cap TM$ is integrable.

We want to produce a sesquilinear map

$$\mu : \Gamma(M.L^S) \times \Gamma(M,L^{S'}) \longrightarrow \Gamma(M/D,\delta_1(M/D))\,,$$

so that

$$(\psi,\psi') \mapsto \int_{M/D} \mu(\psi,\psi')\,,\ \text{ for suitable }\ \psi \in \mathbb{H}^S, \psi' \in \mathbb{H}^{S'}\,,$$

is the germ of a pairing $\mathbb{H}_P^S \times \mathbb{H}_{P'}^{S'} \to \mathbb{C}$ in which we are interested in.

Of central importance for the existence of such a μ is the following lemma

Lemma 17.14. *Let $\varphi : S' \to S$ be an isomorphism of line bundles with $\nabla_X \circ \varphi = \varphi \circ \nabla_X$ for all $X \in \Gamma(M, D^{\mathbb{C}})$. Then there exists a natural sesquilinear map*

$$\tau \Gamma(M, S) \times \Gamma(M, S') \to \Gamma(M, \delta_1(Z^D)\,,$$

such that for all $(\alpha, \alpha') \in \Gamma(M, S) \times \Gamma(M, S')$ satifying $\nabla_X \alpha = 0 = \nabla_X \alpha'$ for all $X \in \Gamma(M, D)^{\mathbb{C}}$ we have $\nabla_X \tau(\alpha, \alpha') = 0$ for all $X \in \Gamma(M, D^{\mathbb{C}})$.

Proof. We have the map $\Gamma(M, S) \times \Gamma(M, S) \to \delta_1(P)\,,\ (\alpha, \alpha') \mapsto \bar{\alpha}\alpha'$ as in Lemma 15.9. Then $\tau'(\alpha, \alpha') := \bar{\alpha}\varphi(\alpha')$ satisfies: When $\nabla_X \alpha = 0 = \nabla_X \alpha'$ for all $X \in \Gamma(M, D^{\mathbb{C}})$ we have $\nabla_X \tau'(\alpha, \alpha') = 0$ for all $X \in \Gamma(M, D^{\mathbb{C}})$. Now we use the result of Proposition 14.17: There exists a natural line bundle isomorphism

$$\tau_P : \delta_{-1}(P) \to \delta_1(Z^D) \quad \text{with} \quad \nabla_X \circ \tau_P = \tau_P \circ \nabla_X$$

for all $X \in \Gamma(M, D^{\mathbb{C}})$. The composition $\tau =: \tau_P \circ \tau'$ provides the result of the lemma. □

Remark 17.15. According to the preparations described in this chapter the typical case where the lemma applies is the following: P and P' are positive complex polarizations and, in particular, isomorphic. If $K_{-1}(P)$ admits a square root S, this square root induces a metaplectic structure on (M, ω) and, hence, a directly related square root S' of $K_{-1}(P')$. Without loss of generality we can assume that S' is the transport of S given by the above mentioned isomorphism. This implies that the induced isomorphism $\varphi : S' \to S$ satisfies the compatibility condition of Lemma 17.14.

Summary:

18 Metaplectic Representation

(incomplete)

In the context of studying and applying the metaplectic group $\mathrm{Mp}(n)$ it is natural to introduce and study the metaplectic representation. This is a distinguished infinite dimensional unitary representation of $\mathrm{Mp}(n)$ and $\mathrm{Mp}^c(n)$ which is interesting in its own right and which is an active area with links to quantum physics, symplectic geometry, differential geometry, index theory, geometric analysis, representation theory and number theory.

In particular there is a direct relation to the infinite dimensional irreducible representation of the CCR relations (cf. Section F.3), i.e. to the induced represention of the Heisenberg representation. In this way the quantization of $T^*\mathbb{C}^n$ and the induced Bargmann representation comes into play. Moreover, the metaplectic representation can be used to obtain concrete details of the group structure of $\mathrm{Mp}(n)$and $\mathrm{Mp}^c(n)$ and it can be applied to see another way how a metaplectic structure determines half-form bundles on positive polarizations.

Last not least, the use of Mp^c opens the way to further geometric quantizations schemes beyond the approaches we have described so far. in particular, since every symplectic manifold admits $\mathrm{Mp}^c(n)$-structures.

18.1 Representation of the Heisenberg Group

We start with the essentially unique (continuous) unitary irreducible representation of the Heisenberg group, which is described in Section F.3 in the context of canonical commutation relations(CCR). Let (V,ω) be a symplectic vector space, i.e. V is a real $2n$-dimensional vector space and ω a constant non-degenerate alternating bilinear form on V. The corresponding Heisenberg group is

$$\mathrm{HS} = \mathrm{HS}(V,\omega) := V \times \mathbb{R}$$

with multiplication

$$(v,s)(w,t) := (v+w, s+t+\frac{1}{2}\omega(v,w))\,.$$

HS is a Lie group and a central extension of the abelien Lie group V by $\mathbb{R}$ given by the exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{HS}(V,\omega) \longrightarrow V \longrightarrow 0\,.$$

Its Lie algebra is the Heisenberg Lie algebra $\mathfrak{hs}$:

$$\mathfrak{hs} = \mathfrak{hs}(V,\omega) := V \times \mathbb{R}$$

with Lie bracket
$$[(v,s),(w,t)] := (0,\omega(v,w))\,.$$

For a fixed dimension $2n$, the Lie groups $\mathrm{HS}(V,\omega)$ are isomorphic to each other as well as the Lie algebras $\mathfrak{hs}(V,\omega)$.

With respect to a symplectic frame of (V,ω), i.e. a basis $(u_j;v_k)$ of V with $\omega(u_j,v_k)=\delta_{jk}, \omega(u_j,u_k)=\omega(v_j,v_k)=0$ we obtain the CCR in the form:
$$[(u_j,0),(v_k,0)] = \delta_{jk}(0,1).$$
With $U_j := (u_j,0), V_k := c(v_j,0), Z := (0,1)$ for a constant $c\in\mathbb{C}$ a variant of the CRR is of the familiar form
$$[U_j,V_k] = c\delta_{jk}Z\,.$$

For every unitary irreducible representation $W : \mathrm{HS}\to \mathrm{U}(\mathbb{H})$ on a separable Hilbert space $\mathbb{H}$ the center $\{0\}\times\mathbb{R}$ of HS will act by multiplicities of the identity $W(0,s) = e^{i\lambda s}\mathrm{id}_{\mathbb{H}}$ inducing the central character $(v,s)\mapsto e^{i\lambda s}$ with parameter λ. We observe:

In case of $\lambda = 0$, the representation comes from an irreducible representation of the abelian group V and thus is one-dimensional of the form $W(v,s) = e^{i\omega(v,w_0)}$.

In case of $\lambda\neq 0$, W is unitarily equivalent to the Schrödinger representation with $(0,s)$ acting as $e^{i\lambda s}\mathrm{id}_{\mathbb{H}}$ (cf. F.48).

One can change the value of the parameter λ by scaling: $W'(v,s) := W_\lambda(cv,c^2s), c > 0$, has parameter $\lambda' = c^2\lambda$. Replacing $W = W_\lambda$ by W^* leads to the parameter $-\lambda$. Altogether, we obtain the result that up to scaling, equivalence and $\dim V$ there is only one infinite dimensional unitary irreducible representation of HS.

In the course of these lecture notes we have encountered CCR already several times. In particular, when describing geometric quantization of the simple phase space $V = T^*\mathbb{R}^n$ with the standard symplectic form ω and different polarizations (see Examplae 10.12): We have on $\mathbb{H} = L^2(\mathbb{R}^n)$ in case of the vertical polarization (Schrödinger representation) the CCR
$$\left[Q^j,P_k\right] = \frac{i}{2\pi}\delta^j_k\,,$$
where
$$Q^j := q^j \ \text{ and } \ P_k := -\frac{i}{2\pi}\frac{\partial}{\partial q^k}\,,\ 1\le j,k\le n\,.$$
In case of the horizontal polarization (Heisenberg representation) the same CCR appear, however with the variables q and p interchanged. In case of the holomorphic polarization (Bargmann representation) the Hilbertspace is the Fock space of holomprphic functions a with CCR
$$[\overline{Z}^j,Z_k] = \frac{1}{\pi}\delta^j_k\,.$$
for
$$Z_k = z_k\,,\ \overline{Z}^j = \frac{1}{\pi}\frac{\partial}{\partial z_j},\ 1\le j,k\le n\,.$$

Integrating the Schrödinger representation which we have just recalled we obtain a unitary representation on $\mathbb{H} = L^2(\mathbb{R}^n)$

$$U_{x,y,s} : \mathbb{H} \to \mathbb{H}\,,\ (v,s) = (x,y,s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} = \mathrm{HS}(\mathbb{R}^{2n}, \omega)\,,$$

which is also called also the Schrödinger representation:

$$U_{x,y,s}\phi(q) := e^{i\lambda s} e^{iqy + i\frac{\lambda}{2}xy} \phi(q + \lambda x)\,, \phi = \phi(q) \in \mathbb{H} = L^2(\mathbb{R}^n)\,.$$

Here, λ is a real parameter and xy, qy is the scalar product. It is easy to check that $U_{x,y,s}$ is a unitary operator[92]. Let us confirm that $(x,y,s) \mapsto U_{x,y,s}$ is homomorphism $\mathrm{HS} \to \mathrm{U}(\mathbb{H})$ by an elementary calculation:

$$\begin{aligned} U_{x,y,s}U_{x',y',s'}\phi(q) &= U_{x,y,s} e^{i\lambda s'} e^{iqy' + i\frac{\lambda}{2}x'y'} \phi(q + \lambda x) \\ &= e^{i\lambda s} e^{iqy + i\frac{\lambda}{2}xy} e^{i\lambda s'} e^{i(q+\lambda x)y' + i\frac{\lambda}{2}x'y'} \phi(q + \lambda x + \lambda x') \\ &= e^{i\lambda(s+s')} e^{iq(y+y') + i\frac{\lambda}{2}xy} e^{i\lambda xy' + i\frac{\lambda}{2}x'y'} \phi(q + \lambda x + \lambda x') \\ &= e^{i\lambda(s+s') + i\frac{\lambda}{2}(xy' - x'y)} e^{iq(y+y') + i\frac{\lambda}{2}(x+x')(y+y')} \phi(q + \lambda(x + x')) \\ &= U_{x+x', y+y', s+s'+\frac{1}{2}(xy' - yx')} \phi(q) \end{aligned} \tag{70}$$

The unitary representation U is irreducible which can be proven as in Proposition F.46.

The infinitesimal version of U, the corresponding Lie algebra representation $\dot{U}$ of $\mathfrak{hs}$, is

$$\dot{U}_{x,y,s}\phi = i\lambda s\phi + iqy\phi + \lambda < \frac{\partial \phi}{\partial q}, x > \ .$$

It is useful to realise the CCR and the Heisenberg group in the Bargmann representation on Fock space. For this we consider on the symplectic space (V, ω) compatible almost complex structures $J : V \to V$ which are positive. Recall that an almost complex structure on V is an $\mathbb{R}$-linear isomorphism

$$J : V \to V \quad \text{with } J^2 = -1 = -\mathrm{id}_V\,.$$

J is called compatible with ω when

$$\omega(v, w) = \omega(Jv, Jw) \ \text{ for } \ (v, w) \in V\,.$$

J is positive if

$$\omega(v, Jw) \ \text{ is positive definite}\,.$$

[92]first defined on all continuous ϕ in $\mathbb{H} = L^2(\mathbb{R}^n)$ and then extended to all of $\mathbb{H}$

In case of a compatible and positive almost complex structure J the real vector space V becomes a complex vector space V_J by the multiplication

$$iv := Jv\,,\; v \in V\,.$$

Moreover, V_J is a Hilbert space with respect to the inner product

$$< v, w >=< v, w >_J := \omega(v, Jv) + i\omega(v, w)\,,\; v, w \in V\,.$$

The corresponding Fock space is

$$\mathbb{F} = \mathbb{F}_J = \mathbb{H}(V, \omega, J) := \{f \in (V_J) \mid \int_V < f(z), f(z) >_J e^{-|z|^2} dz < \infty\}\,.$$

Finally, the Bargmann representation $W = W_J : \mathrm{HS}(V, \omega) \to \mathrm{U}(\mathbb{H}_J)$ with respect to these data is given by

$$(W_J)_{v,s} f(z) = W_{v,s} f(z) = e^{i\lambda s} e^{i\frac{1}{2}<z,v>_J + i\frac{\lambda}{4}<v,v>_J} f(z + \lambda v)\,,\; f \in \mathbb{H}_J\,.$$

W is a homomorphism (we mostly drop the index J in the following):

$$\begin{aligned}
W_{v,s} W_{v',s'} f(q) &= W_{v,s} e^{i\lambda s'} e^{i\frac{1}{2}<z,v'>+i\frac{\lambda}{4}<v',v'>} f(z + \lambda v') \\
&= e^{i\lambda s} e^{i\frac{1}{2}<z,v>+i\frac{\lambda}{4}<v,v>} e^{i\lambda s'} e^{i\frac{1}{2}<z+\lambda v,v'>+i\frac{\lambda}{4}<v',v'>} f(z + \lambda v' + \lambda v) \\
&= e^{i\lambda(s+s')} e^{i<z,v+v'>+i\frac{\lambda}{4}<v,v>+i\frac{1}{2}<\lambda v,v'>+i\frac{\lambda}{4}<v',v'>} f(z + \lambda(v + v')) \\
&= e^{i\lambda(s+s')} e^{i<z,v+v'>+i\frac{\lambda}{4}<v+v',v+v'>+i\frac{\lambda}{4}<v,v'>-i\frac{\lambda}{4}<v',v>} f(z + \lambda(v + v')) \\
&= e^{i\lambda(s+s'+\frac{1}{2}\omega(v,v'))} e^{i<z,v+v'>+i\frac{\lambda}{4}<v+v',v+v'>+} f(z + \lambda(v + v')) \\
&= W_{v+v',s+s'+\frac{1}{2}\omega(v,v')} f(z)\,.
\end{aligned}$$

We have used $i\frac{\lambda}{4} < v, v' > -i\frac{\lambda}{4} < v', v >= i\frac{\lambda}{2}\omega(v, v')$.

W_J is a unitary irreducible representation of HS.

The infinitesimal version is

$$\dot{W}_{v,s} f(z) = i\lambda s f(z) + i\frac{1}{2} < z, v >_J f(z) + \lambda v \frac{\partial f(z)}{\partial z}\,.$$

18.2 Representation of Mp and Mp^c

For a symplectic vector space (V, ω) the symplectic group $\mathrm{Sp}(V, \omega)$ is the Lie group of all invertable real linea maps $g : V \to V$ with $\omega(v, w) = \omega(gv, gw)$ for all $v, w \in V$. $Sp(V, \omega)$ acts on the Heisenberg group $\mathrm{HS}(V, \omega)$ as a group of automorphsims:

$$g \cdot (v, s) := (g(v), s)\,, \quad \text{for } g \in \mathrm{Sp}(V, \omega)\,,\; (v, s) \in \mathrm{HS}(V, \omega)\,.$$

By composing the unitary representation W_J of $\mathrm{HS}(V,\omega)$ on the Fock space $\mathbb{H}(V,\omega,J)$ with an automorphism $g \in \mathrm{Sp}(V,\omega)$ we obtain another representation W_J^g of $\mathrm{HS}(V,\omega)$:

$$W_J^g(v,s) := W_J(g \cdot (v,s)) = W_J(gv,s)\,.$$

This unitary representation is still irreducible and it has the same parameter $W_J^g(0,s) = W_J(0,s) = e^{i\lambda s}$. Hence, according to the Theorem of Stone–von Neumann F.47, there exists a unitary $U \in \mathrm{U}(\mathbb{F}_J)$, where $\mathbb{F}_J = \mathbb{H}(V,\omega,J))$ such that

$$W_J^g = UW_JU^{-1}\,.$$

By the Lemma of Schur the operator U is determined up to a scaler α of norm 1. We consider the group af all occuring U and describe it in the following way.

Definition 18.1.

$$\mathrm{Mp}^c(V,\omega,J) := \{(U,g) \in \mathrm{U}(\mathbb{F}_J) \mid W_J^g = UW_JU^{-1}\}\,.$$

This group turns out to be naturally isomorphic to the previously defined covering group $\mathrm{Mp}^c(n)$, see page 253, but this is not evident from the definitions and will be shown later.

The projection map $\sigma : \mathrm{Mp}^c(V,\omega,J) \to \mathrm{Sp}(V,\omega),, (U,g) \mapsto g$, is a surjective homomorphism. By the Lemma of Schur $\mathrm{Ker}\,\sigma$ is $U(1)$ and we obtain the following exact sequence.

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Mp}^c(H,V,J) \longrightarrow \mathrm{Sp}(V,\omega) \longrightarrow 1$$

In order to see that the unitary representation $\mathrm{Mp}^c(V,\omega,J) \hookrightarrow \mathrm{U}(\mathbb{F}_J)$ induces a representation of the metaplectic group $\mathrm{Mp}(V,\omega,J) \cong \mathrm{Mp}(n)$ one has to understand that $\mathrm{Mp}(V,\omega)$ is contained in $\mathrm{Mp}^c(V,\omega,J)$. This we see with the help of parametrizing $\mathrm{Sp}(V,\omega)$ and $\mathrm{Mp}^c(V,\omega,J)$

18.3 Parametrizing Sp and Mp^c

Summary:

Appendix: Mathematical Supplements

A Manifolds

There are many different classes of manifolds which are investigated in the various fields of mathematics and physics. In these lecture notes we are interested only in differentiable[93] or complex manifolds and in the differentiable bundles over these manifolds.

In this chapter we summarize the basic notions and results for differentiable manifolds. Complex manifolds will be treated in a later section (in Chapter B) as well as the principal bundles and the vector bundles over manifolds (in Chapter D).

A.1 Basic Definitions

Definition A.1 (Manifold)**.** Let M be a Hausdorff space and $n \in \mathbb{N}, n \geq 0$.

- A *manifold*[94] *of dimension* n is M together with a differentiable structure $\mathcal{D}$.
- A *differentiable structure* on M is an equivalence class $\mathcal{D}$ of differentiable atlases $\mathfrak{A}$ on M.
- A *differentiable atlas*[95] ("Atlas") on M is a collection $\mathfrak{A} = (q_\iota : U_\iota \to V_\iota)_{\iota \in J}$ of (n-dimensional) charts of M which are smoothly compatible to each other and which cover M:
$$M = \bigcup_{\iota \in J} U_\iota = M.$$
- An (n-dimensional) *chart* ("Karte") on M is topological map $q : U \to V$ (i.e. q is continuous with a continuous inverse $q^{-1} : V \to U$), where $U \subset M$ is an open subset of M and $V \subset \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$.
- Two such charts $q : U \to V, q' : U' \to V'$ on M are called *smoothly compatible*[96] if the induced map
$$q' \circ q^{-1}|_{q(U \cap U')} : q(U \cap U') \to q'(U \cap U')$$
is smooth (i.e. infinitely often differentiable[97]), or if $U \cap U' = \emptyset$.

[93] i.e. infinitely differentiable, also called smooth

[94] more precisely a differential manifold; but in these notes we will only deal with differentiable manifolds.

[95] In the following mostly called simply atlas, since only differentiable atlases will be considered.

[96] in the following we say simply compatible

[97] We only consider smooth functions and $\mathcal{C}^\infty$ structures in these notes. In general, also $\mathcal{C}^k$-differentiable atlases and $\mathcal{C}^k$-differentiable structures are studied.

- Two atlases $\mathfrak{A}, \mathfrak{B}$ are *equivalent*, if their union is an atlas, i.e. if each chart of $\mathfrak{A}$ is compatible with each chart of $\mathfrak{B}$.

Observation A.2 (Maximal Atlas)**.** Let M be a manifold with its differentiable structure $\mathcal{D}$.

- Any atlas $\mathfrak{A}$ in the equivalence class $\mathcal{D}$ determines a *maximal atlas* $\mathfrak{M}$ given as the collection of all charts on M which are compatible to the charts of $\mathfrak{A}$. Then $\mathfrak{M}$ is maximal in $\mathcal{D}$ with respect the inclusion of atlases.
- The maximal atlas $\mathfrak{M}$ of $\mathcal{D}$ is the union of all the atlases in $\mathcal{D}$.
- As a result one could say that a manifold is a Hausdorff space M together with a maximal atlas $\mathfrak{M}$.
- If we refer to a chart on M then it is always a chart of some atlas determining the differentiable structure of M, in particular, it will be chart of $\mathfrak{M}$.

In general, in these notes a manifold will be metrizable, and therefore paracompact. In most cases the manifold is also assumed to be connected.

Definition A.3 (Smooth Mappings and Functions)**.** A *smooth* (or *differentiable*) *mapping* on an open subset $D \subset M$ of a manifold into another manifold N is a mapping $f : D \to N$ such that for every $a \in D$ there exist charts $\varphi : U \to V \subset \mathbb{R}^n$ on M and $\varphi' : U' \to V' \subset \mathbb{R}^m$ on N with $U \subset D$ and $f(U) \subset U'$ such that

$$\varphi' \circ f \circ \varphi^{-1} : V \to \mathbb{R}^m$$

is smooth. Notation for manifolds M, N:

$$\mathcal{E}(M,N) := \{f : M \to N \mid f, \text{ smooth}\}.$$

$$\mathcal{E}(M) = \mathcal{E}(M,\mathbb{K}),$$

where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Smooth mappings with values in $\mathbb{K}$ are often called smooth functions or simply functions. A diffeomorphism is a smooth map $F : M \to N$ with a differentiable inverse.

Submanifold:

Definition A.4. A *submanifold* ("Untermannigfaltigkeit") N of a manifold M is given by a subset $N \subset M$ such that restrictions of suitable charts on M to N provide an atlas on N: For each $a \in N$ there exists a chart $q : U \to V$ on M such that $a \in U$ and $q(U \cap N) = \{y \in \mathbb{R}^n \mid y_{d+1} = \ldots = y_n = 0\} \cong V \cap \mathbb{R}^d$, where $d \in \mathbb{N}, d \leq n$, providing the chart

$$q|_{U \cap N} : U \cap N \to V \subset \mathbb{R}^d$$

of N.

These charts are automatically compatible to each other and define the structure of a d-dimensional manifold. In particular, open subsets $U \subset M$ are submanifolds.

Proposition A.5. *A subset N of an n-dimensional manifold M defines a submanifold of dimension $d \leq n$ if and only if either*

- *$d = n$ and N is open in M, or*
- *$d < n$, N is contained in an open subset $W \subset M$, and for every $a \in W$ there are an open neighborhood U and functions $f_1, \dots f_{n-d} \in \mathcal{E}(U)$ such that $U \cap N = \{x \in U \mid f_j(0) = 0$ for all $j = 1, \dots, n-d\}$ and $rk(f_1, \dots, f_{n-d}) = n - d$.*[98]

Notation. $n - d$ is called the *codimension* of N.
A closed submanifold N of codimension 1 is called a *hypersurface* ("Hyperfläche").

Observation. With the notation of Proposition A.5 a submanifold is closed in the open subset W of M, but, in general, not in M.

PRODUCT MANIFOLD:

Definition A.6. The *product (manifold)* ("Produktmannigfaltigkeit") of two manifolds M, N is the Hausdorff space $M \times N$ with the differentiable structure given by all the charts

$$q \times q' : U \times U' \to V \times V',$$

where $q' : U \to V$ is a chart on M and $q' : U' \to V'$ is a chart on N.

Proposition A.7. *The product $M_1 \times M_2$ of two manifolds M_1, M_2 together with the projections $p_j : M_1 \times M_2 \to M_j, (x_1, x_2) \mapsto x_j, j = 1, 2$ satisfies the following universal property: Every map $f : M \to M_1 \times M_2$ is smooth if and only if the compositions $p_1 \circ f$ and $p_2 \circ f$ are smooth. And any manifold P with smooth $p_j : P \to M_j$ satisfying the above universal property is isomorphic to $M_1 \times M_2$ (i.e. there is a diffeomorphism $f : P \to M \times N$).*

Slightly more general is the notion of a fibre product:

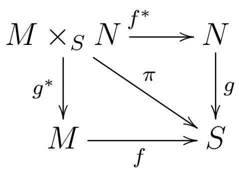
Definition A.8. The *fibre product* ("Faserprodukt") of two mappings $f : M \to S, g : N \to S$ over a third manifold S is the submanifold

$$M \times_S N := \{(x, y) \in M \times N \mid f(x) = g(y)\}$$

of the product $M \times N$, together with its map $\pi : M \times_S N \to S$, $(x, y) \to f(x) = g(y)$.

[98] here we use the notion of rank of a smooth mapping which is explained later.

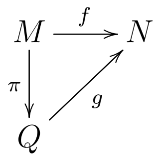
Note, that the two induced maps $f^*(x,y) := x$, $g^*(x,y) =: y$ for $(x,y) \in M \times_S N$ satisfy: $\pi = f^* \circ g = g^* \circ f$, i.e. we have the following commutative diagram



Proposition A.9. *The fibre product $M \times_S N \to S$ of satisfies the following universal property: When the (smooth) mappings $r : Z \to M$, $t : Z \to N$ satisfy $f \circ r = g \circ t : Z \to S$ then there exists a unique $r \times_S t : Z \to M \times_S N$ such that $r = g^* \circ (r \times_S t)$ and $t = f^* \circ (r \times_S t)$, in particular $\pi \circ (r \times_S t) = f \circ r = g \circ t : Z \to S$. And any manifold P with smooth $\hat{f} : P \to M$, $\hat{g} : P \to N$ with the above universal property is isomorphic to $M \times_S N$.*

Quotient Manifold

Definition A.10. A *quotient (manifold)* ("Quotientenmannigfaltigkeit", "Quotient") of a given manifold M with respect to an equivalence relation $\sim$ on M is any manifold Q together with a surjective and smooth map $\pi : M \to Q$ such that the following universal property is satisfied: For every smooth $f : M \to N$ such that f is constant on the equivalence classes of $\sim$ there exists a unique smooth $g : Q \to N$ such that $f = g \circ \pi$.



A quotient manifold is unique up to diffeomorphism.

Note, that the existence of the quotient as a manifold is guarantied only for special equivalence relations $\sim$. In particular, the relation $\sim = \{(x,y) \in M \times M \mid x \sim y\}$ has to be closed as a subset of $M \times M$ to ensure that $M/\sim$ is at least Hausdorff within topological spaces. We come back to quotient manifolds in Section A.4.

A.2 Tangent and Cotangent Bundle

Each curve $x \in \mathcal{E}(I, M)$ defined on an open interval $I \subset \mathbb{R}$ with $0 \in I$ through the point $a := x(0) \in M$ determines a Tangent Vector $[x]_a$ at a: $[x]_a$ is the equivalence class of (germs of) curves in M through a which is given by the equivalence relation

$$x \sim_a y \quad (\text{i.e.} \quad [x]_a = [y]_a) \quad \iff \quad \frac{d}{dt}(q \circ x)|_{t=0} = \frac{d}{dt}(q \circ y)|_{t=0},$$

where $y \in \mathcal{E}(I, M)$ with $y(0) = a$.

Definition A.11. The TANGENT SPACE ("Tangentialraum") at $a \in M$ is the space

$$T_aM := \{[x]_a \mid x \in \mathcal{E}(I, M), x(0) = a\}$$

of all equivalence classes $[x]_a$.

Observation A.12. The set T_aM carries a natural structure of real n-dimensional vector space: For $[x]_a, [y]_a \in T_aM$ (with $x(0) = y(0) = a$) and $\lambda \in \mathbb{R}$ one chooses a chart $q : U \to V \subset R^n$ with $q(a) = 0$ and defines

$$z(t) := q^{-1}\left(q \circ x(t) + \lambda q \circ y(t)\right), t \in I_0\,,$$

for a suitable small open interval I_0 containing 0. Then $[z]_a$ is independent of the choice of the chart q and we set

$$[x]_a + \lambda[y]_a := [z]_a\,.$$

Proposition A.13. *The Tangent Bundle ("Tangentialbündel")*

$$TM = \bigcup_{a \in M} T_aM,$$

together with the projection

$$\tau = \tau_M : TM \to M\,,\ [x]_a \mapsto a,$$

has a natural structure of a $2n$-dimensional manifold where τ is a smooth map and the fibers $\tau^{-1}(a) = T_aM$ are n-dimensional vector spaces.

Proof. For a chart $q : U \to V \subset \mathbb{R}^n$ the corresponding bundle chart $\tilde{q} : \tau^{-1}(U) \to U \times \mathbb{R}^n$ is defined by

$$\tilde{q}([x]_a) := (a, \frac{d}{dt} q \circ x(t)|_{t=0})\,,\ x(0) = a.$$

$\tilde{q}$ is bijective, since $\tilde{q}_a := \tilde{q}|_{T_aM} : T_aM \to \mathbb{R}^n$ is bijective (and linear). For another chart $q' : U' \to V'$ we obtain, with $g := q' \circ q^{-1}$, the change of the bundle charts

$$\tilde{q'} \circ \tilde{q}^{-1}|_{T(U \cap U')} : \tilde{q}(U \cap U') = q(U \cap U') \times \mathbb{R}^n \to \tilde{q'}(U \cap U') = q'(U \cap U') \times \mathbb{R}^n\,,$$

as the map given by

$$(q, v) \mapsto (g(q), Dg(q).v)\,,$$

where $Dg(q).v$ is the Jacobi matrix (or derivative) $Dg(q)$ of g at $q \in q(U \cap U')$ applied to $v \in \mathbb{R}^n$. In particular, this description shows that $\tilde{q'} \circ \tilde{q}^{-1}|_{T(U \cap U')}$ is a diffeomorphism. Now, the topology on $TU := \tau^{-1}(U)$ will be defined by $\tilde{q} : TU \to U \times \mathbb{R}^n$ in such a way that $\tilde{q}$ is a topological map (i.e. continuous with a continuous inverse). For another chart $q' : U' \to V'$ the topologies on $T(U \cap U')$ induced by $\tilde{q}$ and $\tilde{q'}$ agree, since $\tilde{q'} \circ \tilde{q}^{-1}|_{T(U \cap U')}$ is a diffeomorphism, hence in particular, a topological map. Because of the smoothness, the charts $\tilde{q}$ and $\tilde{q'}$ are compatible and thus define a differentiable structure when the topology on TM turns out to be Hausdorff. But the Hausdorff property can indeed be proven quite easily.

The proof is complete, but let us describe the differentiable structure using atlasses. Note, that with respect to an atlas $\mathfrak{A} = (q_j : U_j \to V_j)$ of charts q_j defining the differentiable structure of the manifold M the change of charts is given by $q_i \circ q_j^{-1} : q_j(U_{ij}) \to q_i(U_{ij})$, where $U_{ij} = U_i \cap U_j$, and for the induced bundle charts we obtain the diffeomorphisms $\tilde{q}_i \circ \tilde{q}_j^{-1} : q_j(U_{ij}) \times \mathbb{R}^n \to q_i(U_{ij}) \times \mathbb{R}^n$. Thus, the bundle charts $(\tilde{q}_j)$ form an atlas of the differentiable structure of the tangent bundle TM. □

Observation. For the bundle charts $\tilde{q} : TU = \tau^{-1}(U) \to U \times \mathbb{R}^n$ the following diagram is commutative

$$\begin{array}{ccc} TU & \xrightarrow{\tilde{q}} & U \times \mathbb{R}^n \\ \tau \downarrow & \swarrow \mathrm{pr}_1 & \\ U & & \end{array}$$

where $\mathrm{pr}_1 : U \times \mathbb{R}^n \to U, (x, v) \mapsto x$, is the natural projection. Moreover, for $a \in U$ the induced map $T_aM \to \{a\} \times \mathbb{R}^n \cong \mathbb{R}^n$ is liner over $\mathbb{R}$.

This property essentially implies that $TM \to M$ is a (smooth) vector bundle of rank n (see Section D.1 for an exposition of vector bundles). Moreover, for an atlas of charts (q_j) as at the end of the proof the preceding proposition the transition of bundle charts has the form

$$\tilde{q}_i \circ \tilde{q}_j^{-1} : q_j(U_{ij}) \times \mathbb{R}^n \to q_i(U_{ij}) \times \mathbb{R}^n \,,\ (q, v) \mapsto (q_i \circ q_j^{-1}(q), D(q_i \circ q_j^{-1})(q).v)\,.$$

Consequently, the transition functions $g_{ij} : U_{ij} \to \mathrm{GL}(n, \mathbb{R})$ of the vector bundle TM in the way they are used in Section D.1 and elsewhere are given by $g_{ij}(q) = Dq_i \circ q_j^{-1}(q)\mathrm{GL}(n, \mathbb{R})$.

Observation A.14. For a smooth map $f : M \to N$ between manifolds M, N the Tangent Map ("Tangentialabbildung"), Derivative ("Ableitung") (also called total derivative) or Differential of f at a point $a \in m$ is given by

$$T_af : T_aM \to T_{f(a)}N\,,\ T_aF([x]_a) := [f \circ x]_{f(a)}.$$

T_af is linear. Moreover, the $T_af, a \in M$, fit together to define the Tangent Map (Derivative or Differential)

$$Tf : TM \to TN\,,\ \ v \mapsto T_a(v)\,,\ v = \in \tau^{-1}(a) = T_aM\,,$$

which turns out to be a smooth map. The smoothness can be proven by using the respective bundle charts. Tf is compatible with the projections (i.e. $f \circ \tau_M = \tau_N \circ Tf$) and $\mathbb{R}$-linear in the fibers $T_aM, T_{f(a)}N$. Hence, Tf is a vector bundle (homo-) morphism over f. The compatibility condition can also be expressed by stating that the diagram

$$\begin{array}{ccc} TM & \xrightarrow{Tf} & TN \\ {\scriptstyle \tau_M}\downarrow & & \downarrow{\scriptstyle \tau_N} \\ M & \xrightarrow[f]{} & N \end{array}$$

is commutative: $f \circ \tau_M = \tau_N \circ Tf$.

Cotangent Bundle:

Similarly, one introduces the COTANGENT BUNDLE ("Kotangentenbündel")

$$T^*M := \bigcup_{a \in M} T_a^*M,$$

where

$$T_a^*M := \mathrm{Hom}(T_aM, \mathbb{R}) = (T_aM)^*$$

with the projection $T^*M \to M$ denoted by τ or τ^*. T^*M is a vector bundle of rank n, as well.

Every manifold is endowed not only with the two vector bundles TM ans T^*M, but in addition, with further naturally associated vector bundles. Important for the sequel are the bundles of k-forms for $k = 1, 2, \ldots, n$:

$$\Lambda_a^k M := \bigwedge^k (T_aM) := \{\alpha : (T_aM)^k \to \mathbb{R} \mid \alpha \;\; k\text{−multilinear over } \mathbb{R} \text{ and alternating}\}$$

$$\Lambda^k M := \bigcup_{a \in M} \Lambda_a^k M$$

with smooth projections

$$\tau : \Lambda^k M \to M.$$

The $\Lambda^k M$ are vector bundles of rank $\binom{n}{k}$.

Exercise A.15. Describe the differentiable structure of the real vector bundles $\Lambda^k M$ by bundle charts.

VECTOR FIELDS:

A *vector field* ("Vektorfeld") on the manifold M is a smooth map

$$X : M \to TM\,, \;\; \text{with} \;\; X(a) \in T_aM \;\; \text{for all} \;\; a \in M,$$

i.e. X is a smooth SECTION of the tangent bundle:

$$X \in \mathcal{E}(M.TM)\,, \;\; \text{with} \;\; \tau \circ X = \mathrm{id}_M.$$

We denote the set of all vector fields on M by $\mathfrak{V}(M)$. $\mathfrak{V}(M)$ is a vector space over $\mathbb{R}$ and a module over the commutative ring $\mathcal{E}(M)$ with respect to the following operations for $X, Y \in \mathfrak{V}(M)$ and $f \in \mathcal{M}$:

$$(X+Y)(a) := X(a) + Y(a)\,,\; a \in M \quad \text{and}$$

$$(fX)(a) := f(a)X(a)\,,\; a \in M.$$

It turns out that, in addition, $\mathfrak{V}(M)$ is a Lie algebra over $\mathcal{E}(M)$. This fact explains the notation $\mathfrak{V}(M)$ with a $\mathfrak{V}$ in Gothic type.

Differential Forms

Definition A.16. A differential form ("Differentialform") or simply *form* of degree k is a section in the bundle $\Lambda^k M$:

$$\mathcal{A}^k(M) := \{\alpha \in \mathcal{E}(M, \Lambda^k M) \mid \tau \circ \alpha = \mathrm{id}_M\}$$

The differential forms of degree k are mostly called *k-forms.*

Let $\alpha \in \mathcal{A}^k(M)$ be k-form and $f \in \mathcal{E}(M)$ a smooth function. Then $f\alpha : M \to \Lambda^k(M)$ is defined by $(f\alpha)(a) := f(a)\alpha(a), a \in M$. Certainly, $f\alpha$ is smooth and a section, hence $f\alpha \in \mathcal{A}^k(M)$. Moreover,

$$\mathcal{E}(M) \times \mathcal{A}^k(M) \to \mathcal{A}^k(M)\,,\; (f, \alpha) \mapsto f\alpha\,,$$

is linear over the commutative ring $\mathcal{E}(M)$,

Similarly, for a k-form $\alpha \in \mathcal{A}^k(M)$ and $(X_1, \dots, X_k) \in (\mathfrak{V}(M))^k$ by

$$\alpha(X_1, \dots, X_k)(a) := \alpha(a)(X_1(a), \dots, X_k(a)), a \in M,$$

one obtains a smooth function $\alpha(X_1, \dots, X_k)$. Tha map

$$(\mathfrak{V}(M))^k \to \mathcal{E}(M)\,,\; (X_1, \dots, X_k) \mapsto \alpha(X_1, \dots, X_k)$$

ia k-multilinear over $\mathcal{E}(M)$. It is easy to show:

Proposition A.17. *$\mathcal{A}^k(M)$ is an $\mathcal{E}(M)$-module with respect to the multiplication defined above. Moreover the following map is an isomorphism of $\mathcal{E}(M)$-modules*

$$\mathcal{A}^k(M) \cong \bigwedge^k(\mathfrak{V}(M))\,,\; \omega \mapsto ((X_1, \dots, X_k) \mapsto \omega(X_1, \dots, X_k))\,,$$

identifying the respective $\mathcal{E}(M)$-modules:

$$\mathcal{A}(M) := \mathcal{A}^1(M) \cong (\mathfrak{V}(M))^* := \mathrm{Hom}_{\mathcal{E}(M)}\left(\mathfrak{V}(M), \mathcal{E}(M)\right)\,,$$

$$\mathcal{A}^k(M) \cong \bigwedge^k(\mathfrak{V}(M))\,.$$

Here, for a module W over a commutative ring R with 1 one defines

$$\bigwedge_R^k W = \bigwedge^k W := \{\beta : W^k \to R \mid \beta \ \ k\text{−multilinear over } R \text{ and alternating}\}$$

Remark A.18. The preceding Proposition A.17 suggests an alternative definition of a k-form by

$$\mathcal{A}^k(M) := \bigwedge^k (\mathfrak{V}(M)) ,$$

since every $\mathcal{E}(M)$-multilinear and alternating $\eta : \mathfrak{V}(M)^k \to \mathcal{E}(M)$ induces a section $\eta : M \to \bigwedge^k M$ by $\eta(a)(X_1, \ldots, X_k) = \eta(X_1, \ldots, X_k)(a)$ for $X_j \in \mathfrak{V}(M)$.

Local Expressions

In the following we present local expressions for the vector fields and forms which are used throughout the notes.

Notation A.19. Let M be a manifold of dimension n with its tangent bundle TM and its cotangent bundle T^*M.

1. The Charts on the manifold M defining the differentiable structure of M are mostly written in the following way

$$q : U \to V \ , \ q = (q^1, q^2, \ldots, q^n) ,$$

where $U \in M$ is an open subset in M, $V \in \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$ and q is differentiable with differentiable inverse.

2. The smooth functions $q^j : U \to \mathbb{R}$ (the components of q) are called the (local) Coordinates given by the chart q.

3. A chart $q : U \to V$ provides for each $a \in U$ a natural vector space basis

$$\left(\frac{\partial}{\partial q^1}(a), \frac{\partial}{\partial q^2}(a) \ldots , \frac{\partial}{\partial q^n}(a) \right)$$

of the Tangent Space $T_a M$ of M at a, where

$$\frac{\partial}{\partial q^j}(a) := \left[q^{-1} \left((q(a) + t e_j) \right) \right]_a$$

is given as the tangent vector of the curve $q^{-1}(q(a)+te_j)$, $t \in]-\varepsilon, \varepsilon[$ through $a \in U$ and where $(e_1, \ldots, e_n)$ is the standard unit vector basis of $\mathbb{R}^n$: For convenience, these tangent vectors at a are abbreviated as

$$\partial_j(a) := \frac{\partial}{\partial q^j}(a) , \ \text{ or } \ \frac{\partial}{\partial q^j} \text{ resp. } \partial_j ,$$

when it is clear from the context which coordinates q are used resp. for which point a the expressions are employed.

4. Note that
$$\frac{\partial}{\partial q^j} : U \to TU\,,\ a \mapsto \frac{\partial}{\partial q^j}(a)\,,$$
is a VECTOR FIELD on U. Moreover, every vector field $X : U \to TU$ can be described uniquely by
$$X = X^j \frac{\partial}{\partial q^j} = X^j \partial_j\,,$$
where the coefficients X^j are smooth, X^j can be obtained by
$$X^j(a) = \frac{d}{dt}\left((q^j \circ x)(t)\right)|_{t=t_0}$$
where the curve $x : I \to U, x(t_0) = a$, represents X at a: $X(a) = [x]_a$.

5. Every chart $q : U \to V$ induces a BUNDLE CHART (cf. A.2) $\tilde{q} : TU \to V \times \mathbb{R}^n$ on the TANGENT BUNDLE TM:
$$\tilde{q} = \left(q^1, \dots, q^n, v^1, \dots, v^n\right) : TU \to V \times \mathbb{R}^n\,,\ [x]_a \mapsto \left((q(a), \frac{d}{dt}(q \circ x)|_{t=t_0}\right)\,,$$
when $x(t_0) = a$. Here, v^j acts as
$$v^j([x]_a) = \left(\frac{d}{dt}(q^j \circ x)|_{t=t_0}\right)\,, \quad \text{resp.}$$
$$v^j(X(a)) = X^j(a)\,, \quad \text{for a vector field } X = X^j\partial_j \in \mathfrak{V}(U).$$

6. A chart $q : U \to V$ provides for each $a \in U$ also a natural vector space basis
$$\left(dq^1(a), dq^2(a) \dots, dq^n(a)\right)$$
of the COTANGENT SPACE T^*_aM of M at a, where
$$dq^j(a)\left([x]_a\right) := \left(\frac{d}{dt}(q^j \circ x)|_{t=t_0}\right)$$
Hence, dq^j coincides with v^j (see above). For convenience, $dq^j(a)$ is abbreviated as dq^j when it is clear for which point a the expressions are employed.

7. Note that
$$dq^j : U \to T^*U\,,\ a \mapsto dq^j(a)\,,$$
is a 1-FORM on U. Moreover, every 1-form $\alpha : U \to T^*U$ over U can be described uniquely by
$$\alpha = \alpha_j dq^j\,,$$
where the coefficients α^j are smooth. α_j can be obtained by
$$\alpha_j(a) = \alpha(a)\left(\frac{\partial}{\partial q^j}(a)\right) = \alpha(\partial_j)(a)\,.$$
And $\alpha_j(X) = X^j$ for a vector field $X \in \mathfrak{V}(U)$.

8. Every chart $q : U \to V$ induces a BUNDLE CHART (cf. A.2) $\tilde{q} : TU \to V \times \mathbb{R}^n$ on the COTANGENT BUNDLE T^*M:

$$\tilde{q} = \left(q^1, \dots, q^n, p_1, \dots, p_n\right) : T^*U \to V \times R^n \,,\ \alpha \mapsto \left(q(a), \sum \alpha_j e_j\right) \,,$$

when $\alpha = \alpha_j dq^j(a)$. As a consequence, p_j acts as

$$p_j(\alpha) = \alpha_j \,, \quad \text{if } \alpha = \alpha_j dq^j(a) \text{ in } T^*_a U \,, \quad \text{or}$$

$$p_j(\alpha(a)) = \alpha_j(a) \,, \quad \text{for a 1-field } \alpha = \alpha_j dq^j \in \mathcal{A}(U).$$

A.3 Vector Fields and Dynamical Systems

A DYNAMICAL SYSTEM is essentially an autonomous differential equation of first order on a manifold M. Such a dynamical system will be represented by a a pair (M, X) consisting of a manifold M and a vector field X on M.[99] A special example has been introduced in Section 1.1 with $M = U$ an open subset of $\mathbb{R}^n$ and $X = X_H$ a Hamiltonian vector field (see 1.2) determined by a function H on T^*U.

For a dynamical system (M, X) the corresponding differential equation is

$$\dot{q} = X(q).$$

Here, $\dot{q}$ is the same as the tangent vector $[q]_a$ at a given by the curve $q = q(t)$. Any curve $q : I \to M$ ($I \subset \mathbb{R}$ an open interval) is a solution of the dynamical system (also called integral curve) if it satisfies $\dot{q}(t) = X(q(t))$ for all $t \in I$.

The elementary theory of differential equations of first order establishes the following result

Proposition A.20. *Let (M, X) be dynamical system. For each point $a \in M$ there exists a unique maximal solution $q_a :]t_-(a), t_+(a)[\to M$ with $\dot{q}_a = X(q_a)$ and $q_a(0) = a$ satisfying*

1° $M_t := \{a \in M \mid t \in]t_-(a), t_+(a)[\,\}$ *is open in M with* $\bigcup_{t \in \mathbb{R}} M_t = M$.

2° $\Phi_t : M_t \to M_{-t}\,,\ \Phi_t(a) := q_a(t)$, *is a diffeomorphism with* $\Phi_t^{-1} = \Phi_{-t}$.

3° $M_* = \bigcup_{a \in M} \{a\} \times]t_-(a), t_+(a)[$ *is open in $M \times \mathbb{R}$ and*

$$\Phi : M_* \to M \,,\ (a, t) \mapsto \Phi(a, t) := \Phi_t(a) = q_a(t)$$

is differentiable.

[99] This is the right concept for geometry and elementary analysis. There are more general concepts in other mathematical domains, e.g. ergodic systems in Stochastics.

Notation A.21. The map $\Phi =: \Phi^X$ in the above Proposition A.20 is called the (local) FLOW of the vector field X. X is called COMPLETE if $M_* = M \times \mathbb{R}$ i.e. if $]t_-(a), t_+(a)[= \mathbb{R}$ for all $a \in M$ and $t \in \mathbb{R}$. In this case, $(\Phi_t^X)_{t\in\mathbb{R}}$ is a one parameter group: $\Phi_{s+t}^X = \Phi_s^X \circ \Phi_t^X$ of diffeomorphisms.

In the general case, when X is not complete, (Φ^X) is a local one parameter group.

Note, that on a compact manifold M every vector field is complete.

Proposition A.22. *A one parameter group* $(\Phi_t)_{t\in\mathbb{R}}$ *of diffeomorphisms*

$$\Phi_t : M \to M \, , \, t \in \mathbb{R}$$

(i.e. $\Phi_0 = \mathrm{id}_M$, $\Phi_{s+t} = \Phi_s \circ \Phi_t$ *for* $s, t \in \mathbb{R}$ *with* $(a,t) \mapsto \Phi_t(a))$ *differentiable) induces a unique vector field* X *such that* $\Phi_t^X = \Phi_t$*.*

Proof. The induced vector field is

$$X : \mathcal{E}(M) \to \mathcal{E}(M) \, , \, g \mapsto \left(a \to \frac{d}{dt} g(\Phi_t(a))|_{t=0} \right) .$$

Or, in another description, the tangent vector $X(a) = [\Phi_t(a)]_a \in T_aM$ at $a \in M$ is the tangent vector given by the curve $t \mapsto \Phi_t(a)$ through a. X is called the INFINITESIMAL GENERATOR of (Φ_t). □

Since only local curves are needed for the definition of X the result extends to the local one parameter groups, as well. As a consequence, the vector fields (the dynamical systems) on M can be identified with the local one parameter groups on M.

LIE DERIVATIVE OF VECTOR FIELDS

The concept of the flow of a vector field X on a manifold M enables us to extend the notion of a Lie derivative of functions to the notion of a Lie derivative of vector fields (and, moreover, of differential forms, see (72)). Let $X \in \mathfrak{V}(U)$ a vector field on an open subset U of a manifold M and let $\Phi = \Phi^X$ the local flow of the vector field. As before, let $\Phi_t(a) = \Phi(a,t)$.

In the case of a function $f \in \mathcal{E}(U)$ on an open subset $U \subset M$ one can compare $f(a)$ to a neighbouring $f(\Phi_t(a))$ $(a \in U)$, and define

$$L_X f(a) = \frac{d}{dt} f(\Phi_t(a))|_{t=0} \, .$$

In the case of a vector field $Y : U \to TM$ such a comparison is not available, in general: $Y(a)$ and $Y(\Phi_t(a))$ live in different fibres T_aM and $T_{\Phi_t(a)}M$. Therefore, a suitable isomorphism between these fibres could help. And, indeed, the flow Φ provides such a natural isomorphism: Denote

$$((\Phi_{-t})_* Y)(a) = (T_a\Phi_t)^{-1} Y(\Phi_t(a)) = T_{\Phi_t(a)}\Phi_{-t} Y(\Phi_t(a)) \, ;$$

then

$$(\Phi_{-t})_* : T_{\Phi_t(a)}M \to T_aM$$

is an isomorphism. Now, $Y(a)$ and $(\Phi_{-t})_*Y(\Phi_t(a))$ are both contained in T_aM and can be compared.

Definition A.23. The LIE DERIVATIVE of the vector field Y along the vector field X is

$$L_XY(a) := \frac{d}{dt}((\Phi_{-t})_*Y)(a)|_{t=0}\,,\ a \in U.$$

Proposition A.24. *For a vector field $X \in \mathfrak{V}(M)$, for $f \in \mathcal{E}(M)$ and for $Y \in \mathfrak{V}(M)$ one has*

1. $L_Xf = df(X)$.
2. $L_XY = [X, Y]$.
3. $L_X(fY) = (L_Xf)Y + fL_XY$

Moreover, some natural linearity properties are satisfied.

Proof. 1. is obvious. To show 2. let Ψ be the flow of Y. Then for fixed t:

$$T_{\Phi_t(a)}\Phi_{-t}Y(\Phi_t(a)) = T_{\Phi_t(a)}\Phi_{-t}[\Psi_u]_{\Phi_t(a)} = [\Phi_{-t} \circ \Psi_u \circ \Phi_t)]_{\Phi_t(a)}\,.$$

For $f \in \mathcal{E}(M)$ it follows that

$$(\Phi_{-t})_*Y(a)f = T_{\Phi_t(a)}\Phi_{-t}Y(f(\Phi_t(a)) = \frac{d}{du}f(\Phi_{-t} \circ \Psi_u \circ \Phi_t(a))|_{u=0}\,.$$

Therefore,

$$\begin{aligned} L_XYf(a) &= \frac{d}{dt}((\Phi_{-t})_*Y)f(a)|_{t=0} \\ &= \frac{d}{dt}\left(\frac{d}{du}f(\Phi_{-t} \circ \Psi_u \circ \Phi_t(a))|_{u=0}\right)|_{t=0} \\ &= \frac{d}{dt}\frac{d}{du}\left(f(\Psi_u \circ \Phi_t(a)) - f(\Phi_u \circ \Psi_t(a))\right)|_{u=0}|_{t=0} \\ &= L_{[X,Y]}f(a) \end{aligned}$$

where we use the product rule. This completes the proof of 2., and 3. is again obvious. □

Note, that the Lie derivative can be extended to all tensor fields. The case of differential forms is treated in Section A.5.

A.4 Quotient Manifold

Let M be a manifold with an equivalence relation $R \subset M \times M$ on M and let $\pi : M \to M/R$ be the natural projection onto the quotient, i.e. the set of equivalence classes of R. In several situations one is interested in the quotient M/R as a manifold, the quotient manifold or differential quotient. The pragmatic way to achieve this, is to endow M/R with a differentiable structure and to use this structure for further investigation. In general, little attention is paid to the question, whether this choice yields the quotient manifold, or whether or not the quotient manifold exists at all. One reason for this carelessness might be, that the definition of quotient which is shaped as a universal property is not well adapted to the situation in the study of differentiable manifolds. Quite general, it is known that in many categories the notion of subobject, quotient object or epimorphism is not easy to define in a satisfying matter.

In this section we present some examples in detail, describe results about the existence of the quotient structure and establish criteria which make sure whether or not a candidate, i.e. a differential structure on M/R is indeed a quotient structure.

In order that the quotient manifold exists, M/R has to be a Hausdorff space in the quotient topology. We thus begin with the investigation of topological quotients of spaces M which are merely topological spaces.

A.4.1 Topological Quotients

Definition A.25. Let M be a topological space and R an equivalence relation. A TOPOLOGICAL QUOTIENT with respect to R is a topological space X together with a map $p : M \to X$ with the following properties:

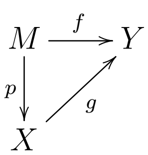
1° The fibres $p^{-1}(x)$ of p are the equivalence classes of R, and p is surjective, i.e. (X, p) describes the equivalence relation exactly,

2° p is continuous,

3° whenever $f : M \to Y$ is continuous and constant on the equivalence classes, the induced map $\hat{f} : X \to Y$, $f = \hat{f} \circ p$, is continuous.

The topology on X is called the QUOTIENT TOPOLOGY.

The last property can be formulated as a universal property: For every commutative diagram



with Y a topological space and f, g maps the following holds:

f is continuous if and only g is continuous.

It is evident, that a topological quotient is unique up to isomorphism, i.e. for another topological quotient $p' : M \to X'$ there exists a unique homeomorphism $h : X \to X'$ with $h \circ p = p'$.

Lemma A.26. *Let M be a topological space with an equivalence relation R and the natural projection $\pi : M \to M/R$. The topological quotient exists. The quotient topology on M/R is the collection of all subsets $V \subset M/R$ for which the inverse image $\pi^{-1}(V)$ is open in M.*

Proof. π is continuous: If $V \subset M/R$ is open, then $\pi^{-1}(V)$ is open by definition. Moreover for any continuous $f : M \to Y$ into another topological space Y the function $\hat{f}(p(a)) := f(a)$, $a \in M$, is continuous: If $W \subset Y$ is open, then $U := f^{-1}(W)$ is open in M. Since $\pi^{-1}(\pi(U)) = U$ the image $V := \pi(U)$ is open by definition. Hence, $V = \pi(U) = \hat{f}^{-1}(W)$ is open, and $\hat{f}$ is continuous. □

Examples A.27. In these lecture notes quotients often appear as orbit spaces. The starting point is a manifold M and a partition of M into orbits of some geometric origin, for instance solution curves of a vector field, integral manifolds of a foliation (cf. Section 9) or equivalence classes of a Lie group action (cf. Section C.5. Already the case of orbits of a Hamiltonian system is interesting.

1. Let M be $\mathbb{R}^n$ with the equivalence relation: $q \sim q'$ (i.e. $(q, q') \in R$) if and only if $\|q\| = \|q'\|$ with respect to the euclidian norm. The equivalence classes are the spheres $\mathbb{S}^n(r) = \{q \in \mathbb{R}^n \mid \|q\|^2 = r\}$, $r \in [0, \infty[$. In the case of $n = 2$ these equivalence classes are the orbits of the Hamiltonian vector field $H(q,p) = \frac{1}{2}(p^2 + q^2)$. The quotient M/R is the collection of spheres $\{\mathbb{S}^n(r) \mid r \in [0, \infty[\}$ with a natural bijection

$$h : M/R \to [0, \infty[\ , \mathbb{S}^n(r) \mapsto r\,.$$

The quotient topology on M/R is the topology, which makes h to a homeomorphism, with respect to the standard topology on the interval $[0, \infty[$: $h^{-1}(W)$ is open for an open subset W, $W \subset [0, \infty[$, since

$$\bigcup \{\mathbb{S}^n(r) \mid r \in W\} = \pi^{-1}(h^{-1}(W))$$

is open in $\mathbb{R}^n$. $h(U)$ is open in the interval for an open subset U in M/R since $\pi^{-1}(U)$ is open in $\mathbb{R}^n$ and therefore $\pi^{-1}(U) \cap ([0, \infty[\times \{0\})$ is open in $[0, \infty[\times \{0\} \cong [0, \infty[$.

In particular, the quotient topology is Hausdorff.

This simple example shows that it is helpful to call $p := h \circ \pi : M \to [0, \infty[$ quotient as well.

2. Let $M = \mathbb{R}^2$ with the differential equation $(\dot{q}, \dot{p}) = (p, 0)$. The solutions (i.e. the motions) are $q(t) = p_0 t + q_0, p(t) = p_0$, $t \in \mathbb{R}$. The set of all orbits O is

$$O = \{\{(t, y) \mid t \in \mathbb{R}\} \mid y \in \mathbb{R}\,,\, y \neq 0\} \cup \{\{(x, 0)\} \mid x \in \mathbb{R}\}\,.$$

Thus the orbits are the vertical lines through $(0,p), p \neq 0$ and the points $(x,0), x\mathbb{R}$. The orbits determine an equivalence relation R with the orbits as the equivalence classes. A natural way to parametrize the orbit space $O = M/R$ seems to be the use of the axes of coordinates $A := \{(x,y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\}$ by $h : O \to A$, $\{(x,0)\} \mapsto (x,0)$, $\{(t,y) \mid t \in \mathbb{R}\}, (y \neq 0) \mapsto (0,y)$. In this way one is tempted to take A with the topology induced from the inclusion $A \subset \mathbb{R}^2$ as the topological quotient.

But $p =:= h \circ \pi : M \to A$ is not continuous. The inverse image $^{-1}(V)$ of the open subset $]r, r+1[\times \{0\} \subset A$ is the set $]r, r+1[\times \{0\} \subset \mathbb{R}^2$ which is not open as a subset of $\mathbb{R}^2$. The quotient topology on $O = M/R$ is strictly weaker than the topology induced on A from $\mathbb{R}^2$.

In particular, the quotient topology is not Hausdorff: Any open neighbourhood V of a point $\{(x,0)\} \in O$ contains an open neighbourhood of the form $V_r(x) := \pi(U_r(x))$ where $U_r(x) =]x-r, x+r[\times]-r, +r[$ for a suitable $r > 0$. This neighbourhood of $\{(x,0)\}$ is $V_r(x) = \{\{(x',0)\} \mid x' \in]x-r, x+r[\setminus \{0\}\} \cup \{\{(t,y) \mid y \neq 0\} \mid y \in]-r,+r[\}$. For a different point $\{(x',0)\} \in O$ any neighbourhood V' of $\{(x',0)\}$ contains an open $V_{r'}(x')$. Let $r \leq r'$. Then $V_r(x) \cap V_r(x') \neq \emptyset$ since this intersection contains $\{0\} \times]-r,+r[\setminus \{(0,0)\}$. Hence, $V \cap V' \neq \emptyset$, i.e. every pair of neighbourhoods V of $\{(x,0)\}$ resp. V' of $\{(x',0)\}$ has a nonempty intersection $V \cap V' \neq \emptyset$.

3. Projective Space: Let M be $\mathbb{K}^{n+1} \setminus \{0\}$ and consider the equivalence relation $z \sim w$ (i.e. $(z,w) \in R$) if and only if there is $\lambda \in \mathbb{K}$ with $z = \lambda w$. The quotient $M/R =: \mathbb{P}^n(\mathbb{K})$ is the set of lines in $\mathbb{K}^{n+1}$ through 0. Here, $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$[100]. Let $\gamma : M \to \mathbb{P}^n(\mathbb{K})$ be the natural projection. The homogeneous coordinates for $z = (z^0, z^1 \ldots, z^n) \in M$ are $[z^0 : z^1 : \ldots : z^n]$. They describe the equivalence classes and γ completely: $\gamma(z) = [z^0 : z^1 : \ldots : z^n]$.

The quotient topology on $\mathbb{P}^n(\mathbb{K})$ can be described as follows: For each $j \in \{0, 1, \ldots, n\}$ let $U_j := \{[z^0 : z^1 : \ldots : z^n] \mid z^j \neq 0\}$ and $H_j := \{w \in \mathbb{K}^{n+1} \mid w^j = 1\}$. The U_j cover $\mathbb{P}^n(\mathbb{K})$. Each H_j is a hypersurface in $\mathbb{K}^{n+1}$ and also an n-dimensional affine subspace of $\mathbb{K}^{n+1}$. H_j is naturally isomorphic to $\mathbb{K}^n$. Now,

$$\varphi_j : U_j \to H_j\,,\ \left[z^0 : z^1 : \ldots : z^n\right] \mapsto \frac{1}{z^j}\left(z^0, z^1, \ldots, z^n\right)$$

is a bijection. Each U_j will be endowed with the topology induced by φ_j. These topologies determine a unique topology on all of $\mathbb{P}^n(\mathbb{K})$ which we call the "chartwise" topology. It consists of all unions of subsets of $\mathbb{P}(\mathbb{K})$ which are open in one of the U_j. In particular, φ_j is a homeomorphism and can be called a topological chart.

We claim that this topology is the quotient topology. First of all, the projection γ is continuous with respect to this topology.

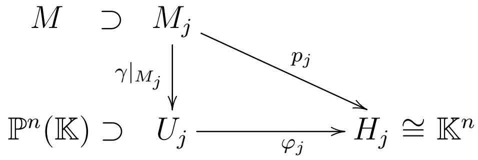
Define M_j by $M_j := \gamma^{-1}(U_j) = \mathbb{K}^{n+1} \setminus \{z^j \neq 0\}$. γ is continuous, since all restrictions $\gamma|_{M_j} : M_j \to U_j$ are continuous which is a consequence of the fact that the "projections"

[100] or other topological fields

$p_j := \varphi_j \circ \gamma|_{M_j} : M_j \to H_j$ given by

$$p_j(z) = \frac{1}{z^j} z$$

are obviously continuous maps.



Moreover, γ is also open in the chartwise topology. For an open subset $U \subset M$ and $j = 0, 1, \ldots, n$ the intersections $U \cap H_j$ are open in H_j, hence $\gamma(U) \cap U_j = \varphi^{-1}(U \cap H_j$ is open for all j. It follows that

$$\gamma(U) = \cup\{\gamma(U) \cap U_j \mid j = 0, 1, \ldots, n\}$$

is open in the chartwise defined topology.

By the universal property it follows that the quotient topology is finer than the chartwise defined topology. It is also coarser: Let $V \subset \mathbb{P}^n(\mathbb{K})$ be open in the quotient topology. Then $\gamma^{-1}(V)$ is open, and by the openness of the projection γ with respect to the chartwise topology, V is open in the chartwise topology. Hence the two topologies coincide.

Finally, it is not difficult to prove that $\mathbb{P}^n(\mathbb{K})$ is Hausdorff. Given two different points $[z_1] \neq [z_2]$ in $\mathbb{P}^n(\mathbb{K})$ by homogeneous coordinates we can assume that $\|z_1\| = \|z_2\| = 1$. In case of $\mathbb{K} = \mathbb{C}$ the two compact subsets $S_k := \{\exp itz_k \mid t \in \mathbb{R}\}$, $k = 1, 2$, of $\mathbb{C}^{n+1}$ have no point in common. In case of $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}$ the same is true for $S_k := \{z_k, -z_k\}$. Hence, there is a positive $r > 0$ such that $\|w_1 - w_2\| \geq 3r$ for all $(w_1, w_2) \in S_1 \times S_2$. Now, the union of open balls $U_k := \bigcup\{B(w_k, r) \mid w_k \in S_k\}$ is open $(k = 1, 2)$ and therefore, $V_k := \gamma(U_k)$ is an open neighbourhood $[z_k]$. By construction $V_1 \cap V_2 = \emptyset$. Otherwise, there would exist a point $[w] \in V_1 \cap V_2$ with $w \in U_k$, $k = 1, 2$. Since $w \in B(w_k, r)$ for suitable $w_k \in S_k$ the distance between w_1 and w_2 would be $\|w_1 - w_2\| \leq \|w_1 - w\| + \|w - w_2\| < 2r$ contradicting $\|w_1 - w_2\| \geq 3r$.

Finally, $\mathbb{P}^n(\mathbb{K})$ is compact, since it is the image of the restriction of γ to the compact sphere $\mathbb{S}^n$ $(\mathbb{K} = \mathbb{R})$ resp. $(\mathbb{K} = \mathbb{C})$.

4. One can generalize the last result to ℓ^2 or $\mathbb{K}^{\mathbb{N}}$ instead of $\mathbb{K}^{n+1}$ and to other (infinite dimensional) sequence spaces by essentially the same arguments. If S is the sequence space[101] there are the natural coordinates $z = (z^j)_{j \in \mathbb{N}}$ of S and $M = S \setminus \{0\}$ as in t he finite dimensional case. The equivalence relation is the same with the lines through 0 as the equivalence classes and the natural projection $\gamma : M \to \mathbb{P}(S)$. We

[101] S is a locally convex space with a Schauder basis

obtain infinitely many coordinate neighbourhoods U_j , $j \in \mathbb{N}$, covering $\mathbb{P}(S)$ with the charts

$$\varphi_j : U_j \to H_j = \{z \in M \mid z^j = 1\} \cong S\,.$$

Moreover, $\gamma^{-1}(U_j) = M_j$ projects to H_j by the continuous maps $p_j(z) = \frac{z}{z^j}$ with $p_j = \varphi_j \circ \gamma|_{M_j}$.

$$\begin{array}{ccccc} M & \supset & M_j & & \\ & & \downarrow{\scriptstyle \gamma|_{M_j}} & \searrow^{p_j} & \\ \mathbb{P}(S) & \supset & U_j & \xrightarrow[\varphi_j]{} & H_j \cong S \end{array}$$

Consequently the chartwise topology is the quotient topology. Moreover the quotient topology is Hausdorff. However it is not compact when S is infinite dimensional.

The final arguments in the third example above works in general and leads to a general criterium for a continuous map being a quotient. We have explained before that in many cases one is interested to know whether a given continuous and surjective map $g : M \to X$ is a quotient, i.e. X carries the quotient topology.

Proposition A.28. *A continuous and surjective map $g : M \to X$ is a quotient*

1. *whenever g is an open map, or*
2. *whenever g has local continuous sections, i.e. for all $x \in X$ there exists an open neighbourhood V and a continuous $s : V \to M$ with $g \circ s = \mathrm{id}_V$.*

Proof. Let $f : M \to Y$ continuous and constant on the equivalence classes, that means on the fibres $f^{-1}(y)\,, y \in Y$. We have to show that $\hat{f} : X \to Y$ is continuous. For an open $W \subset Y$, $f^{-1}(W)$ is open in M hence , if g is open, $g(f^{-1}(W)) = \hat{f}^{-1}(W)$ is open, i.e. $\hat{f}$ is continuous. If f has local continuous sections $s : V \to M$, the composition $f \circ s = \hat{f}|_V : V \to Y$ is continuous, and it follows that $\hat{f}$ is continuous. □

A.4.2 Differentiable Quotients

Similarly, for the differentiable case. Note, that the notion of differential quotient is analogous to that of a topological quotient, see Definition A.10:

Proposition A.29. *A differentiable and surjective map $g : M \to X$ for differentiable manifolds M, X is a differentiable quotient*

1. *whenever g has local differentiable sections, i.e. for all $x \in X$ there exists an open neighbourhood V and a continuous $s : V \to M$ with $g \circ s = \mathrm{id}_V$, or*
2. *whenever f is a submersion, i.e. the derivative $T_a f$ is surjective for all $a \in M$.*

Proof. 1. has the same proof as in the last proposition. And a submersion always has local differentiable sections according to the implicit mapping theorem. □

Examples A.30. We investigate the 4 examples in A.27:

1. We know that $h \circ \pi : M = \mathbb{R}^n \to [0, \infty[$ is a topological quotient and the quotient is Hausdorff. $[0, \infty[$ is not a manifold in any sense, but it is a manifold with boundary $\{0\}$. Deleting 0 in M and in $[0, \infty[$ we obtain a differentiable map $g : M \setminus \{0\} \to]0, \infty[$. It is the differentiable quotient since g has differentiable local sections and is a submersion.

2. The quotient O in example 2. above is not Hausdorff, so there is no chance that the differentiable quotient exists. Note, that the defining differential equation $(\dot{q}, \dot{p}) = (p, 0)$ is the equation of motion of a Hamiltonian system: The Hamiltonian vector field is $X_H(q, p) = (p, 0)$ where $H(q, p) = \frac{1}{2}p^2$. X_H does not define a distribution, since $X_(H)(q, 0) = 0$.

3. The natural projection $\gamma : \mathbb{K}^{n+1} \setminus \{0\} \to \mathbb{P}^n(\mathbb{K})$ is the topological quotient according to the third axample above in A.27, and the quotient is Hausdorff. The projective space $\mathbb{P}^n(\mathbb{K})$ obtains a differentiable structure by the charts $\varphi : U_j \to H_j$ which are (smoothly) compatible to each other. The projection γ is differentiable with respect to this differentiable structure: Since γ is a submersion, $\gamma : \mathbb{K}^{n+1} \setminus \{0\} \to \mathbb{P}^n(\mathbb{K})$ is the differentiable quotient.

In the case of $\mathbb{K} = \mathbb{C}$ $\gamma : \mathbb{K}^{n+1} \setminus \{0\} \to \mathbb{P}^n(\mathbb{K})$ is, moreover, the holomorphic quotient. In particular the charts φ_j are biholomorphic and $\mathbb{P}^n(\mathbb{C})$ is a complex manifold (see Chapter "Complex Analysis" B).

4. For sequence spaces like ℓ^2 or $\mathbb{K}^{\mathbb{N}}$ we obtain in the same manner differentiable resp. holomorphic quotients. However, we need the notion of an infinite dimensional manifold.

Example A.31. Here is an example of a surjective smooth mapping which is not a submersion: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. f is not a submersion, since $f'(x) = 3x^2$ is zero at $x = 0$. Note, that f is an open mapping. Hence, it is a quotient mapping in the topological category, but it is not a differentiable quotient.

In a later section on Lie groups actions and explain in Theorem C.25 the following general result on quotients arising from suitable Lie group action on manifolds:

Proposition A.32. *Suppose G is a Lie group acting smoothly, freely and properly on a manifold M. Then the orbit space M/G is a Hausdorff space, and exists as differential quotient manifold of dimension equal to* $\dim M - \dim G$. *The quotient map $\pi : M \to M/G$ is a submersion.*

A.5 Operations on Differential Forms

We introduce the main operations on forms in order to describe the interplay between the Lie derivative, the exterior derivative and the interior derivative on differential forms and prove the formula

$$\boxed{\begin{aligned} d\eta(X_0, X_1, ..., X_s) &= \sum_{j=0}^{s} (-1)^j L_{X_j}\eta(X_0, ..., \hat{X}_j, ..., X_s) \\ &+ \sum_{0\le i<j\le s} (-1)^{i+j}\eta([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_s) \end{aligned}} \tag{71}$$

which is used as the definition of the exterior derivative d in the first chapter, cf. (5).[102]

Pullback

Definition A.33. Given a smooth map $F : N \to M$ between manifolds N and M, the Pullback

$$F^* : \mathcal{A}^k(M) \to \mathcal{A}^k(N)$$

is defined as follows: For $\eta \in \mathcal{A}^k(M)$ and $Y_j \in \mathfrak{V}(N)$, $j = 1, \dots, k$, we set:

$$F^*\eta(Y_1, \dots Y_k) := \eta\left(TF(Y_1), \dots, TF(Y_k)\right) = \eta(F_*(X_1), \dots, F_*(X_k)) \,,$$

where $TF : TN \to TM$ is the derivative of F (see Observation A.14)[103] and $F_*(X_j) := TF(X_j)$. Pointwise we have for $a \in N$

$$F^*\eta(Y_1, \dots Y_k)(a) = \eta(F(a))\left(T_aF(Y_1), \dots, T_aF(Y_k)\right).$$

Since $TF : \mathfrak{V}(M) \to \mathfrak{V}(N)$ induces a map $\Lambda^k TF : \mathcal{A}^k(M) \to \mathcal{A}^k(N)$ the pullback $F^*\eta$ is the same as $\eta \circ \Lambda^k TF$.

Exterior Product

The wedge product $\wedge : \mathcal{A}^k(M) \times \mathcal{A}^m(M) \to \mathcal{A}^{k+m}(M)$ is given as in multilinear algebra. The wedge product endows

$$\mathcal{A}^\circ(M) := \bigoplus_{i=0}^{\infty} \mathcal{A}^k(M)$$

with the structure of an algebra that is Graded Commutative,
i.e. for $\alpha \in \mathcal{A}^p(M)$, $\beta \in \mathcal{A}^q(M)$ we have

$$\alpha \wedge \beta = (-1)^{|\alpha||\beta|}\beta \wedge \alpha$$

where $|\alpha| = p$, $|\beta| = q$, denote the respective degrees of α and β.

[102]The essential part of this section is taken from an exercise of M. Stankiewicz
[103]Exercise: Describe $F^*\eta$ in local coordinates.

In physics, especially in the context of supersymmetry, this such a graded commutative algebra is often refereed to as a superalgebra.

Lie Derivative

Given a vector field X with the corresponding flow Φ_t (c.f. Proposition A.20), the Lie derivative along X of a differential form is a $\mathbb{K}$-linear map $L_X : \mathcal{A}^p(M) \longrightarrow \mathcal{A}^p(M)$ defined by

$$L_X\omega = \frac{d}{dt}(\Phi_t^*\omega)|_{t=0}. \tag{72}$$

For functions $g \in \mathcal{E}(M)$ this reduces to the known directional derivative $L_X g = Xg = dg(X)$. Moreover, since the wedge product is natural with respect to pullbacks, we obtain

$$L_X(\alpha \wedge \beta) = L_X\alpha \wedge \beta + \alpha \wedge L_X\beta \tag{73}$$

Interior Derivative

The interior derivative i_X along a vector field X is the $\mathcal{E}(M)$-linear map

$$i_X : \mathcal{A}^p(M) \longrightarrow \mathcal{A}^{p-1}(M)$$

defined by inserting X into the first argument of the form, i.e.

$$(i_X\omega)(Y_1, ..., Y_{p-1}) = \omega(X, Y_1, ..., Y_{p-1})$$

The interior derivative of a function is defined to be zero.

The interior derivative is sometimes called interior product; but here we are interested in the property that i_X is a derivation with respect to the graded commutative algebra structure of $\mathcal{A}^\circ(M)$.

Since differential forms are alternating it follows that

Observation A.34.

$$i_X i_Y + i_X i_Y = 0. \tag{74}$$

Moreover i_X satisfies a graded version of the Leibniz rule

$$i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge i_X\beta. \tag{75}$$

which follows from the definition of the wedge product.

Exterior Derivative

The exterior derivative can be defined axiomatically as a $\mathbb{K}$-linear map

$$d : \mathcal{A}^p(M) \longrightarrow \mathcal{A}^{p+1}(M)$$

that satisfies the following properties:

1° For any smooth function f, df is the differential of f.

2° $d^2 = 0$.

3°
$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}(\alpha \wedge d\beta). \tag{76}$$

Equivalently d can be defined locally as in Equation (8) Chapter 1.

An important property of d is that it respects pullbacks, i.e. for any smooth map $F : M \longrightarrow N$ and $\alpha \in \mathcal{A}^\circ(N)$ we have $d(F^*\alpha) = F^*(d\alpha)$.

Superalgebra of Derivations

Although i_X, L_X, d are all defined in quite a different way, one can actually view them as three instances of the same kind of object, namely a derivation on superalgebra.

Definition (Derivation)**.** We say that a linear map $\delta : \mathcal{A}^\circ(M) \longrightarrow \mathcal{A}^\circ(M)$ is a Derivation of degree $|\delta|$ if for any $\alpha, \beta \in \mathcal{A}^\circ(M)$ it satisfies the graded Leibniz rule

$$\delta(\alpha \wedge \beta) = \delta\alpha \wedge \beta + (-1)^{|\alpha||\delta|}\alpha \wedge \delta\beta$$

and $\delta\alpha \in \mathcal{A}^{|\alpha|+|\delta|}(M)$.

From equations (73),(75),(76), one sees that L_X , i_X , d are derivations of degrees respectively 0, -1 and $+1$.

Definition. The Commutator of two derivations δ_1, δ_2 on $\mathcal{A}^\circ(M)$ is defined as

$$[\delta_1, \delta_2] = \delta_1\delta_2 - (-1)^{|\delta_1||\delta_2|}\delta_2\delta_1$$

Proposition. *If δ_1, δ_2 are derivations on $\mathcal{A}^*(M)$ then $[\delta_1, \delta_2]$ is a derivation of degree $|\delta_1| + |\delta_2|$.*

Proof. Clearly, for any $\alpha \in \mathcal{A}^p(M)$, we have that $[\delta_1, \delta_2]\alpha \in \mathcal{A}^{p+|\delta_1|+|\delta_2|}$. Therefore it suffices to check that the graded Leibniz rule holds. This follows from direct computation.

$$\begin{aligned}
\delta_1\delta_2(\alpha \wedge \beta) &= \delta_1(\delta_2\alpha \wedge \beta + (-1)^{|\alpha||\delta_2|}\alpha \wedge \delta_2\beta) \\
&= \delta_1\delta_2\alpha \wedge \beta + (-1)^{|\delta_1|(|\alpha|+|\delta_2|)}\delta_2\alpha \wedge \delta_1\beta \\
&\quad + (-1)^{|\delta_2||\alpha|}\delta_1\alpha \wedge \delta_2\beta + (-1)^{|\alpha|(|\delta_1|+|\delta_2|)}\alpha \wedge \delta_1\delta_2\beta \\
(-1)^{|\delta_1||\delta_2|}\,\delta_2\delta_1(\alpha \wedge \beta) &= (-1)^{|\delta_1||\delta_2|}\delta_2\delta_1\alpha \wedge \beta + (-1)^{|\alpha||\delta_2|+2|\delta_1||\delta_2|}\delta_1\alpha \wedge \delta_2\beta \\
&\quad + (-1)^{|\delta_1||\alpha|+|\delta_1||\delta_2|}\delta_2\alpha \wedge \delta_1\beta \\
&\quad + (-1)^{|\alpha|(|\delta_2|+|\delta_1|)+|\delta_1||\delta_2|}\alpha \wedge \delta_2\delta_1\beta
\end{aligned}$$

In the commutator the mixed terms cancel leaving

$$\begin{aligned}
[\delta_1, \delta_2](\alpha \wedge \beta) &= (\delta_1\delta_2 - (-1)^{|\delta_1||\delta_2|}\delta_2\delta_1)\alpha \wedge \beta \\
&\quad + (-1)^{|\alpha|(|\delta_1|+|\delta_2|)}\alpha \wedge (\delta_1\delta_2 - (-1)^{|\delta_1||\delta_2|}\delta_2\delta_1)\beta
\end{aligned}$$

as required. □

By considering the degrees of i_X, L_X and d, the above proposition and the fact that $[i_X, i_Y]$ and $[d, d]$ both vanish one may suspect that the algebra generated by these derivations should close. And indeed, i_X, L_X and d satisfy the commutation relations as stated below, thereby forming, what is sometimes called a Lie superalgebra.

Theorem A.35. *The following relations hold on* $\mathcal{A}^\circ(M)$

$[i_X, i_Y] =$ $= i_X i_Y + i_Y i_X = 0$	$[i_X, L_Y] =$ $= i_X L_Y - L_Y i_X = i_{[X,Y]}$	$[i_X, d] =$ $= i_X d + d i_X = L_X$
	$[L_X, L_Y] =$ $= L_X L_Y - L_Y L_X = L_{[X,Y]}$	$[L_X, d] =$ $= L_X d - d L_X = 0$
		$[d, d] =$ $= d^2 + d^2 = 0$

Proof. The cases of $[i_X, i_Y]$ and $[d, d]$ have already been considered. Also $[L_X, d] = 0$ clearly holds, because the exterior derivative respects pullback.

To prove the remaining three relations let us notice that it suffices to do so locally. In particular one only needs to check them for the case of functions and exact 1-forms because of the coordinate expression (8) and fact that we are comparing derivations of equal degree.

Let us start with $L_X = [i_X, d]$ also known as Cartan's Magic Formula which has the dorm

$$L_X \eta = d i_X \eta + i_X d\eta \tag{77}$$

for a differential form η. For a function $f \in \mathcal{E}(M)$ we have

$$[i_X, d]f = i_X df = X(f) = L_X f$$

since $i_X f = 0$. Using the above and the fact that $[L_X, d] = 0$ we also get

$$[i_X, d]df = d i_X df = d L_X f = L_X df.$$

In the similar way we prove the remaining two relations:

$$\begin{aligned} [i_X, L_Y]f &= i_X(L_Y f) = 0 = i_{[X,Y]} f \\ [i_X, L_Y]df &= i_X L_Y df - L_Y i_X df \\ &= i_X d(L_Y f) - L_Y(X(f)) = i_X d(Y(f)) - Y(X(f)) \\ &= X(Y(f)) - Y(X(f)) = [X, Y](f) \\ &= i_{[X,Y]} df. \end{aligned}$$

$$\begin{aligned}
[L_X, L_Y]f &= L_X(L_Y f) - L_Y(L_X f) \\
&= X(Y(f)) - Y(X(f)) = [X,Y](f) \\
&= L_{[X,Y]}f \\
[L_X, L_Y]df &= L_X(L_Y(df)) - L_Y(L_X(df)) \\
&= d(L_X(L_Y f)) - d(L_Y(L_X f)) \\
&= d(L_{[X,Y]})f \\
&= L_{[X,Y]}(df)
\end{aligned}$$

This finishes the proof. □

These commutation relations are quite useful, in particular they allow us to prove the formula used to define the exterior derivative in Chapter 1:

Proof of formula (71): The proof proceeds by induction on the degree p of the differential form involved.

1) p=0: For a function, the commutator term of (71) is simply zero and we are left with

$$df(X) = X(f) = L_X(f)$$

2) Induction step: Suppose the formula holds for all differential forms of degree at most $p-1$ and let $\eta \in \mathcal{A}^p(M)$. We can then write

$$\begin{aligned}
d\eta(X_0, ..., X_p) &= i_{X_p}...i_{X_1}i_{X_0}(d\eta) \\
&= -i_{X_p}...i_{X_1}d(i_{X_0}\eta) + i_{X_p}...i_{X_1}(L_{X_0}\eta) \\
&= -d(i_{X_0}\eta)(X_1, ..., X_p) + i_{X_p}...i_{X_1}(L_{X_0}\eta).
\end{aligned}$$

Now, since $i_{X_0}\eta$ is a $(p-1)$-form we can apply our formula which will give us

$$\begin{aligned}
&= -\sum_{j=1}^{p}(-1)^{j-1}L_{X_j}((i_{X_0}\eta)(X_1, ..., \hat{X}_j, ..., X_p)) \\
&- \sum_{1\le i<j\le p}(-1)^{i+j-2}(i_{X_0}\eta)([X_i, X_j], X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p) + i_{X_p}...i_{X_1}(L_{X_0}\eta) \\
&= \sum_{j=1}^{p}(-1)^{j}L_{X_j}(\eta(X_0, X_1, ..., \hat{X}_j, ..., X_p)) \\
&+ \sum_{1\le i<j\le p}(-1)^{i+j}\eta([X_i, X_j], X_0, X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p) + i_{X_p}...i_{X_1}(L_{X_0}\eta)
\end{aligned}$$

where the negative sign in the commutator term vanishes because we exchange X_0 and $[X_i, X_j]$ in the arguments of η. We can now use $[i_X, L_Y] = i_{[X,Y]}$ to commute the Lie derivative in the last term all the way to the left yielding

$$
\begin{aligned}
&= \sum_{j=1}^{p} (-1)^j L_{X_j}(\eta(X_0, X_1, ..., \hat{X}_j, ..., X_p)) \\
&+ \sum_{1 \le i < j \le p} (-1)^{i+j} \eta([X_i, X_j], X_0, X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p) \\
&+ L_{X_0}(\eta(X_1, ..., X_p)) + \sum_{j=1}^{p} i_{X_p} ... i_{[X_j, X_0]} ... i_{X_1} \eta.
\end{aligned}
$$

The second last term can be then absorbed into the first sum and in the last term we move $i_{[X_j,X_0]}$ all the way to the right with $j-1$ swaps gaining a factor of (-1) for each of them and finally we get

$$
\begin{aligned}
&= \sum_{j=0}^{p} (-1)^j L_{X_j}(\eta(X_0, X_1, ..., \hat{X}_j, ..., X_p)) \\
&+ \sum_{1 \le i < j \le p} (-1)^{i+j} \eta([X_i, X_j], X_0, X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p) \\
&+ \sum_{j=1}^{p} (-1)^{j-1} \eta([X_j, X_0], X_1, ..., \hat{X}_j, ..., X_p) \\
&= \sum_{j=0}^{p} (-1)^j L_{X_j}(\eta(X_0, X_1, ..., \hat{X}_j, ..., X_p)) \\
&+ \sum_{1 \le i < j \le p} (-1)^{i+j} \eta([X_i, X_j], X_0, X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p) \\
&+ \sum_{j=1}^{p} (-1)^{j} \eta([X_0, X_j], X_1, ..., \hat{X}_j, ..., X_p) \\
&= \sum_{j=0}^{p} (-1)^j L_{X_j}(\eta(X_0, X_1, ..., \hat{X}_j, ..., X_p)) \\
&+ \sum_{0 \le i < j \le p} (-1)^{i+j} \eta([X_i, X_j], X_0, X_1, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p)
\end{aligned}
$$

which is the desired formula. □

B Complex Analysis

Elementary properties of holomorphic functions in one variable are assumed to be known in the following. General reference for the theory of holomorphic functions in several vasriables, i.e. in Complex Analysis is the book of D. Huybrechts [Huy05].

B.1 Cauchy Integral

A *domain* is a connected open subset $U \subset \mathbb{C}^n$ in $\mathbb{C}^n$. A *polydisc* $D = D(r)$ of *polyradius* $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$ in $\mathbb{C}^n$ is a product of discs, it has the form $D(r) := D_1 \times \ldots \times D_n \subset \mathbb{C}^n$, where $D_j = \{z_j \mid |z_j| < r_j\} = D(r_j)$ is the usual disc in $\mathbb{C}$.

Definition B.1. A HOLOMORPHIC FUNCTION on a domain $U \subset \mathbb{C}^n$ is a function $f : U \to \mathbb{C}$, which is partially holomorphic in the following sense: For each $a \in U$ and for each $j = 1, \ldots, n$ the one variable function $z \mapsto f(a + ze_j)$ is holomorphic in a neighbourhood of $0 \in \mathbb{C}$.

F. Hartogs[104] proved about 100 years ago (published 1906 in Math. Ann.) that a holomorphic function is already continuous. This result is difficult to prove and it seems that it has no relevant implications for the theory of holomorphic functions in several variables. Consequently, one mostly works with the definition that a function is holomorphic if it is continuous and partially holomorphic. From now on, we assume a holomorphic function to be continuous. Let $\mathcal{O}(U)$ be the space of holomorphic functions in this sense. By pointwise operations, $\mathcal{O}(U)$ is in a natural way an algebra over $\mathbb{C}$. We can apply the 1-dimensional CAUCHY INTEGRAL to obtain:

Proposition B.2 (Cauchy Integral). *Let $f \in \mathcal{O}(U)$, and assume $0 \in U$ with $\overline{D} \subset U$ for a polydisc $D = D(r) = D_1 \times \ldots \times D_n$.*

- *Then for $z_j \in D_j$ the Cauchy integral representation*

$$f(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^n} \int_{\partial D_1} \cdots \int_{\partial D_n} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_1 \cdots d\zeta_n \tag{78}$$

  holds true.

- *The partial derivatives have the following description*

$$\frac{\partial^{k_1+\cdots+k_n} f(z_1, \ldots, z_n)}{\partial {z_1}^{k_1} \cdots \partial {z_n}^{k_n}} = \frac{k_1! \cdots k_n!}{(2\pi i)^n} \int_{\partial D_1} \cdots \int_{\partial D_n} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1)^{k_1+1} \cdots (\zeta_n - z_n)^{k_n+1}} \, d\zeta_1 \cdots d\zeta_n.$$

[104] F. Hartogs was university professor at the LMU in Munich

Proof. The integrals are interated integrals in one variable, and they do not depend on the order of evaluating the single integrals since the integrand is continuous.

The first formula can be proven by induction: The result is known for $n = 1$. Let $n > 1$. For fixed $z_n \in D_n$, by the induction hypothesis we can assume

$$f(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^{n-1}} \int_{\partial D_1} \cdots \int_{\partial D_{n-1}} \frac{f(\zeta_1, \ldots, \zeta_{n-1}, z_n)}{(\zeta_1 - z_1) \cdots (\zeta_{n-1} - z_{n-1})} d\zeta_1 \cdots d\zeta_{n-1} \quad (79)$$

For fixed $(\zeta_1, \ldots, \zeta_{n-1}) \in \partial D_1 \times \cdots \times \partial D_{n-1}$ the 1-dimensional Cauchy integral is

$$f(\zeta_1, \ldots, \zeta_{n-1}, z_n) = \frac{1}{2\pi i} \int_{\partial D_n} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_n - z_n)} d\zeta_n.$$

Inserting this equality in the formula (79) we receive the result.

Since the integrand is continuous we can exchange integration and differentiation to obtain the second result. □

The Cauchy integral representation of holomorphic functions exhibits a special kind of mean value property. The value $f(0)$ of $f \in \mathcal{O}(U)$ is the average of the values on the product $\partial D_1 \times \ldots \times \partial D_n$ (if $\overline{D} \subset U$): For $z = 0$ and with respect to new variables $\zeta_j = r_j e^{i\theta_j}$, $d\zeta_j = ire^{i\theta_j} d\theta_j$, we obtain

$$f(0) = \left(\frac{1}{2\pi}\right)^n \int_0^{2\pi} \cdots \int_0^{2\pi} f(r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})\, d\theta_1 \ldots d\theta_n \,. \quad (80)$$

This result leads to

Proposition B.3 (Mean Value). *For a holomorphic function $f \in \mathcal{O}(U)$ and $w \in U$ with $w + \overline{D(r)} \subset U$ the value $f(w)$ is the average of the values of f at $w + D(r)$:*

$$f(w) = \frac{1}{\pi r_1^2} \cdots \frac{1}{\pi r_n^2} \int_{w+D(r)} f(z) dz\,,$$

where dz is Lebesgue integration over $\mathbb{C}^n$ in this situation.

Proof. We set $w = 0$ without loss of generality. The integral $\int_{D(r)} f(z)dz$ is, with respect to new variables ρ, θ and $dz_j = \rho_j d\rho_j d\theta_j$,

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \int_0^{r_1} \cdots \int_0^{r_n} f(\rho_1 e^{i\theta_1}, \ldots, \rho_n e^{i\theta_n})\, d\theta_1 \ldots d\theta_n \rho_1 d\rho_1 \ldots \rho_n d\rho_n \,.$$

Inserting (80) yields

$$\left(\frac{1}{2\pi}\right)^n \int_{D(r)} f(z) dz = \int_0^{r_1} \cdots \int_0^{r_n} f(0)\, \rho_1 d\rho_1 \ldots \rho_n d\rho_n \,.$$

The result follows from

$$\int_0^{r_1} \cdots \int_0^{r_n} \rho_1 d\rho_1 \dots \rho_n d\rho_n = \frac{1}{2} r_1^2 \cdots \frac{1}{2} r_n^2 .$$

□

The Cauchy integral can be used to define holomorphic functions by integrals:

Proposition B.4. *For continuous* $g : \partial D_1 \times \ldots \times \partial D_n \to \mathbb{C}$ *and* $(z_1, \ldots, z_n) \in D_1 \times \ldots \times D_n$ *the integral*

$$f(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^n} \int_{\partial D_1} \cdots \int_{\partial D_n} \frac{g(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} d\zeta_1 \cdots d\zeta_n$$

defines a holomorphic function $f : D_1 \times \ldots \times D_n \to \mathbb{C}$.

Proof. f is certainly continuous (the integrand is continuous in the variables ζ_j and z_j). Again by interchanging integration and differentiation one deduces that the function is partially holomorphic. □

B.2 Power Series

Proposition B.5 (Power Series Development)**.** *Let* $f \in \mathcal{O}(U)$. *For each* $a \in U$ *there exists a sequence of homogeneous polynomials* $P^k = P^k f(a) \in \mathbb{C}_{(k)}[z_1, \ldots, z_n]$ *such that for every polydisc* D *such that* $a + \overline{D} \subset U$

$$f(a+z) = \sum_{k \in \mathbb{N}} P^k(z)$$

for all $z = (z_1, \ldots, z_n) \in D$. *The convergence is absolute and uniform on the polydisc* D.

Proof. We may assume $a = 0$ and use the formula (78): For $\zeta_j \in \partial D_j$ and $z_j \in D_j$ (i.e. $|z_j| < |\zeta_j| = r_j$) the fraction

$$\frac{1}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \quad \text{has a series development}$$

$$\frac{1}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} = \sum_{j_1, \ldots, j_n \in \mathbb{N}} \frac{z_1^{j_1} \cdots z_n^{j_n}}{\zeta_1^{j_1+1} \cdots \zeta_n^{j_n+1}} ,$$

which converges uniformly on compact subsets K contained in the polydisc $D = D_1 \times \ldots \times D_n$. Therefore, summation and integration can be interchanged to yield, using the abbreviation $f(\zeta)d\zeta := f(\zeta_1, \ldots, \zeta_n)d\zeta_1 \cdots d\zeta_n$:

$$\begin{aligned}
\int_{\partial D_1} \cdots \int_{\partial D_n} \frac{f(\zeta)d\zeta}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} &= \int_{\partial D_1} \cdots \int_{\partial D_n} \sum_{j_1,\ldots,j_n \in \mathbb{N}} \frac{z_1^{j_1} \cdots z_n^{j_n}}{\zeta_1^{j_1+1} \cdots \zeta_n^{j_n+1}} f(\zeta)d\zeta \\
&= \sum_{j_1,\ldots,j_n \in \mathbb{N}} \int_{\partial D_1} \cdots \int_{\partial D_n} f(\zeta) \frac{z_1^{j_1} \cdots z_n^{j_n}}{\zeta_1^{j_1+1} \cdots \zeta_n^{j_n+1}} d\zeta \\
&= \sum_{j_1,\ldots,j_n \in \mathbb{N}} \left(\int_{\partial D_1} \cdots \int_{\partial D_n} \frac{f(\zeta)d\zeta}{\zeta_1^{j_1+1} \cdots \zeta_n^{j_n+1}} \right) z_1^{j_1} \cdots z_n^{j_n}
\end{aligned}$$

Let

$$c_{j_1,\ldots,j_n} := \frac{1}{(2\pi i)^n} \int_{\partial D_1} \cdots \int_{\partial D_n} \frac{f(\zeta)}{\zeta_1^{j_1+1} \cdots \zeta_n^{j_n+1}} \, d\zeta$$

be the coefficients in the last formula. Using (78) we obtain

$$\begin{aligned}
f(z) &= \frac{1}{(2\pi i)^n} \int_{\partial D_1} \cdots \int_{\partial D_n} \frac{f(\zeta)d\zeta}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} \\
&= \sum_{j_1,\ldots,j_n} c_{j_1,\ldots,j_n} z_1^{j_1} \cdots z_n^{j_n} \\
&= \sum_{k=0}^{\infty} \sum_{j_1+\ldots+j_n=k} c_{j_1,\ldots,j_n} z_1^{j_1} \cdots z_n^{j_n} \\
&= \sum_{k=0}^{\infty} P^k(z) \, ,
\end{aligned}$$

if $P^k(z) := \sum_{j_1+\ldots j_n=k} c_{j_1,\ldots,j_n} z_1^{j_1} \cdots z_n^{j_n}$. P^k is a homogeneous polynomials of degree k, which proves the proposition. □

From the proof one can deduce the following results:

Corollary B.6. *The coefficients of the homogeneous polynomials $P^k f(a)$ are suitable sums of higher partial derivatives according to Proposition* B.2. *In fact, the $P^k f$ are of the form*

$$P^k f(a)(z) = \sum_{j_1+\ldots+j_n=k} \frac{1}{j_1! \cdots j_n!} \frac{\partial^{j_1+\cdots+j_n} f(a)}{\partial {z_1}^{j_1} \cdots \partial {z_n}^{j_n}} z_1^{j_1} \cdots z_n^{j_n}. \tag{81}$$

Hence the power series development is the Taylor expansion.

Furthermore, the mappings $P^k f : U \to \mathbb{C}_{(k)}[z_1, \ldots, z_n]$ are holomorphic.

The following result is in sharp contrast to the theory of smooth functions:

Proposition B.7 (Identity Theorem)**.** *Let $f, g \in \mathcal{O}(U)$ holomorphic functions on the domain $U \subset \mathbb{C}^n$.*

1. If $P^k f(a) = P^k g(a)$ holds for all $k \in \mathbb{N}$ at one point $a \in U$ the two functions agree in all of U.

2. If $f|V = g|V$ for a nonempty open subset $V \subset U$ the two functions agree in all of U.

Proof. 1. If $P^k f(b) = P^k g(b)$ holds for all $k \in \mathbb{N}$ at $b \in U$ the two functions agree in a neighbourhood of b according to Proposition B.5. Hence the set $W := \{b \in U \mid P^k f(b) = P^k g(b) \text{ for all } k \in \mathbb{N}\}$ is open and $W \neq \emptyset$ because of $a \in W$. By the continuity of all $P^k f, P^k g)$ the intersection $W = \bigcap_{k \in \mathbb{N}} \{b \in U \mid P^k f(b) = P^k g(b)\}$ is closed as well. Therefore, $W = U$ because U is connected (we assume U to be a domain).

2. The assumption implies $P^k f(a) = P^k g(a)$ for all $k \in \mathbb{N}$ and for any $a \in V$. The result follows from 1. □

Corollary B.8. *$\mathcal{O}(U)$ is an integral domain (i.e. has no zero divisors).*

Proposition B.9 (Open Mapping)**.** *Every nonconstant $f \in \mathcal{O}(U)$ is an open mapping.*

Proof. We have to show that $f(W)$ is open in $\mathbb{C}$ for any open subset $W \subset U$. Let $a \in W$ and let D be a polydisc with $V := a + D \subset W$. There exists $b \in V$ with $f(a) \neq f(b)$ according to Proposition B.7. The function $h(\zeta) := f((1-\zeta)a + \zeta b)$ is holomorphic in the unit disc $\Delta = \{\zeta \mid |\zeta| < 1\}$ and not constant. Therefore, h is open as a holomorphic function in one variable, and $h(\Delta)$ is an open neighbourhood of $f(a)$. As a result, $f(V)$ is a neighbourhood of $f(a)$ contained in $f(W)$ which proves that $f(W)$ is open. □

Proposition B.10 (Maximum principle)**.** *If a holomorphic $f \in \mathcal{O}(U)$ attains its maximum in a point $a \in U$, i.e. $|f(a)| = \max\{|f(z)|, z \in U\}$, then f is constant.*

Proof. Otherwise, f would be open, in particular $f(U)$ would be an open subset of $\mathbb{C}$. But for an open $W \subset \mathbb{C}$ the maximum of the $|z|$, $z \in W$, is not attained: In every neighbourhood V of a point $z \in \mathbb{C}$ there e exists a point $w \in V$ with $|z| < |w|$. □

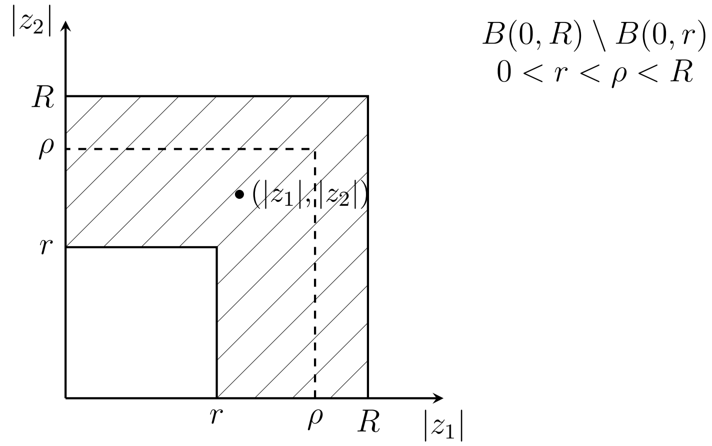
B.3 Hartogs' Extension Theorem

The extension theorem considers certain configurations of two open subsets $V, W \subset \mathbb{C}^n$, $V \subset W$ and $V \neq W$, such that every holomorphic function $f : V \to \mathbb{C}$ has a holomorphic continuation $\tilde{f} : W \to \mathbb{C}$: $\tilde{f}$ is holomorphic and $\tilde{f}|_V = f$. As a consequence, the restriction mapping $\mathcal{O}(W) \to \mathcal{O}(V)$, $g \mapsto g|_V$ is an isomorphism of vector spaces. This property of holomorphic functions in more than one variable is in strong contrast to smooth functions on open subsets of $\mathbb{R}^n$ as well as with holomorphic functions in one variable.

We prove a special case of the extension theorem of Hartogs. Let $\| \;\|$ be a norm on $\mathbb{C}^n$ and $B(a,r) := \{z \in \mathbb{C}^n \mid \|a - z\| < r\}$ the open ball of radius r around a with respect to this norm.

Proposition B.11 (Kugelsatz). *Let U be domain such that $B(a,R) \setminus B(a,r) \subset U$ for some $a \in U$ and for R, r with $0 < r < R$. Then every holomorphic function $f \in \mathcal{O}(U)$ has a unique* Holomorphic Continuation[105] *to $U \cup B(a,R)$, i.e. there is a holomorphic $\tilde{f} \in \mathcal{O}(U \cup B(a,R))$ such that $\tilde{f}|_U = f$.*

Proof. We prove the statement for the supnorm and for $n = 2$ for simplicity. We can assume $a = 0$. Let $(z_1, z_2) \in B(0,R) \setminus B(0,r)$ with $r < |z_j| < \rho < R$, $j = 1, 2$. This configuration is illustrated in the following sketch of the absolute space:



For fixed z_1 and varying z_2 the Cauchy integral in one variable yields

$$f(z_1, z_2) = \frac{1}{2\pi i} \int_{\partial D_2(\rho)} \frac{f(z_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2 \,,$$

where $D_2(\rho)$ is the open disc of radius ρ. For each fixed $\zeta_2 \in \partial D_2(\rho)$ we obtain in the same way

$$f(z_1, \zeta_2) = \frac{1}{2\pi i} \int_{\partial D_1(\rho)} \frac{f(\zeta_1, \zeta_2)}{\zeta_1 - z_1} d\zeta_1 \,.$$

Inserting one formula in the other yields

$$f(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{\partial D_1(\rho)} \int_{\partial D_2(\rho)} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2 \,. \tag{82}$$

[105] Also called analytic continuation.

This is again the Cauchy integral as presented above. The clou is, that the integral gives sense not only for the $z = (z_1, z_2)$ with $r < |z_j| < \rho$ but for all $z = (z_1, z_2)$, $|z_j| < \rho$, i.e. for all $z \in B(0, \rho)$. Therefore, (82) defines a holomorphic function $\tilde{f}(z)$ for $z \in B(0, \rho)$ by Proposition B.4, i.e. $\tilde{f} \in \mathcal{O}(B(0, \rho))$ which agrees with f on $U \cap B(0, \rho)$ and thus defines a holomorphic continuation to $U \cup B(0, R)$.

□

Corollary B.12. *A holomorphic function in 2 or more variables can have no isolated singularity, i.e. if $a \in U$ is a point in the domain U and f is holomorphic in $U \setminus \{a\}$, then f can be continued holomorphically to all of U.*

B.4 Sequences of Holomorphic Functions

The convergence behaviour of sequences of holomorphic functions in several variables is in many aspects the same as for holomorphic functions in one variable. The following is an important result which is easy to prove using the Cauchy integral representation. Note, that a corresponding result for smooth function does not hold.

Proposition B.13 (Weierstrass). *Let (f_k) be a sequence of holomorphic functions $f_k \in \mathcal{O}(U)$ which converges uniformly on compact subsets of U to a function f, Then f is holomorphic.*

Proof. Because of the uniform convergence on compacta the limit function is continuous. Hence, for $z = (z_1, \dots, z_n) \in U$ and $\overline{D(r)} \subset U$

$$
\begin{aligned}
f(z) = \lim f_k(z) &= \lim \frac{1}{(2\pi i)^n} \int_{\partial D_1} \cdots \int_{\partial D_n} \frac{f_k(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_1 \cdots d\zeta_n \\
&= \frac{1}{(2\pi i)^n} \int_{\partial D_1} \cdots \int_{\partial D_n} \frac{\lim f_k(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_1 \cdots d\zeta_n \, ,
\end{aligned}
$$

since (f_k) converges uniformly on $\partial D_1 \times \ldots \times \partial D_n$ and, therefore, integration and limit can be interchanged. As a consequence,

$$
f(z) = \frac{1}{(2\pi i)^n} \int_{\partial D_1} \cdots \int_{\partial D_n} \frac{f(\zeta_1, \dots, \zeta_n)}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)} \, d\zeta_1 \cdots d\zeta_n \, ,
$$

and f is holomorphic according to Proposition B.4. □

Note, that all partial derivatives of f_k converge – uniformly on compact subsets – to the corresponding partial derivative of f. Likewise, $P^m f_k$ converges to $P^m f$.

We present an application of this result which is useful in the context of geometric quantization of the phase space $T^*\mathbb{R}^n = \mathbb{C}^n$ (simple phase space) with respect to the holomorphic polarization. In this application a standard representation space of

Quantum Mechanics is described, the Bargmann space, also called Fock space: It is the space

$$\mathbb{F} := \{f \in \mathcal{O}(\mathbb{C}^n) \mid \int_{\mathbb{C}^n} |f(z)|^2 \exp(-\pi z\bar{z})dz < \infty\},$$

where $z\bar{z} := |z|^2 = \sum_1^n z_j\overline{z_j}$ and dz is Lebesgue integration over $\mathbb{C}^n$. Let us denote the density in the above integral by $k(z) = \exp(-\pi z\bar{z})$, and $d\mu(z) := k(z)dz$ the corresponding measure on $\mathbb{C}^n$.[106] The Bargmann space $\mathbb{F}$ is a subspace of the Hilbert space $L^2(\mathbb{C}^n, d\mu)$ of functions on $\mathbb{C}^n$ which are square integrable with respect to $d\mu$. In this way, $\mathbb{F}$ is a prehilbert space and a normed space with the norm

$$\|f\| = \|f\|_\mu = \sqrt{\int |f(z)|^2 d\mu(z)}\,.$$

Proposition B.14. *The Bargmann space $\mathbb{F}$ is complete and, hence, a closed subspace of the Hilbert space $L^2(\mathbb{C}^n, d\mu)$. In particular, $\mathbb{F}$ is a Hilbert space. The Hilbert space $\mathbb{F}$ will be denoted by $\mathbb{H}_P$ with respect to the holomorphic polarization P in the context of representation spaces determined by polarizations (e.g. in Example 10.9 and Example 10.13).*

Proof. According to Proposition B.3, for every holomorphic function $f \in \mathcal{O}(\mathbb{C}^n)$ the value $f(w)$ at a point $w \in \mathbb{C}^n$ is the average of the values $f(z)$, $z \in w + D(r)$, where $D(r)$ is any polydisc. If we assume all radii r_j to be equal, $r_j = \rho$, we obtain

$$f(w) = \left(\frac{1}{\pi\rho}\right)^n \int_{w+D(r)} f(z)dz = \left(\frac{1}{\pi\rho}\right)^n \int_{w+D(r)} \frac{1}{k(z)} f(z)k(z)dz\,.$$

Applying the Cauchy-Schwarz inequality to the right hand side leads to

$$|f(w)| \le \left(\frac{1}{\pi\rho}\right)^n \sup_{z\in D} k(z)^{-1} \|\chi_D\|_\mu \|f\|_\mu\,,$$

where the abbreviation $D = w + D(r)$ is used and where χ_X is the indicator function of X.

In particular, this inequality implies that the evaluation $\hat{w} : \mathbb{F} \to \mathbb{C}$, $f \mapsto f(w)$, is continuous.

The inequality can be extended to be uniform over compact subsets: Let $K \subset \mathbb{C}^n$ be a compact subset. Then

$$L := \bigcup_{w\in K} w + \overline{D(r)} = K + \overline{D(r)}$$

[106] The result below is true for more general densities k.

is compact as well and contained in a big polydisc $D(R) \subset \mathbb{C}^n$. For each $w \in K$ the inequality now reads.

$$|f(w)| \leq \left(\frac{1}{\pi\rho}\right)^n \sup_{z\in D(R)} k(z)^{-1} \left\|\xi_{D(R)}\right\|_\mu \|f\|_\mu . \tag{83}$$

Now let (f_k) be a Cauchy sequence in the Bargmann space $\mathbb{F}$ with respect to the norm $\| \ \|_\mu$. With $C := \left(\frac{1}{\pi\rho}\right)^n \sup_{z\in D(R)} k(z)^{-1} \left\|\chi_{D(R)}\right\|_\mu$ we obtain, by (83), for each $w \in K$

$$|f_k(w) - f_m(w)| \leq C \|f_k - f_m\|_\mu$$

and conclude: $(f_k(w))$ is a Cauchy sequence in $\mathbb{C}$ converging to a value $f(w)$ defining a function $f : \mathbb{C}^n \to \mathbb{C}$. And the convergence $f_k \to f$ is uniform on K. Hence, by Proposition B.13, f is holomorphic. Moreover, the last inequality implies, that f is in $\mathbb{F}$ and that $f_k \to f$ in the norm $\| \ \|_\mu$, i.e. $f_k \to f$ in $\mathbb{F} \subset L^2(\mathbb{C}^n, d\mu)$. □

To get to know more about the Bargmann space, we observe that $\neq \mathcal{O}(\mathbb{C}^n)$. As an example, the function

$$f(z) = \exp(\frac{\pi}{2} z_1^2)$$

is not contained in $\mathbb{F}$ since $\overline{f(z)} f(z) = \exp(\pi(x_1^2 - y_1^2))$ and hence

$$\int \bar{f} f d\mu(z) = \int_{\mathbb{C}} \exp(-2\pi y_1^2) dz_1 \int_{\mathbb{C}^{n-1}} \exp(-\pi \overline{z'} z') dz' = \infty,$$

where $x_1 = \mathrm{Re} z_1$, $y_1 = \mathrm{Im} z_1$ and $z' = (z_2, \ldots, z_n)$. Of course, $\mathbb{F}$ is not empty. The constants are in $\mathbb{F}$ as well as all complex polynomials, since the monomials $z^j := z_1^{j_1} \ldots z_n^{j_n}$ are in $\mathbb{F}$ for multiindices $j = (j_1, \ldots, j_n)$. In fact, introducing polar coordinates for each variable z_k and using Fubini's theorem it is easy to show

Lemma B.15.

$$\left\|z^j\right\|_\mu < \infty , \quad \text{and} \quad \left\langle z^j, z^k \right\rangle = 0 \ \text{ if } \ j \neq k .$$

As a consequence,

$$\mathcal{P} = \mathbb{C}[z_1, \ldots, z_n] \subsetneq \mathbb{F} \subsetneq \mathcal{O}(\mathbb{C}^n).$$

We conclude this subsection with

Proposition B.16. *The space of polynomials $\mathcal{P}$ is dense in $\mathbb{F}$. As a consequence the normed monomials*

$$m_j(z) := \frac{z^j}{\left\|z^j\right\|_\mu} , \ j \in \mathbb{N}^n,$$

form an orthonormal basis of the Hilbert space $\mathbb{F}$.

Proof. Let $f \in \mathbb{F}$. As a holomorphic function f has the power series expansion

$$f(z) = \sum_{k\in\mathbb{N}} P^f k(z) = \sum_{j\in\mathbb{N}^n} c_j z^j \, ,$$

where $c_j \in \mathbb{C}$ (cf. Proposition B.5). The convergence is uniform with respect to every polydisc $P(r)$, $r \in \mathbb{R}_+$. We want to show that the series converges in norm, as well. By the preceding lemma we know that the terms $c_j z^j$ (no summation!) are orthogonal in $\mathbb{F}$. Therefore, in order that $\sum c_j z^j$ converges to f in the norm of $\mathbb{F}$ it is sufficient, that the sequence $(|c_j| \, \|z^j\|)$ is square summable.

Restricting to a polydisc $D(r)$, the $c_j z^j$ are again orthogonal in $L^2(D(r), d\mu)$ (same proof as for the preceding lemma). The convergence of $\sum c_j z^j$ is uniform on $D(r)$, which implies, that summation and integration can be interchanged in

$$\|f\|^2_{L^2(D(r),d\mu)} = \int_{D(r)} \bar{f} f d\mu = \int_{D(r)} \sum \overline{c_j z^j} c_j z^j d\mu = \sum |c_j|^2 \int_{D(r)} \overline{z^j} z^j d\mu$$

to obtain

$$\|f\|^2_{L^2(D(r),d\mu)} = \sum |c_j|^2 \left\|z^j\right\|^2_{L^2(D(r),d\mu)} \, .$$

For $r \to \infty$ we conclude (monotone convergence)

$$\|f\|^2_{L^2(\mathbb{C}^n,d\mu)} = \sum |c_j|^2 \left\|z^j\right\|^2_{L^2(\mathbb{C}^n,d\mu)} \, ,$$

which is enough to assure the convergence $\sum c_j z^j \to f$ in the norm of $\mathbb{F}$. □

This result has the following interpretation.

Remark B.17. Let $\mathbb{V} := \mathbb{C}^{n\vee} \subset \mathbb{F}$ the space of complex linear functionals on $\mathbb{C}^n$ with the induced Hilbert space structure and let $\mathbb{V}^{\odot k}$ its k-fold symmetric tensor product. Then $\mathbb{F}$ can be identified with the symmetric Fock space of $\mathbb{V}$

$$\bigoplus_{k=0}^{\infty} \mathbb{V}^{\odot k} \, .$$

Moreover, the operators $P^k : \mathbb{F} \to \mathbb{V}^{\odot k}$ are the projections of the above decompositon of $\mathbb{F}$.

Remark B.18. It is easy to show that for a complex polynomial g the multiplication operator $M_g : \mathcal{P} \to \mathcal{P} \subset \mathbb{F}$, $f \mapsto gf := M_g(f)$ is a closed operator with domain $\mathcal{P}$ in the Hilbert space $\mathbb{F}$ (see Definition F.7) and densely defined.

B.5 Complex Manifolds

Most of the basic notions of smooth manifolds (see Section A.1) carry over to the holomorphic case.

A map $F : V \to V'$ between open subsets $V \subset \mathbb{C}^n$ and $V' \subset \mathbb{C}^m$ is holomorphic by definition, if for every $a \in V$ and $b \in \mathbb{C}^n$, the map

$$z \mapsto F(a + zb)$$

is a holomorphic map in one variable in a neighbourhood of 0 (with values in $\mathbb{C}^m$). This condition is equivalent to the property that all the components $F_k : V \to \mathbb{C}$ of $F = (F_1, \dots, F_m)$ are holomorphic in the sense Section B.1; but V is no longer assumed to be connected, in general. As before, by the result of Hartogs, F is continuous.

Definition B.19. *M is a* Complex Manifold *of dimension n, if it is a smooth manifold of real dimension $2n$ where an atlas of the differentiable structure is specified which determines the complex structure of M and which consists of* Holomorphic Charts *$(\varphi_j)_{j \in I}$ which are holomorphically compatible to each other. This means*

$$\varphi_j : U_j \to V_j \subset \mathbb{C}^n, \;\; j \in I, V_j \text{ open in } \mathbb{C}^n \,,$$

are diffeomorphisms, where $(U_j)_{j \in I}$ is an open cover of M and $V_j \subset \mathbb{C}^n$ is open, such that the transition maps:

$$\varphi_k \circ \varphi_j^{-1} : \varphi_j(U_j \cap U_k) \to \varphi_k(U_j \cap U_k)$$

are biholomorphic.

This atlas determines the complex structure by defining general holomorphic charts: A holomorphic chart on the complex manifold is a continuous map $\varphi : U \to V$ from an open $U \subset M$ to an open $V \subset \mathbb{C}^n$ such that all

$$\varphi \circ \varphi_j^{-1} : \varphi_j(U \cap U_j) \to \varphi(U \cap U_j) \,, \, j \in I \,,$$

are biholomorphic.

A mapping $F : M \to N$ between complex manifold is holomorphic if the mappings $\varphi' \circ F|_U \circ \varphi^{-1} : V \to U'$ are holomorphic for all holomorphic charts $\varphi : U \to V \subset \mathbb{C}^n$ of M and all holomorphic charts $\varphi' : U' \to V' \subset \mathbb{C}^m$ with $F(U) \subset U'$.

$$\begin{array}{ccc} U & \xrightarrow{F|_U} & U' \\ {\scriptstyle \varphi^{-1}} \uparrow & & \downarrow {\scriptstyle \varphi'} \\ V & \dashrightarrow & V' \end{array}$$

$\mathcal{O}(M, N)$ denotes the set of holomorphic maps $F : M \to N$.

Submanifolds, Products, Quotients

The notion of submanifolds, product manifolds and quotient manifolds are completely analogous to the real case (see Section A.1). The universal property of quotients, however, need some care since there exists complex manifolds which have no global holomorphic functions except for the locally constant functions.

Observation B.20. In fact, the maximum principle (Proposition B.10) implies that for a complex manifold M, which is connected and compact, the space of holomorphic functions is $\mathcal{O}(M) \cong \mathbb{C}$.

Holomorphic Vector Bundles

The notion of holomorphic vector bundle carries over as well (c.f. Definition D.1): $\pi : E \to M$ is a holomorphic vector bundle of rank r over a complex manifold M, when E is a complex manifold such that π is holomorphic and when each fibre $E_a := \pi^{-1}(a)$, $a \in M$, has the structure of a r-dimensional complex vector space. Moreover, there is an open cover $(U_j)_{j \in I}$ of M with biholomorphic maps (the local trivializations) $\psi_j : E_{U_j} := \pi^{-1}(U_j) \to U_j \times \mathbb{C}^r$ such that

1. $\pi|_{E_{U_j}} = pr_1 \circ \psi_j$, i.e. the following diagram is commutative

$$\begin{array}{ccc} E_{U_j} & \xrightarrow{\psi_j} & U_j \times \mathbb{C}^r \\ {\scriptstyle \pi|_{E_{U_j}}} \downarrow & \swarrow {\scriptstyle pr_1} & \\ U_j & & \end{array}$$

2. the restrictions $E_a \to \mathbb{C}^r$, $v \to pr_2(\psi_j(v))$, are $\mathbb{C}$-linear.

Tangent and Cotangent Bundles

In particular, for a n-dimension complex manifold M the tangent bundle $\tau : TM \to M$ is a complex manifold of dimension $2n$ and a holomorphic vector bundle of rank n, and the same is true for the cotangent bundle $\tau^* : T^*M \to M$. Some facts about these bundles are summarized:

Each curve $\gamma \in \mathcal{O}(D, M)$ defined on an open disc $D := \{z \in \mathbb{C} \mid |z| < r\}$ through the point $a := \gamma(0) \in M$ determines a (complex) Tangent Vector $X = [\gamma]_a$ at a to M: $[\gamma]_a$ is the equivalence class of (germs of) curves in M through a which is given by the equivalence relation

$$\gamma \sim_a \beta \quad \Longleftrightarrow \quad \frac{d}{dz}(\varphi \circ \gamma)|_{z=0} = \frac{d}{dz}(\varphi \circ \beta)|_{z=0},$$

where $\beta \in \mathcal{O}(D, M)$ with $\beta(0) = a$ and $\varphi : U \to V$ is a holomorphic chart with $a \in U$. The equivalence relation is independent of the holomorphic chart φ. The tangent vector $X = [\gamma]_a$ is denoted also by

$$\frac{d}{dz}\gamma|_{z=0} \;\text{ or simply }\; \dot{\gamma}(0)\,.$$

Every holomorphic chart $\varphi : U \to V$ induces a Bundle Chart (cf. A.2) $\tilde{\varphi} : TU \to V \times \mathbb{C}^n$ on the tangent bundle TM by which the complex structure on the tangent bundle is determined:

$$\tilde{\varphi} = \left(z^1, \ldots, z^n, w^1, \ldots, w^n\right) : TU \to V \times \mathbb{C}^n\,,\ [\gamma]_a \mapsto \left((\varphi(a), \frac{d}{dz}(\varphi \circ \gamma)|_{z=0}\right)\,,$$

when $\gamma(0) = a$. Here, w^j acts in the following way

$$w^j([\gamma]_a) = \left(\frac{d}{dz}(z^j \circ \gamma)|_{z=0}\right)\,.$$

The holomorphic sections of the two bundles are the holomorphic vector fields resp. holomorphic one forms.

Differential Forms

Let $\mathfrak{X}(U)$ be the $\mathcal{O}(U)$-module of the holomorphic vector fields on an open subset $U \subset M$ of a complex manifold. As in the smooth case the $\mathcal{O}(U)$-module $\Omega(U)$ of holomorphic one forms and the $\mathcal{O}(U)$-module of holomorphic vector fields $X(U)$ are in duality: The $\mathcal{O}(U)$-bilinear form

$$\langle\ ,\ \rangle : \Omega(U) \times \mathfrak{X}(U) \longrightarrow \mathcal{O}(U)\,,\ (\alpha, X) \mapsto \alpha(X)\,,$$

is non degenerate and defines the isomorphisms

$$\mathfrak{X}(U) \to \Omega(U)^*\,,\ X \mapsto (\alpha \mapsto \alpha(X) = \langle \alpha, X\rangle\ ,$$

$$\Omega(U) \to \mathfrak{X}(U)^*\,,\ \alpha \mapsto (X \mapsto \alpha(X) = \langle \alpha, X\rangle\ .$$

The holomorphic k-forms are the k-multilinear and alternating maps

$$\mathfrak{X}(U)^k \to \mathcal{O}(U)$$

over $\mathcal{O}(U)$, and the $\mathcal{O}(U)$-module of holomorphic k-forms is denoted by $\Omega^k(U)$ with $\Omega^0(U) = \mathcal{O}(U)$, and $\Omega^1(U) = \Omega(U)$.

In local coordinates given by a holomorphic chart $\varphi = (z^1, \ldots, z^n) : U \to V \subset \mathbb{C}^n$ a holomorphic vector field $X \in \mathfrak{X}(U)$ can be represented by $X = X^j\partial_j$, where ∂_j is the holomorphic vector field ${\partial_j}_a = [\varphi^{-1}(\varphi(a) + ze_j)]_a$ and $X_j \in \mathcal{O}(U)$. This vector field acts on holomorphic functions f as the Lie derivative

$$L_X f = X^j L_{\partial_j} f = X^j \frac{\partial f}{\partial z^j}\,.$$

In particular,

$$L_{\partial_j} f = \frac{\partial f}{\partial z^j} := \left.\frac{d}{dz}\right|_{z=0} f(\varphi^{-1}(\varphi(a) + ze_j))\,,$$

for holomorphic $f \in \mathcal{O}(U)$, and it seems natural to denote ∂_j by

$$\frac{\partial}{\partial z^j},$$

analogous to the case of a differentiable manifold, where a coordinate x^j generates the vector field

$$\frac{\partial}{\partial x^j}.$$

However this notation is reserved for a slightly more general definition and leads to a vector field which acts not only on holomorphic but also on differentiable functions (see below (84)).

A typical 1-form is $df \in \Omega$ defined by $df(X) = L_X f$, $X \in \mathfrak{X}(U)$, for $f \in \mathcal{O}(U)$. In particular, dz^j is dual to ∂_j with $dz^j(\partial_k) = \delta^j_k$.

A k-form $\eta \in \Omega^k(U)$ has the representation

$$\eta = \sum_{j_1+\ldots+j_n=k} \eta_{j_1,\ldots,j_n} dz^{j_1} \wedge \ldots dz^{j_n},$$

with

$$\eta_{j_1,\ldots,j_n} = \frac{1}{k!}\eta(\partial_{j_1},\ldots,\partial_{j_n}) \in \mathcal{O}(U).$$

Underlying Smooth Structure

A complex manifold considered as a differentiable manifold has differential forms which are not holomorphic. They have additional graduations coming from the complex structure of the manifold and, in addition, from the complexification of the tangent bundle. This graduation will be described in detail in the following.

The results are best motivated by investigating the local situation first.

So, let $U \subset \mathbb{C}^n$ be an open subset with standard holomorphic coordinates $(z^1,\ldots,z^n)$ and related real coordinates x^j, y^j satisfying $z^j = x^j + iy^j$. Let us regard the real tangent space T_aU, $a \in U$, considered as the tangent space of dimension $2n$ with respect to the differential structure. Then

$$\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j}\right), \; 1 \leq j \leq n,$$

is a natural basis of T_aM over $\mathbb{R}$. Recall that

$$\left.\frac{\partial}{\partial x^j}\right|_a = [a + te_j]_a \;, \text{ and } \left.\frac{\partial}{\partial y^j}\right|_a = [a + tie_j]_a \;,$$

where e_j is the standard basis of $\mathbb{C}^n$ defining the complex (and holomorphic) coordinates z^j: $z = z^j e_j$ for $z \in \mathbb{C}^n$. The

$$\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j}$$

are smooth sections in the tangent bundle and hence they are (smooth) vector fields which are not holomorphic vector fields!

The multiplication by i giving $\mathbb{C}^n$ the structure of a complex vector space carries over to the tangent space T_aU: Applied to the basic vector fields

$$\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^j}$$

we obtain

$$i\left.\frac{\partial}{\partial x^j}\right|_a = [a + tie_j]_a = \left.\frac{\partial}{\partial y^j}\right|_a, \quad \text{and} \quad i\left.\frac{\partial}{\partial y^j}\right|_a = \left[a + ti^2 e_j\right]_a = [a - te_j]_a = -\left.\frac{\partial}{\partial x^j}\right|_a.$$

As a consequence, the multiplication by i in the tangential space T_aU is the real isomorphism determined by

$$\frac{\partial}{\partial x^j} \mapsto \frac{\partial}{\partial y^j}, \quad \frac{\partial}{\partial y^j} \mapsto -\frac{\partial}{\partial x^j}.$$

Definition B.21. Let V be a real vector space. An ALMOST COMPLEX STRUCTURE on a V is a $\mathbb{R}$-linear map $J : V \to V$ with $J^2 = J \circ J = -\mathrm{id}_V$.

An ALMOST COMPLEX STRUCTURE on a differentiable manifold M is a section $J \in \Gamma(M, \mathrm{End}\,(TM))$ satisfying $J \circ J = -\mathrm{id}_{TM}$. (M, J) is called an almost complex manifold.

It is easy to see that an almost complex structure requires V to be even dimensional when V is finite dimensional. In Section 9.5 on Kähler polarizations the concept of an almost complex structure is introduced and investigated (cf. Definitions 9.19, 9.23 ff.).

Examples B.22.

1. When V is a complex vector space, then the multiplication with i induces an almost complex structure $v \mapsto iv$ since $i^2 v = -v$. Conversely, a given almost complex structure $J : V \to V$ on a real vector space defines the structure of a complex vector space on V by $(\alpha + i\beta)v := \alpha v + \beta J(v)$, $\alpha, \eta \in \mathbb{R}$, $v \in V$.

2. As we have just seen above, the real tangent space T_aU for U open in $\mathbb{C}^n$, $a \in U$ admits the almost complex structure J_a, $a \in U$ given by

$$J_a : T_aU \to T_aU, \frac{\partial}{\partial x^j} \mapsto \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^j} \mapsto -\frac{\partial}{\partial x^j},$$

which is independent of the choice of the holomorphic chart.

3. The corresponding section $J \in \Gamma(U, \mathrm{End}\,(TU))$ is an almost complex structure on U. Therefore, every complex manifold is an almost complex manifold. The converse is only true for integrable almost complex manifolds (see Theorem 9.25).

4. The symplectic involution (c.f. Section 1.1.2) defining the symplectic structure in Chapter 1 is an almost complex structure.

Now, let V be a real vector space V with an almost complex structure J and let $V^{\mathbb{C}}$ be its complexification $V \otimes \mathbb{C}$. Then J has a linear continuation $J^{\mathbb{C}} : V^{\mathbb{C}} \to V^{\mathbb{C}}$ by $J(v \otimes \lambda) = J(v) \otimes \lambda$. $J^{\mathbb{C}}$ is an almost complex structure of $V^{\mathbb{C}}$, so that on V^C we have the two different (almost) complex structures given by $J^{\mathbb{C}}$ and by the multiplication with $i : v \otimes \lambda \mapsto v \otimes i\lambda$.

$V^{\mathbb{C}}$ decomposes into the eigenspaces of $J^{\mathbb{C}}$ for the eigenvalues $i, -i$ (see (43)). This decomposition carries over to the tangent space $TM^{\mathbb{C}}$ of an almost complex manifold (see also in Section 9.5):

Proposition B.23. *Let (M, J) be an almost complex manifold. The continuation $J_a^{\mathbb{C}} : T_aM \otimes \mathbb{C} \to T_aM \otimes \mathbb{C}$ has the two complementary eigenspaces: $T_a^{(1,0)}M$ with eigenvalue i and $T_a^{(0,1)}M$ with eigenvalue $-i$.*

In case of a complex manifold M the decomposition can be described with the aid of local holomorphic coordinates z^j in an open subset $U \subset M$. The complexified tangent space $T_aU \otimes \mathbb{C}$ has the basis

$$\left(\frac{\partial}{\partial x^j} \otimes 1 \, , \; \frac{\partial}{\partial y^j} \otimes 1 \right)$$

over $\mathbb{C}$. We define

$$\frac{\partial}{\partial z^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i \frac{\partial}{\partial y^j} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i \frac{\partial}{\partial y^j} \right). \tag{84}$$

We obtain

$$T_a^{(1,0)}U = \mathrm{Ker}\,(J^{\mathbb{C}} - i) = \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z^j} \,\middle|\, j = 1, \dots, n \right\} ,$$

$$T_a^{(0,1)}U = \mathrm{Ker}\,(J^{\mathbb{C}} + i) = \mathrm{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}^j} \,\middle|\, j = 1, \dots, n \right\} .$$

Remark B.24. The decomposition $T_aM \otimes \mathbb{C} = T_a^{(1,0)}M \oplus T_a^{(0,1)}M$ holds true in all of M and remains true for the bundles:

$$TM^{\mathbb{C}} = T^{(1,0)}M \oplus T^{(0,1)}M$$

with the obvious definition for the holomorphic bundle structure on the direct sum of the bundles $T^{(1,0)}U$, $T^{(0,1)}U$.

The operator J induces J^* by $J^*(\mu) := \mu \circ J$ for $\mu \in T^*M$. $J^* \circ J^* = -\mathrm{id}$. In the same way as before we obtain:

$$T_a^*M^{\mathbb{C}} = T_a^{*(1,0)}M \oplus T_a^{*(0,1)}M$$

Corollary B.25. *The analogous decomposition holds for the cotangent bundle with respect to* J^**:*

$$T^*U \otimes \mathbb{C} = T^{*(1,0)}U \oplus T^{*(0,1)}U$$

with

$$T_a^{*(1,0)}U = \mathrm{Ker}\,(J^* - i) = \mathrm{span}_{\mathbb{C}}\{dz^j \mid j = 1, \ldots, n\}\,,$$
$$T_a^{*(0,1)}U = \mathrm{Ker}\,(J^* + i) = \mathrm{span}_{\mathbb{C}}\{d\bar{z}^j \mid j = 1, \ldots, n\}.$$

Here, for a differentiable $f \in \mathcal{E}(U)$ *the form* $df \in \Gamma(U, T^*U^{\mathbb{C}})$ *by*

$$df(\frac{\partial}{\partial z^j}) = \frac{\partial}{\partial z^j} f\,,\; df(\frac{\partial}{\partial \bar{z}^j}) = \frac{\partial}{\partial \bar{z}^j} f\,.$$

Definition B.26. We define as bundles and spaces

$$\bigwedge^{r,s} T^*U := \bigwedge^{r} T^{*(1,0)}U \wedge \bigwedge^{s} T^{*(0,1)}U$$

for $r, s \in \mathbb{N}\,,\; r + s \leq n$

$$\mathcal{A}^{r,s}(V) := \Gamma(V, \bigwedge^{r,s} T^*U)\,,\; V \subset U\,,\; V \text{ open.}$$

The k-forms $\eta \in \mathcal{A}^{r,s}(V)\,,\; k = r + s$, are called differential forms of degree (r, s) or (r, s)-forms.

Note, that $\mathcal{A}^k(V) = \bigoplus_{r+s=k} \mathcal{A}^{r,s}(V)$. In local coordinates

$$\mathcal{A}^{r,s}(V) = \left\{ \sum_{j_1,\ldots j_r;\bar{j}_1,\ldots \bar{j}_s} \eta_{j_1,\ldots j_r;\bar{j}_1,\ldots \bar{j}_s} dz^{j_1} \wedge \cdots \wedge dz^{j_n} \wedge d\bar{z}^{\bar{j}_1} \wedge \cdots \wedge d\bar{z}^{\bar{j}_n} \} \,\middle|\, \eta_{j_1,\ldots j_n;\bar{j}_1,\ldots \bar{j}_r} \in \mathcal{E}(V, \mathbb{C}) \right\}\,.$$

C Lie Theory

The concept of a Lie group is important in many areas in physics, since essentially all SYMMETRIES are formulated with the aid of Lie groups and correspondingly the INFINITESIMAL SYMMETRIES are formulated by using Lie algebras[107]. In mathematics Lie groups are equally important in particular in Differential and Algebraic Geometry as well as in Number Theory.

In this chapter we will present the basics of Lie groups and Lie algebras. Our approach will be introductory, we start from the basic definition of a Lie group, then specialize to matrix Lie groups (Section C.1), and explain several examples, in particular by semidirect products or central extensions. Afterwards, we study Lie algebras, which can be defined over general fields, a priori without emphasizing the connection to Lie groups (Section C.3). The two concepts are related as described in Section C.4, namely: to every Lie group, there is a corresponding Lie algebra. Conversely for every finite dimensional Lie algebra L over $\mathbb{R}$ there exists a Lie group whose Lie algebra is L[108]. Note, that this correspondence is not one-to-one. There might be several Lie groups with the same Lie algebra.

Symmetry often leads to an identification of elements which are related directly by the symmetry. In this way orbits are created in the space on which the symmetry group acts, and the orbit space is indeuced. If the symmetry is given by a Lie group G acting on a differentiable manifold M it is of great interest to have a natural differentiable structure on the corresponding ORBIT SPACE M/G, where natural means that it is the quotient structure. Under quite general assumptions on the action such a result is known, as we explain in the Section C.5 of this chapter.

C.1 Lie Groups

Definition C.1. A LIE GROUP G is a group, which at the same time is a manifold, such that the multiplication

$$\mu : G \times G \to G\,,\ (f,g) \mapsto \mu(f,g) := fg,$$

and the inversion

$$j : G \to G\,,\ f \mapsto j(f) = f^{-1},$$

are smooth maps. A LIE GROUP HOMOMORPHISM between Lie groups G, G' is a group homomorphism $h : G \to G'$ which is smooth.

Of course, $\mathbb{R}^n$ with the addition as group operation is a Lie group. The same is true for $\mathbb{R}^\times$ and $\mathbb{C}^\times$ with respect to multiplication. A typical Lie group homomorphism is $\exp : \mathbb{C} \to \mathbb{C}^\times$ from the additive group $\mathbb{C}$ of complex numbers onto the multiplicative group $\mathbb{C}^\times$: $\exp(z+w) = \exp z \exp w$. Moreover:

[107] This is the topic of our book [Sch95].
[108] Theorem of Ado

Examples C.2.

1. **Circle Group.** The circle group $\mathrm{U}(1) := \{z \in \mathbb{C} \mid |z| = 1\}$ with the multiplication inherited from $\mathbb{C}$ is an abelian group. Evidently, U(1) as a group is isomorphic to the group of rotation matrices

$$\mathrm{SO}(2) := \left\{ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}.$$

As a subset of $\mathbb{C}^1 \cong \mathbb{R}^2$ the group U(1) inherits a natural smooth manifold structure from $\mathbb{R}^2$, namely as a submanifold it is the circle $\mathbb{S}^1$. The multiplication

$$\mu : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \, , \; \mu(e^{i\varphi}, e^{i\psi}) = e^{i(\varphi+\psi)} \, ,$$

and the inversion

$$j : \mathbb{S}^1 \to \mathbb{S}^1, j(e^{i\varphi}) = e^{-i\varphi} \, ,$$

are clearly smooth, hence U(1) is a Lie group. Notice, that $\mathbb{C}^\times \cong \mathbb{R}_+ \times \mathrm{U}(1)$ as Lie groups.

2. **Special Unitary Group** SU(2). Important in Quantum Mechanics is the special unitary group SU(2):

$$\mathrm{SU}(2) := \{\mathrm{A} \in \mathrm{U}(2) \mid \det \mathrm{A} = 1\}.$$

$$\mathrm{U}(2) := \{A \in \mathbb{C}(2)^{109} \mid \langle Az, Aw \rangle = \langle z, w \rangle \text{ for all } \; z, w \in \mathbb{C}\},$$

where $\langle z, w \rangle := \bar{z}^1 w^1 + \bar{z}^2 w^2$ is the Hermitean scalar product on $\mathbb{C}^2$.

SU(2) can also be written as:

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \middle| \, |z|^2 + |w|^2 = 1. \right\}.$$

The equation $|z|^2 + |w|^2 = 1$ shows that SU(2) is a manifold diffeomorphic to the 3-sphere $\mathbb{S}^3$ in $\mathbb{C}^2 \cong \mathbb{R}^4$. One can easily see that matrix multiplication is smooth and – by using Cramer's rule – the same is true for building the inverse. Hence, SU(2) is a Lie group.

3. **General Linear Group.** The group $\mathrm{GL}(n, \mathbb{R})$ of all real invertible $(n \times n)$-matrices is an open subset of the $\mathbb{R}-$vector space of all $(n \times n)$-real matrices $\mathbb{R}(n) \cong \mathbb{R}^{n^2}$. This follows from:

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathbb{R}(n) : \det A \neq 0\} \, ,$$

[109] $\mathbb{K}(n)$ denotes the algebra of $(n \times n)$-matrices with coeffincients in $\mathbb{K}$.

since the map det : $\mathbb{R}(n) \to \mathbb{R}$ is continuous. The GENERAL LINEAR GROUP $\mathrm{GL}(n,\mathbb{R})$ has the structure of a differentiable manifold as open subset of $\mathbb{R}^{n^2}$. The matrix multiplication

$$\mu : \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \longrightarrow \mathrm{GL}(n,\mathbb{R}),\ (A,B) \longmapsto AB,$$

is polynomial of degree 2 in the coefficients a_i^k, b_j^i of the matrices $A = (a_i^k)$ and $B = (b_j^i)$:

$$AB = \sum_{i=1}^{n} a_i^k b_j^i.$$

Therefore, matrix multiplication is smooth (even analytic). Similarly, using Cramer's rule, one can see that the inversion is also smooth and analytic. Therefore, $\mathrm{GL}(n,\mathbb{R})$ is a Lie group. In the same way, one can show that $\mathrm{GL}(n,\mathbb{C})$, the group of invertible $n \times n$ complex matrices is a Lie group. $\mathrm{GL}(n,\mathbb{C})$ is, in addition, a complex manifold, and the group operations are holomorphic mappings.

4. **Special Linear Group:**

$$\mathrm{SL}(n,\mathbb{R}) := \{\mathrm{A} \in \mathbb{R}(n) : \det \mathrm{A} = 1\}$$
$$\mathrm{SL}(n,\mathbb{C}) := \{\mathrm{A} \in \mathbb{C}(n) : \det \mathrm{A} = 1\}$$

To show that the special linear groups are Lie groups one only has to check, that they are submanifolds of the general linear groups. This is not difficult since they are the zero sets of the differentiable function det. However, one can use a general result on closed subgroups of a Lie group to deduce the Lie property as we explain in the following.

Let us define:

Definition C.3. A MATRIX LIE GROUP or simply a matrix group is a closed subgroup G of $\mathrm{GL}(n,\mathbb{C})$, $n \in \mathbb{N}$.

Of course, $\mathrm{GL}(n,\mathbb{C})$ itself is a matrix group. Moreover, $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{K})$, $\mathrm{U}(1), \mathrm{SU}(2), \mathrm{SO}(3)$ are all matrix groups.

With some work, one can show that every matrix group is a Lie group. Since the inversion and multiplication are smooth, it is enough to show the following result (cf. in [RuS13], for example).

Proposition C.4. *A closed subgroup of a Lie group is a submanifold of the Lie group and thus a Lie group. In particular, a matrix Lie group* $G \subset \mathrm{GL}(n,\mathbb{C})$ *is a Lie group and a differentiable submanifold of* $\mathbb{R}^{4n^2}$.

A large class of geometrically induced matrix Lie groups is given by the following result:

Lemma C.5. *Let $B \in \mathbb{K}(n)$ be a non-degenerate $(n \times n)$-matrix. Then*

$$\mathrm{O}_B(n, \mathbb{R}) = \{A \in \mathbb{R}(n) \mid A^\top BA = B\}$$

in case of $\mathbb{K} = \mathbb{R}$, *resp.*

$$\mathrm{O}_B(n, \mathbb{C}) = \{A \in \mathbb{C}(n) \mid \overline{A}^\top BA = B\},$$

is a matrix Lie group.

Proof. For $A, A' \in \mathrm{O}_B(n, \mathbb{K})$ the equality $(AA')^\top BAA' = A'^\top A^\top BAA' = A'^\top BA' = B$ holds true. Furthermore, A is invertible because of $0 \neq \det B = \det A^\top \det B \det A$. Finally, $A^\top BA = B$ implies $(A^{-1})^\top BA^{-1} = B$ (resp. $(\overline{A}^{-1})^\top BA^{-1} = B$). As a result $\mathrm{O}_B(n, \mathbb{K})$ is a subgroup of $\mathrm{GL}(n\mathbb{K})$. $\mathrm{O}_B(n, \mathbb{R})$ is closed since it is the zero set of the continuous map $A \mapsto A^\top BA - B$. Hence, $\mathrm{O}_B(n, \mathbb{K})$ is a matrix group and Lie group. □

We list a couple of matrix groups of the form described in the Lemma:

Examples C.6.

1. **Orthogonal Group:**

$$\mathrm{O}(n) := \left\{A \in \mathbb{R}(n) \mid A^\top A = 1\right\} = \mathrm{O}_1(n, \mathbb{R})$$

is the Orthogonal Group. The orthogonal matrices $A \in \mathrm{O}(n)$ leave invariant the Euclidean scalar product $\langle\ ,\ \rangle$ on $\mathbb{R}^n$. $\mathrm{O}(n)$ is compact, since the coefficients of the matrices $A \in \mathrm{O}(n)$ are bounded: $|A^i_j| \leq 1$ and since $\mathrm{O}(n)$ is closed in $\mathbb{C}(n)$. It is easy to see that $|\det A| = 1$ for all $A \in \mathrm{O}(n)$. The Special Orthogonal Groupspecial orthogonal group

$$\mathrm{SO}(n) := \{A \in \mathrm{O}(n) \mid \det A = 1\}$$

is connected, and it is a proper subgroup of $\mathrm{O}(n)$. $\mathrm{O}(n)$ is not connected, since $\{A \in \mathrm{O}(n, \mathbb{R}) \mid \det A = -1\}$ is open and closed. $\mathrm{O}(n, \mathbb{R}) = \mathrm{SO}(n, \mathbb{R}) \cup \{-A \mid A \in \mathrm{SO}(n, \mathbb{R})\}$.

2. **Unitary Group.**

$$\mathrm{U(n)} := \left\{A \in \mathbb{C}(n) \mid \overline{A}^\top A = 1\right\} = \mathrm{O}_1(n, \mathbb{C})$$

is the Unitary Group, the operators $A \in \mathrm{U}(n)$ respect the hermitian scalar product. The special unitary group is

$$\mathrm{SU}(n) := \{A \in \mathrm{U}(n) : \det A = 1\}.$$

The groups $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are connected and compact.

3. **Generalized Orthogonal Group.** For $p, q \in \mathbb{N}, n = p + q$, one defines using the bilinear form
$$B(x,y) := \sum_{j=1}^{p} x^j y^j - \sum_{j=p+1}^{n} x^j y^j \,.$$
The matrix Lie groups $\mathrm{O}_B(n,\mathbb{R})$ are called the GENERALIZED ORTHOGONAL GROUPS and they are denoted by $\mathrm{O}(p,q)$. The corresponding special orthogonal groups $\mathrm{SO}(p,q)$ are defined as the connected components of the identity of $\mathrm{O}(p,q)$.

4. **Symplectic Group.** Let $I \in \mathbb{R}(2n)$ be the matrix
$$I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$
where 0 and 1 are $n \times n$ matrices (see Section 1.1). The corresponding matrix group $\mathrm{O}_I(2n,\mathbb{R})$ is the SYMPLECTIC GROUP:
$$\mathrm{Sp}(n) := \left\{ A \in \mathbb{R}(2n) : A^\top I A = I \right\} \,. \tag{85}$$

A matrix Lie group of the form $\mathrm{O}_B(n,\mathbb{K})$ can be understood as a symmetry group, namely as the group of mappings $\mathbb{K}^n \to \mathbb{K}^n$ preserving the vector space structure and the structure given by B. Many more Lie groups occur as symmetry groups, for instance the Euclidean group $E(n)$, which is the group of all differentiable maps $\mathbb{R}^n \to \mathbb{R}^n$ preserving the euclidean scalar product and the orientation, in short the isometries. Similar to the Galilean group and the Poincaré group, the Euclidean group has a description as a semidirect group.

Definition C.7. Let G, H be Lie groups and $\sigma : G \to \operatorname{Aut} H$[110] a group homomorphism such that $G \times H \to H$, $(g,h) \mapsto \sigma(g)(h)$, is smooth. The SEMIDIRECT PRODUCT of G over H (denoted by $G \ltimes H$ or $G \ltimes_\sigma H$) is the group with the underlying set $G \times H$ and the group operation
$$((g,h)(g',h')) \mapsto (gg', (\sigma(g)h')h) \,.$$

It is not difficult to check that a semidirect product $G \ltimes H$ of Lie groups is a Lie group. The underlying manifold is the product $G \times H$ of the two manifolds. The group operation $((g,h)(g',h')) \mapsto (gg', (\sigma(g)h')h)$ is smooth since σ and the group operations on G and H are smooth. The inversion $(g,h) \mapsto (g^{-1}, \sigma(g^{-1})h^{-1})$ is smooth as well. Finally, the associativity can directly be checked.

Examples C.8.

1. The product $E = G \times H$ of Lie groups is a special case of a semidirect product where $\sigma(g) = \mathrm{id}_H$.

[110] $\operatorname{Aut} H$ is the automorphism group of H, i.e. the group of Lie group isomorphisms

2. The Euclidean group $E(n)$ is (isomorphic) to the semidirect product $\mathrm{SO}(n) \ltimes \mathbb{R}^n$ with respect to the inclusion $\sigma : \mathrm{SO}(n) \subset \mathrm{GL}(n, \mathbb{R}) \subset \mathrm{Aut}(\mathbb{R}^n)$, where $\mathbb{R}^n$ is considered as the abelian Lie group.

3. The affin linear group $\mathrm{Aff}(\mathbb{R}^n)$ can be described as the semidirect product $\mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$.

4. The Galilei group Γ is isomorphic to the semidirect product of the Euclidean group $E(3)$ and the translation group $V \cong \mathbb{R}^4$: $\Gamma \cong E(3) \ltimes V$. The action $\sigma : E(3) \to \mathrm{Aut}\, V$ is defined by $\sigma(g)(q,t) = (Aq + tv, t)$ for $g = (A, w) \in \mathrm{SO}(3) \times \mathbb{R}^3$ and $(q,t) \in \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$. Eventually, $\Gamma \cong (\mathrm{SO}(3) \ltimes \mathbb{R}^3) \ltimes \mathbb{R}^4$.

5. The Poincaré group P is isomorphic to the semidirect product of the Lorentz group $\mathrm{O}(3,1)$ and the translation group $\mathbb{R}^{3+1}$: $P \cong \mathrm{O}(3,1) \ltimes \mathbb{R}^4$.

C.2 Extensions of Lie Groups

A Lie group E has a representation as a semidirect product $E = G \ltimes H$ if and only if there is an exact sequence

$$1 \longrightarrow H \stackrel{\iota}{\longrightarrow} E \stackrel{\pi}{\longrightarrow} G \longrightarrow 1$$

of Lie group homomomorphisms such that there exists a splitting $s : G \to E$, i.e. a Lie group homomorphism with $\pi \circ s = \mathrm{id}_G$.

Such a sequence is an extension of groups in the following sense:

Definition C.9. An Extension E of G by the group H is given by an *exact sequence* of group homomorphisms

$$1 \longrightarrow H \stackrel{\iota}{\longrightarrow} E \stackrel{\pi}{\longrightarrow} G \longrightarrow 1 .$$

Exactness of the sequence means that the kernel of every map in the sequence equals the image of the previous map. Hence the sequence is exact if and only if ι is injective, π is surjective, and $\mathrm{Ker}\, \pi = \mathrm{Im}\, \iota$.

The extension is called central if H is abelian and its image $\mathrm{im}\, \iota$ is in the center of E, that is

$$a \in H, b \in E \Rightarrow \iota(a)\, b = b\, \iota(a) .$$

In case all groups are Lie groups we have (central) extensions of Lie groups.

Note, that H is written multiplicatively and 1 is the neutral element even if H is supposed to be abelian.

There are extensions with abelian H which are not central, as e.g. in the above examples of semidirect products.

Examples C.10.

1. A trivial extension has the form
$$1 \longrightarrow H \stackrel{\iota}{\longrightarrow} E = G \times H \stackrel{\pi}{\longrightarrow} G \longrightarrow 1\,.$$
with $\iota(h) = (1, h)$ and $\pi = pr_1$. More generally, a central extension is called TRIVIAL if it is isomorphic to a trivial central extension, and this is, in turn, equivalent to the existence of a splitting $s : G \to E$, $\pi \circ s = \mathrm{id}_G$.

2. Given $k \in \mathbb{N}$, $k \geq 2$, a nontrivial central extension is the sequence
$$1 \longrightarrow \mathbb{Z}/k\mathbb{Z} \stackrel{\iota}{\longrightarrow} E = \mathrm{U}(1) \stackrel{\pi}{\longrightarrow} \mathrm{U}(1) \longrightarrow 1\,,$$
with $\pi(z) = z^k$ and $\iota([n]) = \exp 2\pi\frac{n}{k}$. A splitting $s : \mathrm{U}(1) \to \mathrm{U}(1)$ would correspond to a global root of $z \mapsto z^k$.

3. A familiar central extension is
$$1 \longrightarrow \mathrm{U}(1) \stackrel{\iota}{\longrightarrow} \mathrm{U}(n) \stackrel{\pi}{\longrightarrow} \mathrm{SU}(n) \longrightarrow 1\,.$$

4. The universal cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$ is a central extension
$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \stackrel{\iota}{\longrightarrow} \mathrm{SU}(2) \stackrel{\pi}{\longrightarrow} \mathrm{SO}(3) \longrightarrow 1\,,$$
as well as the corresponding cover $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}(3, 1)$:
$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \stackrel{\iota}{\longrightarrow} \mathrm{SL}(2, \mathbb{C}) \stackrel{\pi}{\longrightarrow} \mathrm{SO}(3, 1) \longrightarrow 1\,.$$

5. In general, the universal covering E of a Lie group G is again a Lie group and the projection $\pi : E \to G$ gives rise to an extension of Lie groups
$$1 \longrightarrow H = \operatorname{Ker} \pi \stackrel{\iota}{\longrightarrow} E \stackrel{\pi}{\longrightarrow} G \longrightarrow 1\,,$$

6. The METALINEAR GROUP $\mathrm{ML}(n, \mathbb{C})$ is the following 2-1-covering of the general linear group $\mathrm{GL}(n, \mathbb{C})$: $\mathrm{ML}(n, \mathbb{C}) := \{(z, A) \in \mathbb{C} \times \mathrm{GL}(n, \mathbb{C}) \mid z^2 = \det A\}$. It is a Lie group as a closed subgroup of a Lie group: $\mathrm{ML}(n, \mathbb{C}) \subset \mathrm{GL}(n+1, \mathbb{C})$. The map $\pi : \mathrm{ML}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$, $(z, A) \mapsto A$, is a Lie group homomorphism. For $A \in \mathrm{GL}(n, \mathbb{C})$ there are exactly two roots $z, -z$ of $\det A$: $z^2 = (-z)^2 = \det A$. Hence $\pi^{-1}(A) = \{(z, A), (-z, A)\}$. In case if $A = \mathrm{id}_{\mathbb{C}^n} = \mathrm{id}$ we have $\det A = 1$ and $1^2 = (-1)^2 = \det A$, Hence, The kernel of π is $\{(1, \mathrm{id}), (-1, \mathrm{id})\} \cong \mathbb{Z}/2\mathbb{Z}$, and we obtain the exact sequence of Lie groups:
$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathrm{ML}(n, \mathbb{C}) \longrightarrow \mathrm{GL}(n, \mathbb{C}) \longrightarrow 1\,.$$
The natural homomorphism $\chi : \mathrm{ML}(n, \mathbb{C}) \to \mathbb{C}^\times$, $(z, A) \mapsto z$, can be viewed as a root of the determinant on $\mathrm{GL}(n, \mathbb{C})$, since it satisfies $\chi^2 = \det \circ \pi(A)$.

7. The HEISENBERG GROUP HS_n is defined as the central group extension of the abelian group $\mathbb{R}^{2n}$ by $\mathbb{R}$: HS_n can be described by $\mathrm{HS}_n := \mathbb{R} \times \mathbb{R}^{2n}$ with the group composition

$$(r,p,q) \cdot (r',p',q',) := \left(r + r' + \frac{1}{2}(p \cdot q' - q \cdot p'), p + p', q + q' \right) ,$$

where $p, q, p', q' \in \mathbb{R}^n$, $r, r' \in \mathbb{R}$. The corresponding exact sequence of Lie groups is

$$1 \longrightarrow \mathbb{R} \longrightarrow \mathrm{HS}_n \longrightarrow \mathbb{R}^{2n} \longrightarrow 1 .$$

A variant of the Heisenberg group is the polarized version $\mathrm{HS}_n^{\mathrm{pol}} = \mathrm{U}(1) \times \mathbb{R}^n$ with exact sequence

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{HS}_n^{\mathrm{pol}} \longrightarrow \mathbb{R}^{2n} \longrightarrow 1 .$$

A coordinate free version is the following: Let (V, ω) be a symplectic vector space of dimension n. The Heisenberg group $\mathrm{HS}(V, \omega)$ is $\mathbb{R} \times V$ with the group law

$$(r,v)(r',v') = (r + r' + \frac{1}{2}\omega(v,v'), v + v') .$$

Notice, that HS_n is not abelian although $\mathbb{R}^{2n}$ and $\mathbb{R}$ are abelian.

8. An important example in the context of quantization of symmetries is the following: Let $\mathbb{H}$ be a Hilbert space and let $\mathbb{P} = \mathbb{P}(\mathbb{H})$ be the projective space of one-dimensional linear subspaces of $\mathbb{H}$ with the natural projection $\gamma : \mathbb{H} \to \mathbb{P}(\mathbb{H})$. $\mathbb{P}$ is the space of states in quantum physics, that is the quantum mechanical phase space (see Appendix F). The group $\mathrm{U}(\mathbb{H})$ of unitary operators on $\mathbb{H}$ turns out to be e central extension, as we explain in the following.

   By definition, the group $\mathrm{U}(\mathbb{P}) = \mathrm{U}(\mathbb{P}(\mathbb{H}))$ of projective unitary operators or quantum symmetries consists of the bijective mappings $V : \mathbb{P} \to \mathbb{P}$ which are induced by unitary operator $U \in \mathrm{U}(\mathbb{H})$ by $V(\gamma(\phi)) := \gamma(U(\phi))$, $\phi \in \mathbb{H}$. Let us denote $V =: \widehat{\gamma}(U)$. Then $\widehat{\gamma} : U(\mathbb{H}) \to \mathrm{U}(\mathbb{P})$ is surjective with kernel $\{\lambda\, \mathrm{id}_{\mathbb{H}} \mid \lambda \in \mathrm{U}(1)\} \cong \mathrm{U}(1)$ and yields in a natural way a nontrivial central extension of the group $\mathrm{U}(\mathbb{P})$ of (unitary) projective transformations on $\mathbb{P}$ by $\mathrm{U}(1)$:

$$1 \longrightarrow \mathrm{U}(1) \xrightarrow{\iota} \mathrm{U}(\mathbb{H}) \xrightarrow{\widehat{\gamma}} \mathrm{U}(\mathbb{P}) \longrightarrow 1 .$$

   $\mathrm{U}(\mathbb{H})$ is a topological group [111] with respect to the strong topology (see [Sch95]). The strong topology on $\mathrm{U}(\mathbb{H})$ is the topology which is generated by the subsets $V(T, \phi, \varepsilon) := \{S \in \mathrm{U}(\mathbb{H}) \mid \|S\phi - T\phi\| < \varepsilon\}$: The open subsets of

[111] i.e. the group multiplication and the inversion are continuous. Note, that in the infinite dimensional case, no Lie group (or differentiable) structure on $\mathrm{U}(\mathbb{H})$ is known. However, in finite dimensions, the Hilbert space is isomorphic to $\mathbb{C}^n$, $\mathbb{H} \cong \mathbb{C}^n$, and $\mathrm{U}(\mathbb{H}) \cong \mathrm{U}(n)$ with a Lie group quotient $\mathrm{U}(n) \to \mathbb{P}\mathrm{U}(n) \cong \mathrm{U}(\mathbb{P})$. In particular, the sequence is an exact sequence of Lie groups.

U($\mathbb{H}$) are arbitrary unions of finite intersections of sets in the subbase $\mathcal{B} = (V(T, \phi, \varepsilon) \mid T \in \mathrm{U}(\mathbb{H}), \phi \in \mathbb{H}, \varepsilon > 0)$. Note, that the strong topology is weaker than the operator norm topology when $\mathbb{H}$ is infinite dimensional. In case of finite dimension the strong topology on $\mathrm{U}(\mathbb{H}) \cong \mathrm{U}(n)$ is simply the usual topology induces from $\mathbb{C}(n)$. U($\mathbb{P}$) becomes a topological group by the quotient topology with respect to $\hat{\gamma} : \mathrm{U}(\mathbb{H}) \to \mathrm{U}(\mathbb{P})$. This topology is called the strong topology as well.

We have introduced these topologies on $\mathrm{U}(\mathbb{H}), \mathrm{U}(\mathbb{P})$ in order to state, that the above exact sequence consists of continuous maps, and therefore is an exact sequence (and a central extension) of topological groups.

Remark C.11. In the light of the last examples it is possible to explain to which extent a classical symmetry given by a Lie group G can induce a quantum symmetry. Let G act on a symplectic manifold (M, ω) by symplectomorphisms, i.e. there is an action $\Psi : M \times G \to M$ such that $q \mapsto \Psi(q, g) := \Psi_g(q)$ is a symplectomorphism. If $\mathrm{Symp}(M, \omega)$ denotes the group of diffeomorphisms which are symplectomorphisms, the action induces a group homomorphism $\Psi : G \to \mathrm{Symp}(M, \omega)$, $g \mapsto \Psi_g$. Assume now, that a Lie algebra $\mathfrak{R} \subset \mathcal{E}(M)$ of classical observables has been quantized yielding a subalgebra $q(\mathfrak{R})$ of self-adjoint operators on a Hilbert space $\mathbb{H}$. The best one can hope is that the action Ψ transforms into a projective unitary representation $\rho : G \to \mathrm{U}(\mathbb{P}(\mathbb{H}))$[112]. In general, this property can not be deduced from physics or mathematics. But of course, it can be formulated as a postulate.

Since in Quantum Mechanics calculations are essentially carried through in the Hilbert space $\mathbb{H}$ associated to the quantum mechanical phase space $\mathbb{P}$ one is interested to lift the representation $\rho : G \to \mathrm{U}(\mathbb{P})$ to a unitary representation $G \to \mathrm{U}(\mathbb{H})$ i.e. a group homomorphism which is continuous with respect to the strong topology on U($\mathbb{H}$)[113]. Such a lift will not exist in general, but it exists up to a central extension. In fact, there exist a surjective Lie group homomorphism $\pi : \hat{G} \to G$ with $\iota : \mathrm{U}(1) \cong \mathrm{Ker}\,\pi \to \hat{G}$ in the center of G and a unitary representation $\hat{\rho} : \hat{G} \to G$ such that the following diagram commutes (see, for instance [Sch08]):

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{U}(1) & \longrightarrow & \hat{G} & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \hat{\rho}} & & \downarrow{\scriptstyle \rho} & & \\ 1 & \longrightarrow & \mathrm{U}(1) & \longrightarrow & \mathrm{U}(\mathbb{H}) & \xrightarrow{\hat{\gamma}} & \mathrm{U}(\mathbb{P}) & \longrightarrow & 1 \end{array}$$

According to a theorem of Bargmann, for a simply connected Lie group G with trivial cohomology $H^2(\mathrm{Lie}\,G, \mathbb{R}) = 0$ the upper row can be replaced by a trivial central extension and therefore, the representation $\rho : G \to \mathrm{U}(\mathbb{P})$ has a direct lift $\tilde{\rho} : G \to \mathrm{U}(\mathbb{H})$

[112] ρ is a projective unitary representation if ρ is a group homomorphism and a continuous map with respect to the strong topology on U($\mathbb{P}$) defined above.

[113] often enough it is simply assumed that such a representations exists.

with $\rho = \tilde{\rho} \circ \hat{\gamma}$: $\tilde{\rho}(g) = \hat{\rho}(1, g)$, $g \in G$. The following diagram is commutative:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{U}(1) & \longrightarrow & \mathrm{U}(1) \times G & \xrightarrow{\pi} & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \hat{\rho}} & \swarrow{\scriptstyle \tilde{\rho}} & \downarrow{\scriptstyle \rho} & & \\
1 & \longrightarrow & \mathrm{U}(1) & \longrightarrow & \mathrm{U}(\mathbb{H}) & \xrightarrow{\hat{\gamma}} & \mathrm{U}(\mathbb{P}) & \longrightarrow & 1
\end{array}$$

Note, that $G = \mathrm{SU}(2)$ and $G = \mathrm{SL}(2, \mathbb{C})$ are simply connected and satisfy $H^2(\mathrm{Lie}\, G, \mathbb{R}) = 0$.

As a consequence, in the case of $G = \mathrm{SO}(3)$ the upper row in the diagram can be replaced by

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \mathrm{SU}(2) \xrightarrow{\pi} \mathrm{SO}(3) \longrightarrow 1\,,$$

and analogously in the case of $G = \mathrm{SO}(3, 1)$ by

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\iota} \mathrm{SL}(2, \mathbb{C}) \xrightarrow{\pi} \mathrm{SO}(3, 1) \longrightarrow 1\,.$$

These properties explain why it is reasonable to call SU(2) the quantum mechanical rotation group or $\mathrm{SL}(2, \mathbb{C})$ the corresponding Lorentz group.

C.3 Lie Algebras

Definition C.12. A Lie Algebra L over a field k is a k-vector space together with the map $[\ ,\ ] : L \times L \to L$ (the Lie Bracket) with the following properties: For all $X, Y, Z \in L$ and $\lambda \in k$ the Lie bracket satisfies

1. $[X + \lambda Y, Z] = [X, Z] + \lambda\, [Y, Z]$
2. $[X, Y] = -\,[Y, X]$
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$

A Lie algebra homomorphism between Lie algebras L, L' is a linear map $h : L \to L'$ with $h([X, Y]) = [h(X), h(Y)]$, $X, Y \in L$.

Examples C.13.

1. **Abelian Lie Algebra**. Every $k-$vector space L with $[X, Y] = 0$ for all $X, Y \in L$ is a Lie algebra over k, the so called trivial or abelian Lie algebra.
2. **Cross product**. $\mathbb{R}^3$ with $[X, Y] := X \times Y$ (cross product) is a three-dimensional Lie algebra over $\mathbb{R}$.

3. **Endomorphism Algebra**. Let V be a k-vector space and let $L = \mathrm{Hom}\,(V,V) := \mathrm{End}\,V$ be the k-vector space of k-linear maps (endomorphisms) from V to V. With the "commutator" $[X,Y] := X \circ Y - Y \circ X$ for $X, Y \in \mathrm{End}V$ as the Lie bracket, this defines a Lie algebra structure as can easily be shown be direct calculation. The triple $(\mathrm{End}\,V, \circ, [\,,])$ is called the ENDOMORPHISM ALGEBRA.

4. **Matrix Algebra**. In case of $k = \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $V = \mathbb{K}^n$ the endomorphism algebra consists of matrices $X \in \mathbb{K}^{n\times n}$ and the Lie subalgebras of $\mathrm{End}\,\mathbb{K}^n$ are called matrix algebras. $\mathrm{End}\,\mathbb{K}^n$ is denoted by $\mathfrak{gl}(n,\mathbb{K})$. Several matrix algebras are described further below in Examples C.18.

5. **The Lie algebra of vector fields**. Let M be a manifold. For smooth vector fields X on M let $L_X : \mathcal{E}(M) \to \mathcal{E}(M)$ be the Lie derivative. To vector fields X, Y there is a unique vector field Z, for which $L_Z = L_X \circ L_Y - L_Y \circ L_X$. Let $Z =: [X,Y]$. Therefore the $\mathbb{R}-$ vector space of vector fields on M becomes an infinite-dimensional Lie algebra $\mathfrak{V}(M)$. In a local chart q, we have:

$$[X,Y]^k = X^\mu \frac{\partial Y^k}{\partial q^\mu} - Y^\mu \frac{\partial X^k}{\partial q^\mu} \,.$$

6. **The Poisson algebra**. Let (M,ω) be the phase space of Hamiltonian Mechanics, namely a manifold M with symplectic form ω. Then $\mathcal{E}(M)$ with the Poisson bracket is a Lie algebra. Moreover, the Hamiltonian vector fields $\mathfrak{Ham}(M)$ form a Lie algebra, and the map $F \mapsto -X_F$ is a surjective Lie algebra homomorphism (cf. Corollary 1.31 and Observation 1.35).

Example C.14. (Heisenberg Algebra) Let V be a finite-dimensional vector space over $\mathbb{R}$ with a non-degenerate (constant) 2-form ω, i.e. a symplectic vector space. The HEISENBERG ALGEBRA is the central (Lie algebra) extension $\mathfrak{hs} = \mathfrak{hs}(V,\omega)$ of the abelian Lie algebra V with respect to the 2-form ω: Its underlying vector space is $\mathbb{R} \times V$ and the Lie bracket is

$$[(r,X),(s,Y)] := (rs\omega(X,Y), 0).$$

The projection $p = pr_1 : \mathbb{R} \times V \to V$ is a surjective Lie algebra homomorphism and the injection $j : \mathbb{R} \to V$, $t \mapsto (t,0)$, is an injective Lie algebra homomorphism with $\mathrm{Im}\,j = \mathrm{Ker}\,p$. As a result we have the exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{hs}(V,\omega) = \mathbb{R} \times V \longrightarrow V \longrightarrow 0\,.$$

Moreover, the elements $(r,0)$ are central in $\mathfrak{hs}(V,\omega)$: $[(r,0),(t,X)] = 0$ for all $(t,X) \in \mathfrak{hs}(V,\omega)$.

There exists a symplectic frame, i.e. a basis $(e_j),(f_k)$, $1 \le j,k \le n$, of V, such that $\omega(e_j,f_k) = \delta_{jk}$ and $\omega(e_j,e_k) = \omega(f_j,f_k) = 0$. The elements $Z := (1,0), Q_j = (1,e_j)$, $P_k = (1,f_k)$ satisfy the canonical commutation relations (CCR)

$$[P_k, Z] = [Q_j, Z] = 0\,,\ [P_j, Q_k] = -\delta_{jk} Z\,.$$

In case of $V = \mathbb{R}^n$ and the standard symplectic form on $\mathbb{R}^n$ the Heisenberg algebra is also denoted by $\mathfrak{hs}_n$ (or $\mathfrak{hs}_{2n+1}$).

One can introduce a so-called central charge $c \in \mathbb{R}$ changing the above Lie bracket to $c \cdot (0, rs\omega(X,Y))$ resp. $-c \cdot \delta_{jk}$. The result is an isomorphic algebra and the change can also be achieved by replacing ω by $c \cdot \omega$.

Remark C.15. The Heisenberg algebra yields an abstract form of the canonical commutation relations (CCR) which is an important concept of Quantum Mechanics. In Section F.3 of the Appendix F on Quantum Mechanics we study the representations of $\mathfrak{hs}_n$ and the corresponding Heisenberg group HS_n in a Hilbert space, thus realizing the canonical commutation relations. Up to unitary equivalence, the Heisenberg Lie group has only one irreducible unitary representation and this is the Schrödinger representation. In particular, $\mathfrak{hs}_n$ is not matrix algebra.

After having seen the abstract definition and concrete examples of both abelian and non-abelian Lie algebras, let us discuss the connection between Lie algebras and Lie groups. We will do this in two steps: first we discuss it for the simpler case of matrix Lie groups, then turn to abstract Lie groups.

C.4 The Lie Algebra of a Lie Group

For any matrix $X \in \mathbb{C}(n)$ the exponential series

$$e^{tX} = \sum_{\nu=0}^{\infty} \frac{1}{\nu!}(tX)^\nu \,, \; t \in \mathbb{R} \,,$$

converges in $\mathbb{C}(n)$, and e^{tX} is invertible with inverse e^{-tX}. Furthermore, the exponential map $e : \mathbb{R} \times \mathbb{C}(n) \to \mathrm{GL}(n,\mathbb{C})$ is smooth[114].

Let us begin with the Lie group $G = \mathrm{GL}(n,\mathbb{C})$. The exponential map e^{tX} induces the so-called FUNDAMENTAL FIELD, namely the left-invariant vector field $\tilde{X} \in \mathfrak{V}(G)$ defined by

$$\tilde{X}(A) := [Ae^{tX}]_A = \frac{d}{dt}\, Ae^{tX}\big|_{t=0} = AX \,,\; A \in G \,.$$

In fact, $t \mapsto Ae^{tX}$, $t \in \mathbb{R}$, is a curve γ in G through $A = \gamma(0)$ and determines the tangent vector $[\gamma]_A = \tilde{X}(A) \in T_AG$ in the tangent space T_AG at $A \in G$. Using the trivialisation $T\mathrm{GL}(n,\mathbb{C}) \cong \mathrm{GL}(n,\mathbb{C}) \times \mathbb{C}(n)$, the fundamental field can be described in the simple form $\tilde{X} : G \to TG$, $A \mapsto (A, AX)$. Hence, the fundamental field can be considered to be constant and we conclude

Assertion C.16. *The Lie bracket of $\tilde{X}, \tilde{Y}$ defined in $\mathfrak{V}(G) \cong \mathcal{E}(G, \mathbb{C}(n))$ coincides with the Lie bracket induced by the commutator in* $\mathrm{End}\,\mathbb{C}^n = \mathbb{C}(n)$: $[\tilde{X}, \tilde{Y}] = \widetilde{[X,Y]}$.

[114]even analytic!

The flow of the vector field $\tilde{X}$ (i.e. the solution of the differential equation $\dot{\Phi} = \tilde{X}(\Phi)\,, \Phi(A,0) = A$, cf. Notation A.21) is $\Phi(A,t) := Ae^{tX}\,, (A,t) \in G \times \mathbb{R}$. In particular, the vector field $\tilde{X}$ is complete.

$\tilde{X}$ is called left-invariant, since for the left multiplication $L_g : G \to G\,, A \mapsto gA\,,\ A, g \in G$, the following invariance condition holds:

$$\tilde{X} \circ L_g = TL_g \circ \tilde{X}\,.$$

This invariance gives rise to an alternative definition of the vector field $\tilde{X}$, namely

$$\tilde{X}(g) = T_e L_g(X)\,,\ g \in G\,.$$

The main result of this section is:

Proposition C.17. *Let $G \subset \mathrm{GL}(n,\mathbb{C})$ be a matrix Lie group. Then*

$$\mathfrak{g} := \mathrm{Lie}\, G := \left\{X \in \mathbb{C}(n) \mid \forall t \in \mathbb{R} : e^{tX} \in G\right\}$$

is a Lie algebra over $\mathbb{R}$ with the Lie bracket $[X,Y] = X \circ Y - Y \circ X, X, Y \in \mathfrak{g}$, i.e. Lie G *is a Lie subalgebra of the endomorphism algebra $\mathbb{C}(n)$. Moreover,*

$$\mathrm{Lie}\, G = \{\dot{\gamma}(0) : \gamma \text{ curve in } G \text{ with } \gamma(0) = \mathrm{id}_{\mathbb{C}^n} = \mathrm{e}\ \}\,.$$

Therefore, $\mathfrak{g} = \mathrm{Lie}\, G$ can be identified with the tangent space at the identity element of G.

Proof. We first show that $\mathrm{Lie}\, G = T_e G$. For each matrix $X \in \mathrm{Lie}\, G$ the curve $t \mapsto e^{tX}$ in G through $e = e^0$ induces a tangent vector $[e^{tX}]_0$ at identity $e \in G$ which can be identified with $\frac{d}{dt} e^{tX}|_{t=0} = X$. Hence, $\mathrm{Lie}\, G \subset T_e G$.

Conversely, each tangent vector $X \in T_e G$ determines a left-invariant vector field $\tilde{X}$ on G (even on all of $\mathrm{GL}(n,\mathbb{C})$) by $\tilde{X}(g) := T_e L_g(X)$. The differential equation $\dot{\gamma} = \tilde{X}(\gamma)$ has a locally unique solution $\gamma : I \to G$ on an open interval I containing 0 such that $\gamma(0) = e$. Since the flow $\Phi(A,t) = Ae^{tX}$ is also a solution of $\dot{\gamma} = \tilde{X}(\gamma)$ with $\gamma(0) = e$ the two curves agree on I, which implies $e^{tX} = \gamma(t) \in G$ for $t \in I$. It follows $e^{tX} \in G$ for all $t \in \mathbb{R}$, which implies $X \in \mathrm{Lie}\, G$. Thus $T_e G \subset \mathrm{Lie}\, G$.

$T_e G$ obtains a Lie algebra structure through the left-invariant vector fields. Given $X, Y \in T_e G$ the fundamental fields $\tilde{X}$ and $\tilde{Y}$ determine the Lie bracket $[\tilde{X}, \tilde{Y}] \in \mathfrak{V}(G)$. Then $Z := [\tilde{X}, \tilde{Y}](e) \in T_e G$ is well-defined as the bracket $Z = [X,Y]$ of X, Y. It is easy to check that in this way, $T_e G$ becomes a Lie algebra. Furthermore, by the assertion C.16 the Lie bracket $[X,Y] = [\tilde{X}, \tilde{Y}](e) \in T_e G$ for X, Y coincides with the commutator of X, Y in $\mathrm{End}\, \mathbb{C}^n$. In particular, this shows that $\mathfrak{g}$, as defined in the proposition, is a subalgebra of $\mathbb{C}(n)$. □

We now come to the general case: the notion of the Lie algebra $\mathrm{Lie}\, G$ assigned to a given abstract Lie group G. Once again, we focus on the tangent space at the identity T_eG of G and we define for each tangent vector $X \in T_eG$ the left-invariant vector field $\tilde{X}$ to X as:

$$\tilde{X}(g) := T_e L_g(X),$$

where $g \in G$ and where $L_g : G \to G$ is the left multiplication in G. $\tilde{X}$ is a vector field on the manifold G, and for two such left-invariant vector fields $\tilde{X}$ and $\tilde{Y}$ the Lie bracket $[\tilde{X}, \tilde{Y}] \in \mathfrak{V}(G)$ is a vector field on G. As a consequence, the tangent vector given by $[X, Y] := [\tilde{X}, \tilde{Y}](e) \in T_eG$ is well-defined. By construction, the tangent space T_eG with this bracket $[\,,\,]$ becomes a Lie algebra. This Lie algebra is called the Lie algebra of the Lie group G and is denoted with $\mathfrak{g}$ or $\mathrm{Lie}\, G$.

Examples C.18.

1. The Lie algebra of $\mathrm{GL}(n, \mathbb{R})$ is $\mathfrak{gl}(n, \mathbb{R}) \cong \mathbb{R}(n) \cong \mathrm{End}\, \mathbb{R}^n$.

2. The Lie algebra of $\mathrm{SL}(n, \mathbb{R})$ is $\mathfrak{sl}(n, \mathbb{R}) = \{X \in \mathbb{R}(n) \mid \mathrm{tr} X = 0\}$.

3. The Lie algebra of $\mathrm{O}(n)$ is $\mathfrak{o}(n) = \{X \in \mathbb{R}(n) \mid X^\top + X = 0\}$. Moreover, $\mathrm{Lie}\, \mathrm{SO}(n) = \mathrm{Lie}\ \mathrm{O}(n) = \mathfrak{o}(n)$.

4. The Lie algebra of $\mathrm{U}(n)$ is $\mathfrak{u}(n) = \{X \in \mathbb{C}(n) \mid \overline{X}^\top + X = 0\}$.

5 The Lie algebra of $\mathrm{SU}(n)$ is $\mathfrak{su}(n) = \{X \in \mathbb{C}(n) \mid \overline{X}^\top + X = 0 \text{ and } \mathrm{tr} X = 0\}$. Moreover, $\mathrm{Lie}\, \mathrm{SO}(3) = \mathrm{Lie}\, \mathrm{SU}(2)$.

6. The Lie algebra of $\mathrm{O}_B(n, \mathbb{R})$ is $\mathfrak{o}_B(n, \mathbb{R}) = \{X \in \mathbb{R}(n) \mid X^\top B + BX = 0\}$.

7. The Lie algebra of the Heisenberg group HS_n is the Heisenberg algebra $\mathfrak{hs}_n$.

Proposition C.19. *The assignment $G \mapsto \mathrm{Lie}\, G$ for Lie groups G is categorial in the following sense: Every Lie group homomorphism $h : G \to G'$ induces a natural Lie algebra homomorphism $\mathrm{Lie}\, h : \mathrm{Lie}\, G \to \mathrm{Lie}\, G'$ given by $\mathrm{Lie}\, h := T_e h : T_eG \to T_{e'}G'$. Any further Lie group homomorphism $h' : G' \to G''$ satisfies $\mathrm{Lie}\, h' \circ h = \mathrm{Lie}\, h' \circ \mathrm{Lie}\, h$.*

Note, that – according to result of Cartan – a continuous homomorphism between Lie groups is already smooth.

Similar to the exponential series e^X for matrices X, in the case of an abstract Lie group G, there is the EXPONENTIAL MAP

$$\exp : \mathrm{Lie}\, G \to G\,,\ X \mapsto \exp X\,,\ X \in T_eG\,.$$

To define $\exp X$ for $X \in T_eG$, we start with a solution of the autonomous differential equation $\dot{\gamma} = \tilde{X}(\gamma)$, through e, $\gamma(0) = e$. Such a curve exists because of the existence and uniqueness theorem for ordinary differential equations. It is, in general, not assured, that such a solution curve can be defined on all $\mathbb{R}$. However, for left-invariant vector fields $\tilde{X}$ on a Lie group this can be done because of the invariance, as we show in the following:

First, let the curve γ be defined only on $]-\varepsilon,\varepsilon[$ with $\dot{\gamma}=\tilde{X}(\gamma)$ and $\gamma(0)=e$. For any $g\in G$ the curve $\varphi_g(t):=g\gamma(t)\,,\ t\in]-\varepsilon,\varepsilon[$, satisfies $\dot{\varphi}_g=\tilde{X}(\phi_g)$. In fact, using $\dot{\gamma}(t)=[\gamma(t+s)]_{\gamma(t)}$, we have:

$$\begin{aligned}\dot{\varphi}_g(t)&=[g\gamma(t+s)]_{\gamma(t)}=T_{\gamma(t)}L_g\left([\gamma(t+s)]_{\gamma(t)}\right)=T_{\gamma(t)}L_g(\dot{\gamma}(t))=T_{\gamma(t)}L_g(\tilde{X}(\gamma(t))\\&=T_{\gamma(t)}L_g\circ T_eL_{\gamma(t)}(X)=T_e\left(L_g\circ L_{\gamma(t)}\right)(X)=T_eL_{g\gamma(t)}(X)=\tilde{X}(g\gamma(t))=\tilde{X}\left(\varphi_g(t)\right).\end{aligned}$$

Thus, we can define for $g:=\gamma\left(\frac{1}{2}\varepsilon\right)$:

$$\gamma_1(t):=\begin{cases}\gamma(t)\ \text{for}\ -\varepsilon<t<\varepsilon\\ \varphi_g\left(t-\frac{1}{2}\varepsilon\right)\ \text{für}\ -\varepsilon+\frac{1}{2}\varepsilon<t<\varepsilon+\frac{1}{2}\varepsilon\end{cases}$$

γ_1 is a well-defined smooth curve since on $]-\frac{1}{2}\varepsilon,\varepsilon[$ the values $\gamma(t)$ and $\varphi_g(t)=\gamma(\frac{1}{2}\varepsilon)\gamma(t-\frac{1}{2}\varepsilon)=\gamma(t)$ agree. Hence γ_1 is solution of $\dot{\gamma}=\tilde{X}(\gamma)$ on the interval $]-\varepsilon,\varepsilon+\frac{1}{2}\varepsilon[$. By repetition of the argument γ can be extended as a solution to all of $\mathbb{R}$. This solution corresponds to the exponential series e^{tX} and is denoted by $\exp tX$.

In particular, $\exp X$ is a well-defined group element $\exp X\in G$ and determines the so-called Exponential Map $\exp:\mathfrak{g}\to G$. exp is smooth and locally invertible. As a consequence, exp provides local charts for the differentiable structure of G. In particular, there is a neighbourhood $U\subset G$ of $e\in G$ and a neighbourhood $V\subset \operatorname{Lie} G$ such that the restriction $\exp|_V:V\to U$ is diffeomorphism: Near e the Lie group looks like the flat neighbourhood in the linear space $\operatorname{Lie} G$, in other words, at U the Lie group appears infinitesimally as V (modulo exp).

We have the following relation between the exponential mapping and the induced Lie algebra homomorphism:

Proposition C.20. *A Lie group homomorphism $h:G\to G'$ satisfies:*

$$h\circ\exp=\exp\circ\operatorname{Lie} h\,,$$

in other words, the following diagram is commutative:

$$\begin{array}{ccc}G & \xrightarrow{h} & G'\\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp}\\ \mathfrak{g} & \xrightarrow[\operatorname{Lie} h]{} & \mathfrak{g}'\end{array}$$

We conclude this section with

Proposition C.21. *The tangent bundle TG of a Lie group is trivial.*

Proof. In fact, the map $G\times\mathfrak{g}\to TG\,,\ (g,X)\mapsto\tilde{X}(g)$, is an isomorphism of vector bundles. □

C.5 Lie Group Action

Definition C.22. A (left) action of a group G on a manifold M is a map $\Psi : G \times M \to M$ such that the maps $\Psi_g : M \to M$, $\Psi_g(a) := \Psi(g,a) = ga$, satisfy

$$\Psi_g \circ \Psi_h = \Psi_{gh}\,,\ \Psi_e = \mathrm{id}_M\,,\ g,h \in G\,.$$

When G is a Lie group, such a group action Ψ is called a Lie Group Action if Ψ is differentiable.

Equivalently, a Lie group action of G on M consists of a group homomorphism $\Psi : G \to \mathrm{Diff}(M)$, such that the induced map $G \times M \to M$ is smooth. A smooth manifold endowed with a Lie group action of G is also called a G-manifold.

In an analogous way one defines the notion of a right Lie group action, which is used, for instance, in the context of principal fibre bundles (see Section D.2).

The left action of a Lie group will often be denoted as ga instead of $\Psi(g,a) = \Psi_g(a)$.

Definition C.23. Let $\Psi : G \times M \to M$ be a Lie group action. The Isotropy Group at $a \in M$ is the subgroup $G_a := \{g \in G \mid ga = a\}$. The Orbit through $a \in M$ is $M_a := \{ga \mid g \in G\}$.

The action Ψ is said to be

1. Transitive if for each pair $(a,b) \in M \times M$ there exists $g \in G$ with $ga = b$. i.e. all orbits M_a are $M = M_a$.
2. Effective (or faithful) if $\Psi_g = \mathrm{id}_M$ implies $g = e$, i.e. if $\Psi : G \to \mathrm{Diff}(M)$ is injective.
3. Free if all Ψ_g, $g \in G \setminus \{e\}$, have no fixed points, i.e. $\Psi_g(a) \neq a$ for all $a \in M$. Equivalently, for all $a \in M$ the isotropy groups G_a are trivial.
4. Proper if $\tilde{\Psi} : G \times M \to M \times M\,,\ (g,a) \mapsto (a, ga)$ is a proper mapping, that is, the inverse images $\tilde{\Psi}^{-1}(K)$ of compact subsets $K \subset M \times M$ are compact. Equivalently, if for all sequences $(g_n)\ inG, (p_n) \in M$ such that (p_n) and $g_n p_n)$ converge the sequence (g_n) has a convergent subsequence.

Note, that the isotropy groups are closed subgroups of G, hence they are Lie groups.

Examples C.24. Let G be a Lie group and let M be a manifold. The following are examples of Lie group actions of g on M:

1. The trivial action $\Psi_g = \mathrm{id}_M\,,\ g \in G$.

2. The action of G on itself by left multiplication, right multiplication or by conjugation. The action is transitive and free.

3. The action of a given Lie subgroup $H \subset G$ on G by left multiplication or by conjugation. For instance, U(1) acting on SU(2).

4. The action of a matrix Lie group $G \subset \mathrm{GL}(n, \mathbb{K})$ on $\mathbb{K}^n$.

5. The action of the group $\mathbb{R}$ on M given by the flow Φ of a complete vector field on M.

6. The action of the multiplicative group $\mathbb{K}^\times$ on $V \setminus \{0\}$ for a $\mathbb{K}$-vector space V ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$).

7. The adjoint action of G on its Lie algebra $\mathfrak{g}$. Similarly, the coadjoint action on $\mathfrak{g}^*$.

8. The group action of G on a principal fiber bundle $\pi : P \to M$ over a manifold M (cf. Section D.2). Here, the action is a right action of the Lie group G on P. The isotropy groups are trivial and the orbits are the fibres $P_a = \pi^{-1}(a)$, $a \in M$.

9. Hamiltonian group action on a symplectic manifold.

The following important theorem completes the discussion on quotient manifolds in Section A.4. A proof can be found e.g. in [RuS13].

Theorem C.25. *The orbit space M/G of a proper and free Lie group action exists as a quotient manifold and the quotient map $\pi : M \to M/G$ is a submersion.*

Examples C.26. 1. $P^n(\mathbb{R}) \cong \mathbb{R}^{n+1}/\mathbb{R}^\times$. $P^n(\mathbb{C}) \cong \mathbb{C}^{n+1}/\mathbb{C}^\times$. $P^n(\mathbb{C}) \cong S^{n+1}/\mathrm{U}(1)$.

2. $\mathbb{S}^2 \cong \mathrm{SU}(2)/\mathrm{U}(1)$, the Hopf fibration.

3. Let G be a compact Lie group. For every $\mu \in \mathfrak{g}^*$ the coadjoint orbit satisfies $G/G_\mu \cong M_\mu$. The theorem applies since the action of the isotropy group G_μ on G action is free and proper.

4. In the case of a principal fibre bundle $\pi : P \to M$ with structure group G: the orbit space P/G is isomorphic to M.

5. For a closed subgroup H of a Lie group G the right multiplication is free and proper. The quotient $G \to G/H$ yields a principal fibre bundle with structure group H.

6. Another important general example is the associated fibre bundle $P \times_G F$ to a principal fibre bundle P (see Section D.3). In this situation the Lie group G acts from the right on the principal fibre bundle P and from the left on the fibre F. These actions induce on $P \times F$ the right action $((p, x), g) \mapsto (pg, g^{-1}x)$, which turns out to be free and proper, as shown in Section D.3. By the theorem the orbit space $(P \times F)/G$ exists as a manifold. The orbit space is the associated fibre bundle $P \times_G F \cong (P \times F)/G$.

D Fibre Bundles

This section adds some concepts generalizing line bundles. The aim is to present line bundles and connections within the framework of vector bundles and principal fibre bundles with their associated fibre bundles and to make them available for the later chapters in these notes beginning with the chapter on half-density quantization. The line bundles are special vector bundles, namely those with 1-dimensional fibres. Also, vector bundles are in close connection with principal fibre bundles. To discover these correlations we describe the relationship between vector bundles, principal fibre bundles and their associated bundles.

D.1 Vector Bundles

Definition D.1. A VECTOR BUNDLE of rank $r \in \mathbb{N}$, $r \geq 1$, over a manifold M, is given by a total space E and a (smooth) map $\pi : E \to M$ such that:

1. For each $a \in M$, $E_a = \pi^{-1}(a)$ is an r-dimensional vector space over $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$),
2. E is locally trivial, i.e. there exists an open cover $(U_j)_{j \in I}$ of M and diffeomorphisms
$$\psi_j : E_{U_j} := E|_{\pi^{-1}(U_j)} \to U_j \times \mathbb{K}^r$$
with
   (a) the diagram
$$\begin{array}{ccc} E_{U_j} & \xrightarrow{\psi_j|_{E_{U_j}}} & U_j \times \mathbb{K}^r \\ {\scriptstyle \pi|_{\pi^{-1}(U_j)}}\big\downarrow & \swarrow{\scriptstyle pr_1} & \\ U_j & & \end{array}$$
   is commutative: $pr_1 \circ \psi_j|_{E_{U_j}} = \pi|_{\pi^{-1}(U_j)}$[115]
   (b) For all $b \in U_j$, the following induced map
$$(\psi_j)_b : E_b \xrightarrow{\psi_j|_{E_b}} \{b\} \times \mathbb{K}^r \xrightarrow{pr_2} \mathbb{K}^r$$
   is a homomorphism (in fact an isomorphism) of vector spaces over $\mathbb{K}$.

As before with line bundles, a vector bundle $\pi : E \to M$ is determined by transition functions $(g_{jk})_{j,k \in I}$ with respect to any open cover $(U_j)_{j \in I}$. The g_{jk} are defined by the trivializations (ψ_j) through the condition
$$\psi_j \circ \psi_k^{-1}(a, y) = (a, g_{jk}(a).y), \quad (a, y) \in U_{jk} \times \mathbb{K}^r,$$

[115] pr_1, pr_2 denote the natural projections $pr_1 : W \times V \to W$, $(x, y) \mapsto x$ resp. $pr_2 : W \times V \to V$, $(x, y) \mapsto y$ for a product $W \times V$.

where $g_{jk}(a).y$ stands for applying the matrix

$$g_{jk}(a) := (\psi_j)_a \circ (\psi_k)_a^{-1} \in \mathrm{GL}(r, \mathbb{K})$$

to the vector $y \in \mathbb{K}^r$. Note, that in the case of $r > 1$ the g_{jk} have their values in the group $G = \mathrm{GL}(r, \mathbb{K})$ instead of $\mathbb{K}^\times = GL(1, \mathbb{K}) = \mathbb{K} \setminus \{0\}$ in the case of line bundles:

$$g_{jk} \in \mathcal{E}(U_{jk}, G).$$

The transition functions satisfy the cocycle condition:

$$\text{(C)} \qquad \boxed{g_{ij}(a) \cdot g_{jk}(a) \cdot g_{ki}(a) = \mathrm{id}_{\mathbb{K}^r}}$$

for $a \in U_{ijk} := U_i \cap U_j \cap U_k$, where "$\cdot$" denotes matrix multiplication.

Conversely, every cocycle (g_{jk}) , $g_{jk} \in \mathcal{E}(U_{jk}, \mathrm{GL}(r, \mathbb{K}))$, yields a vector bundle of rank r. This can be proven in the same way as for the case $r = 1$, see Proposition 3.9.

Observation D.2. The direct sum, the tensor product, taking the dual, the space of endomorphisms of vector spaces carries over to the vector bundles. Thus, we obtain from two vector bundles E and F the bundles

$$E \oplus F, E \otimes F, E^\vee, \mathrm{End}\,(E, F), \bigwedge^k E, ...$$

Here $E^\vee$ is the dual bundle of E with fibre the dual $(E_a)^* = \mathrm{Hom}_{\mathbb{K}}(E_a, \mathbb{K})$ of the fibre E_a of E at $a \in M$. In particular, if (g_{jk}), resp. (h_{jk}) is a cocycle corresponding to E, resp. F, then

(g_{jk}^{-1}) is the cocycle for $E^\vee$,

$(g_{jk} \oplus h_{jk})$ is the cocycle for $E \oplus F$,

$(g_{jk} \otimes h_{jk})$ is the cocycle for $E \otimes F$,

$\left(g_{jk}^{-1} \otimes h_{jk}\right)$ is the cocycle for $\mathrm{End}\,(E, F)$,

$(g_{jk} \wedge \ldots \wedge g_{jk})$ is the cocycle for $\bigwedge^k E$.

In the case of a complex manifold M and $\mathbb{K} = \mathbb{C}$ one also studies holomorphic vector bundles, where all occurring maps are holomorphic.

D.2 Principal Fibre Bundles and Frame Bundles

Closely related to vector bundles are the principal fibre bundles with structure group being a Lie group G. Recall, that a right Lie group action of a group G on a manifold P is a (smooth) map $\Psi : P \times G \to P$, such that with the notation

$$pg := \Psi(p, g), p \in P, g \in G,$$

the associativity on the right

$$(pg)h = p(gh)\,, h \in G,$$

holds. In contrast to the notion of a left action (see Section C.5) the induced maps $\Psi_g : M \to M$, $p \mapsto pg$, satisfy the opposite of the homomorphism property, namely

$$\Psi_g \circ \Psi_h = \Psi_{hg}\ ,\ g, h \in G\,.$$

On a product $P := M \times G$ one has the natural right action, also called trivial action:

$$P \times G \to P\ ,\ \ ((a, g), h) \mapsto (a, gh)\,,\ a \in M\ ,\ g, h \in G.$$

Such a product bundle is the local version of a principal fibre bundle with structure group G. In general:

Definition D.3. A PRINCIPAL FIBRE BUNDLE ("Hauptfaserbündel") with STRUCTURE GROUP ("Strukturgruppe") G is a manifold P (the TOTAL SPACE ("Totalraum") together with a smooth projection map $\pi : P \to M$ and a differentiable RIGHT ACTION $\Psi : P \times G \to P$, with the following properties:

1° The action respects the projection, i.e. for all $(p, g) \in P \times G$ one has $\pi(pg) = \pi(p)$, and the action $G \to P_a := \pi^{-1}(a), g \to pg$, is a diffeomorphism for each $p \in P_a$, and

2° there is an open cover (U_j) of M with LOCAL TRIVIALIZATIONS

$$\psi_j : P_{U_j} := \pi^{-1}(U_j) \to U_j \times G$$

satisfying $\pi|P_{U_j} := pr_1 \circ \psi_j$ and

$$\psi_j(pg) = \psi_j(p)g,\ g \in G, p \in P_{U_j}.$$

In particular, with $\psi_j(p) = (a, h)$: $\psi_j(pg) = (a, h)g = (a, hg)$.

Similar to vector bundles, principal fibre bundles are determined by transition functions (g_{jk}) , $g_{jk} \in \mathcal{E}(U_{jk}, G)$, with respect to an open cover (U_j) of M. They are defined by the analogous condition as above:

$$\psi_j \circ \psi_k^{-1}(a, h) = (a, g_{jk}(a).h))\,,\ (a, h) \in U_{jk} \times G.$$

The condition implies $g_{jk}(a) := pr_2 \circ \psi_j \circ \psi_k^{-1}(a, 1) \in G$, where 1 is the unit in the group G and with this definition we confirm

$$\psi_j \circ \psi_k^{-1}(a, h) = \psi_j \circ \psi_k^{-1}(a, 1)h = (a, pr_2 \circ \psi_j \circ \psi_k^{-1}(a, 1))h = (a, g_{jk}(a))h = (a, g_{jk}(a).h).$$

As before, the cocycle condition is satisfied for the transition functions $g_{jk} : U_{jk} \to G$:

$$\text{(C)} \qquad \boxed{g_{ij}(a) \cdot g_{jk}(a) \cdot g_{ki}(a) = 1}$$

for $a \in U_{ijk}$, where 1 is the unit in the group G and the "$\cdot$" denotes the group multiplication, but $\cdot$ mostly is omitted.

Again, similar to the statement in Proposition 3.9 any cocycle with values in G yields a principal fibre bundle.

Construction D.4 (Frame Bundle)**.** A vector bundle $\pi : E \to M$ induces a principal fibre bundle $R(E) \to M$ with structure group being the general linear group $G = \mathrm{GL}(r, \mathbb{K})$, the so called FRAME BUNDLE ("Reperbündel") $R(E)$ of $E \to M$.

In the case of a complex line bundle $\pi : L \to M$ over M the corresponding principal fibre bundle (which is used in Section 4.2) can be obtained simply by deleting the zero section: The map, $a \to 0_a$, where 0_a is the zero element in L_a defines a section $z : M \to L$ the so called zero section. Let $L^\times := L \setminus z(M)$. Then the restriction of π to $L^\times$ defines a principal fibre bundle $L^\times$ over M with structure group $\mathbb{C}^\times = \mathrm{GL}(1, \mathbb{C})$. $L^\times$ has the "same" transition functions as L.

In the general case the frame bundle $R(E)$ of $E \to M$ can be constructed as follows: The total space $R(E)$ is fibrewise the set $R_a(E)$ of all ordered vector space bases $b = (b_1, \dots, b_r)$ of E_a: $R(E) := \bigcup_{a \in M} R_a(E)$. For $g \in G = \mathrm{GL}(r, \mathbb{K})$ one defines the right action of G on $R(E)$ by

$$(bg)_\alpha := g^\beta_\alpha b_\beta, \quad (1 \leq \rho, \sigma \leq r).$$

Then $bg = ((bg)_1, \dots, (bg)_r) \in R_a(E)$ with $(bg)h = b(gh)$ for $g, h \in \mathrm{GL}(r, \mathbb{K})$, and using elementary linear algebra, we see that the map $G \to R_a(E), g \mapsto bg$, is bijective.

$R(E)$ obtains its topological and differential structure from the local trivializations

$$\psi : E_U \to U \times \mathbb{K}^r$$

of the vector bundle E, as we see in the following:

To each $a \in U$ there corresponds a special basis $\hat{e}(a)$ of E_a depending on ψ: $\hat{e}(a) = \hat{e}_\psi := (\psi^{-1}(a, e_1), \dots, \psi^{-1}(a, e_r))$, where $(e_1, \dots, e_r) \in (\mathbb{K}^r)^r$ is the standard basis of $\mathbb{K}^r$, defined by $e_\sigma = (\delta^\rho_\sigma)$. Now, for every $b \in R_a(E)$, $a \in U$, there exists a unique matrix$\hat{\psi}(b) \in \mathrm{GL}(r, \mathbb{K})$ such that

$$b = \hat{e_\psi}(a)\hat{\psi}(b), .$$

This construction leads to the definition of the map

$$\psi^R : R(E)_U \to U \times G \,, \; b \mapsto \Big(\pi(b), \hat{\psi}(b)\Big) \,.$$

ψ^R is bijective with inverse

$$\left(\psi^R\right)^{-1}(a,g) = \hat{e}(a)g\,,\ (a,g) \in U \times G.$$

In particular,

$$\psi^R(bh) = \Big(\pi(ah), \hat{\psi}(bh)\Big) = \Big(\pi(a), \hat{\psi}(b)h\Big) = \psi^R(b)h.$$

Finally, the topology and the differentiable structure on $R(E)_U$ (and hence on $R(E)$) will be defined by requiring that all these ψ^R are diffeomorphisms.

This construction immediately yields that the transition functions of E are also transition functions of $R(E)$: Let g_{jk} be transition functions for E coming from local trivializations $\psi_j : E_{U_j} \to U_j \times \mathbb{K}^r$. Then

$$\psi_j^R \circ (\psi_k^R)^{-1}(a,1) = \psi_j^R(\hat{e}_{\psi_k}(a)) = (a, \hat{\psi}_j(\hat{e}_{\psi_k}(a))\,,$$

which imples that $g_{jk}^R := \hat{\psi}_j(\hat{e}_{\psi_k}) \in \mathcal{E}(U_{jk}$ are transition functions for $R(E)$. Moreover, because of $\hat{e}_{\psi_j}(a)\hat{\psi}_j(\hat{e}_{\psi_k}(a)) = \hat{e}_{\psi_k}(a)$ and $\hat{e}_{\psi_j}(a)g_{jk}(a) = \hat{e}_{\psi_k}(a)$ it follows $g_{jk} = \hat{\psi}_j(\hat{e}_{\psi_k}) = g_{jk}^R$. Thus, up to isomorphism, $R(E)$ can as well be defined as the principal fibre bundle given by (g_{jk}) directly.

In the case of a holomorphic vector bundle, the transition functions are holomorphic (note, that $\mathrm{GL}(r,\mathbb{C})$ is a complex Lie group). Therefore, $R(E)$ can be endowed with a complex structure and thus becomes a holomorphic principal fibre bundle.

Definition D.5. It is evident how to define homomorphisms of vector bundles over M and similarly morphisms (also called homomorphisms) of principal fibre bundles: For two such bundles $\pi : P \to M$ and $\pi' : P' \to M$ with structure group G a morphism[116] is a smooth map

$$\Theta : P \to P'$$

respecting the projections and right actions on P, P', i.e. $\pi' \circ \Theta = \pi$, described by the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\Theta} & P' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[\mathrm{id}_M]{} & M\,, \end{array}$$

and $\Theta\Psi(p,g) = \Psi'(\Theta(p)g)$, or $\Theta(pg) = \Theta(p)g\,,\ (p,g) \in P \times G$, described by the commutative commutative diagram

$$\begin{array}{ccc} P \times G & \xrightarrow{\Psi} & P \\ {\scriptstyle\Theta\times\mathrm{id}_G}\downarrow & & \downarrow{\scriptstyle\Theta} \\ P' \times G & \xrightarrow{\Psi'} & P'\,. \end{array}$$

The last property is sometimes called equivariance.

[116] More general definitions are possible, for instance allowing G to change or M to change

Such a morphism of principal bundles determines, as in the case of line bundles over M, local maps $h_j \in \mathcal{E}(U_j, G)$ satisfying

(I) $g'_{jk} = h_j g_{jk} h_k^{-1}$.

Vice versa, a family (h_j) with (I) determines a morphism $P \to P'$.

Example D.6. In particular, a principal fibre bundle P over M is called to be TRIVIAL, if there exists a morphism

$$\Theta : P \to M \times G,$$

which has an inverse Θ^{-1} as a morphism (which means here that Θ is a diffeomorphism) and thus establishing an isomorphism of principal fibre bundles. Such a morphism Θ is given by a map $\theta : M \to G$ satisfying $\theta(pg) = \theta(p)g$ such that $\Theta(p) = (\pi(p), \theta(p))$, $p \in P$.

When transition functions g_{jk} for P are given, then P is trivial if and only if there exist local functions $h_j \in \mathcal{E}(M, G)$ such that

$$g_{jk} = h_j {h_k}^{-1}.$$

In the context of bundles, one studies the SECTIONS of the bundles. In case of a vector bundle $\pi : E \to M$ the set of sections over an open $U \subset M$,

$$\Gamma(U, E) := \{s \in \mathcal{E}(U, E) \mid \pi \circ s = \mathrm{id}_U\}$$

is an $\mathcal{E}(U)$-module in a natural way. For a principal fibre bundle $\pi : P \to M$ the set of sections over an open $U \subset M$,

$$\Gamma(U, P) := \{\sigma \in \mathcal{E}(U, P) \mid \pi \circ \sigma = \mathrm{id}_U\}$$

comes with a natural right G-action

$$\Gamma(U, P) \times G \to \Gamma(U, P), (\sigma, g) \mapsto \sigma g,$$

where $\sigma g(a) := \sigma(a)g$.

As an example, the

$$\hat{e} : U \to R(E)$$

in the construction of the frame bundle above is a section.

Note, that a principal fibre bundle $\pi : P \to M$ admitting a global section $\sigma \in \Gamma(M, P)$ is already trivial:

$$\Theta_\sigma : P \to M \times G\,,\ \sigma(a)g \mapsto (a, g)$$

defines a trivialization with inverse

$$\Theta_\sigma^{-1} : M \times G \to P\,,\ (a, g) \mapsto \sigma(a)g\,.$$

In this way, every local section $\sigma : U \to P$ provides a local trivialization: $\Theta_\sigma : P_U \to U \times G$

$$\Theta_\sigma(p) = (\pi(p), \theta(p)),\ \ p \in P_U,$$

where $\theta(p) \in G$ is the unique group element with $p = \sigma(\pi(p))\theta(b)$, and where $\theta \in \mathcal{E}(P_U, G)$ satisfies $\theta(pg) = \theta(p)g$.

D.3 Associated bundles

How to come back from principal fibre bundles to vector bundles? Consider a principal fibre bundle $\pi : P \to M$ with structure group G and let F be a manifold (the general fibre) with a differentiable left action on F by G, denoted

$$\Lambda : G \times F \to F\,,\ (g,x) \mapsto gx := \Lambda(x).$$

(cf. Section C.5 for the notion of a Lie group action). Being a Lie group action includes that the associativity in the following form holds: $h(gx) = (hg)x\,,\ g,h \in G, x \in F$.

We define $P \times_G F := P \times F/\sim$, where the equivalence relation is

$$(p,x) \sim (p',x') \iff \exists g \in G : (p',x') = \left(pg, g^{-1}x\right).$$

Note, that the equivalence classes are the orbits of the following right action on $P \times F$ induced by the left action Λ and the given right action on the principal fibre bundle P:

$$(P \times F) \times G \to P \times F\,,\ \left((p,x),g\right) \mapsto (pg, g^{-1}x)\,.$$

The result is a fibre bundle $P \times_G F \to M$

- with the projection $\pi_F : P \times_G F \to M\,,\ [(p,x)] \to \pi(p)$,
- with general fibre F, where for $a \in M$ and $p_0 \in P_a$ the map $[(p_0,x)] \mapsto x\,,\ x \in F$ is a diffeomorphism $F_a := \pi_F^{-1}(a) \cong F$,
- with structure group G,
- and with the right action $(P \times_G F) \times G \to P \times_G P$ where $([(p,x)],g) \to [(p,g^{-1}x)]$ for $[(p,x)] \in P \times_G F\,,\ g \in G$.

$P \times_G F$ is called the Associated Fibre Bundle.

We have to ensure that the quotient exists as a differentiable manifold. This can be done by applying the result on the existence of the orbits space X/G of a Lie group action on a manifold X (cf. Theorem C.25): Since the quotient $P \times_G F := P \times F/\sim$ is the orbit space $(P \times F)/G$ with respect to the above mentioned right Lie group action of G on $P \times F$ it is enough, according to the theorem, to check that the action Φ is free and proper.

Another argumentation to show that the quotient manifold exists is to use the local trivializations of $P \to M$ to verify that they yield local trivializations of $\pi_F : P \times_G F \to M$ and that the trivializations glue together to obtain the desired manifold structure of the quotient. Yet another proof uses the transition functions induced by the original bundle $\pi : P \to M$ and the Lie group action on F, see Proposition D.9 below.

The sections $s : U \to P \times_G F$ of the associated bundle are defined as before: s is a section if s is smooth and $\pi \circ s = \mathrm{id}_U$. Let $\Gamma(U, P \times_G F)$ denote the $\mathcal{E}(U)$-module of sections of $P \times_G F$ over U. The sections can be described by functions on $P_U \to F$ with an equivariance or invariance property:

Proposition D.7. $\Gamma(U, P \times_G F)$ *is isomorphic to the* $\mathcal{E}(U)$*-module*

$$\mathcal{E}_G(P_U, F) := \{f \in \mathcal{E}(P_U, F) \mid \forall g \in G : f(pg) = g^{-1}f(p)\}\,.$$

Proof. For $f \in \mathcal{E}_G(P_U, F)$ we define $s_f(a) := [(p, f(p))]\,,\ a \in U$, where $\pi(p) = a$. $s_f(a)$ is well-defined, since for another $p' \in P_a$ there exists a unique $g \in G$ with $p' = pg$. Therefore, $(p', f(p')) = (pg, f(pg)) = (pg, g^{-1}f(p)$, by the invariance property of f, and consequently $(p', f(p')) \sim (p, f(p))$. It is easy to see that s_f is a section, and $\mathcal{E}_G(P_U, F) \to \Gamma(U, P \times_G F)\,,\ f \mapsto s_f$, is linear over $\mathcal{E}(U)$ and injective. Finally the surjectivity follows from the inverse construction: a section $s : U \to P \times_G F\,,\ s(a) = [(p, x)]$ determines a map $s^\sharp(p) := x$ with $s^\sharp(pg) = g^{-1}x = g^{-1}s^\sharp(p)$. This $s^\sharp$ is well-defined and satisfies $s^\sharp \in \mathcal{E}_G(P_U, F)$ with $s_{s^\sharp} = s$. □

ASSOCIATED VECTOR BUNDLE

We now concentrate on the special case of a vector space $F = \mathbb{C}^r$ as the general fibre and on the left action on F given by a Lie group representation $\rho : G \to \mathrm{GL}(r, \mathbb{C})$, i.e. ρ is a smooth homomorphism. The induced left action on $F = \mathbb{C}^r$ is

$$gx = \rho(g)x\,,\ x \in \mathbb{C}^r\,,\ g \in G\,.$$

$\rho(g)x$ defines indeed a left action: $(hg)x = \rho(hg)x = \rho(h)\rho(g)x = \rho(h)(gx) = h(gx)$. On the basis of such a representation one obtains a vector bundle $P \times_G \mathbb{C}^r = P \times_\rho \mathbb{C}^r$ of rank r, also denoted by E_ρ. The last result leads to the following.

Corollary D.8. $\Gamma(M, E_\rho)$ *is isomorphic to the* $\mathcal{E}(U)$*-module*

$$\mathcal{E}_\rho(P, F) := \{f \in \mathcal{E}(P, F) \mid \forall g \in G : f(pg) = \rho(g)^{-1}f(p)\}\,.$$

Let us compare the transition functions $g_{jk} \in \mathcal{E}(U_{jk}, G)$ of the principal fibre bundle $\pi : P \to M$ with structure group G and the corresponding transition functions of the associated vector bundle E_ρ induced by a representation $\rho : G \to \mathrm{GL}(r, \mathbb{C})$.

The result is: $\rho(g_{jk}) \in \mathcal{E}(U_{jk}, \mathrm{GL}(r, \mathbb{C}))$ can serve as the transition functions of E_ρ and a corresponding result holds for the case of a general associated fibre bundle. We describe this in some detail.

Proposition D.9. *Let* $\Lambda : G \times F \to F$ *be a left action on a manifold* F *in the form of the induced map* $\rho : G \to \mathrm{Diff}(F)$ *(where* $\mathrm{Diff}(F)$ *is the group of diffeomorphisms* $F \to F$*) given by*

$$\rho(g)(x) := \Lambda(g, x) = gx\,,\ (g, x) \in G \times F.$$

Then ρ *is a group homomorphism. Moreover, the associated bundle* $P \times_G F$ *has as suitable transition functions* g^F_{jk} *the functions* $\rho(g_{jk}) : U_{jk} \to \mathrm{Diff}(F)$ *where* $g_{jk} \in \mathcal{E}(U_{jk}, G)$ *are transition functions of* P.

Proof. In order to show this result we compare in the situation of a local trivialization $\psi : P_U \to U \times G$ of P over an open $U \subset M$ the quotients

$$P_U \times F \to P_U \times_G F \text{ and } \eta^F : (U \times G) \times F \to (U \times G) \times_G F.$$

Because of

$$\eta^F((a,h),x) := [((a,h),x)] = \{((a,hg),\rho(g^{-1})x) : g \in G\},$$

for $(a,h) \in U \times G$ and $x \in F$ this equivalence class has a unique representative of the form $((a,1),\rho(h)x) \in (U \times G) \times F$. Using this, we identify $(U \times G) \times_G F$ with $U \times F$ and note that the quotient map now is

$$\eta^F : (U \times G) \times F \to U \times F\,,\ ((a,h),x) \mapsto (a,\rho(h)x).$$

As a result, a suitable trivialization of $P_U \times_G F$ is

$$\psi^F : P_U \times_G F \to U \times_G F\,,\ [(p,x)] \mapsto (\pi(p),\rho(h(p))x),$$

if $\psi(p) = (\pi(p),h(p))$, $h : P_U \to G, p \in P_U$. Now, we conclude

$$\rho_j^F \circ \left(\rho_k^F\right)^{-1}(a,x) = \rho_j^F\left(\left[\left(\rho_k^{-1}(a,1),x\right)\right]\right) = \left(a,\rho\left(\psi_j \circ \psi_k^{-1}(a,1)x\right)\right).$$

This equality yields the desired result

$$g_{ij}^F = \rho(g_{ij}) : U_{ij} \to \mathrm{Diff}(F),$$

respectively

$$\rho(g_{ij}) : U_{ij} \to \mathrm{GL}(r,\mathbb{C})$$

in the case of $\rho : G \to \mathrm{GL}(r,\mathbb{C})$. □

We finally present an example of the process of creating vector bundles as associated fibre bundles induced by a representation – an example which we already know.

Example D.10. M is the projective space $\mathbb{P}^n(\mathbb{C})$ of complex dimension n. On M we consider the dual $H = T^\vee$ of the tautological line bundle $T \to M$, i.e. the hyperplane bundle. H is determined by the transition functions

$$g_{jk}(z_0 : \cdots : z_n) = \frac{z_k}{z_j},\ \ (z_0 : \cdots : z_n) \in U_{jk} = U_j \cap U_k$$

with respect to the homogeneous coordinates $(z_0 : \cdots : z_n)$.

The corresponding principal fibre bundle of H, the frame bundle of H, is $H^\times \to M$ with structure group $\mathbb{C}^\times = \mathrm{GL}(1,\mathbb{C})$. This bundle is determined by the same transition functions g_{jk}.

Now, to each representation $\rho_m : \mathbb{C}^\times \to GL(1,\mathbb{C}), z \mapsto z^m$, where $m \in \mathbb{Z}$, we obtain the associated vector bundle E_{ρ_m}, which is a line bundle with the transition functions

$$\rho_m(g_{jk}) = g_{jk}^m = \left(\frac{z_k}{z_j}\right)^m$$

according to our proposition. Hence, for $m \in \mathbb{Z}$ the associated line bundle E_{ρ_m} is equivalent to our previously defined line bundle $H(m) = H^{\otimes m}$, see Construction 3.18 in Section 3.3.

D.4 Principal Connection

In Chapter 4 we present at least 5 different ways to define the concept of a connection on a line bundle $L \to M$: In form of a collection of covariant derivatives ∇_X in 4.1, through local connection forms α_j on the base manifold M in 4.3, through a global connection form α on the frame bundle $L^\times$ in 4.7, through a horizontal subbundle H of $TL^\times$ in 4.8, and through a vertical projection on the tangent bundle $TL^\times$ in 4.9. We now introduce connections in principal fibre bundles. In this way, we can regard connections on line bundles in the framework of general connections. Some parts become more complicated, but others look simpler in the general case.

Let $\pi : P \to M$ be a principal fibre bundle with structure group G (as in Section D.2). G is a Lie group and we restrict, for simplicity to matrix groups, i.e. to closed subgroups of the general linear group $\mathrm{GL}(m, \mathbb{C})$ for some $m \in \mathbb{N}$. We have a right action of G on P, that is a differentiable map

$$\Psi : P \times G \to P\,,\ (p, g) \longmapsto \Psi(p, g),$$

(mostly written in the form $pg := \Psi(p, g)$) satisfying $pe = p$ ($e \in G$ is the unit) and $(pg)h = p(gh)$ for all $g, h \in G$ and $p \in P$. The action is compatible with π, i.e. $\pi(pg) = \pi(p)$ for all $g \in G$ and $p \in P$, and it is free, meaning that the map $G \ni g \longmapsto pg \in P$ is a diffeomorphism onto the fibre $P_a = \bar{\pi}^{-1}(a)\,,\ a = \pi(p)$. Most importantly, the projection $\pi : P \to G$, is locally trivial: Each $a \in M$ has an open neighbourhood $U \subset M$ with a diffeomorphism

$$\psi : P_U = \bar{\pi}^{-1}(U) \to U \times G$$

which respects π and the right action: $pr_1 \circ \psi = \pi|_{P_U}$ and $\psi(pg) = \psi(p)g$ for all $p \in P_U$ and $g \in G$ - where the action on $U \times G$ is the standard right action: $(a, h)g = (a, hg)$. Finally, let n denote the dimension of M and k the dimension of G.

Let $\mathfrak{g} = \mathrm{Lie}\, G$ denote the Lie algebra of G and $\exp : \mathfrak{g} \to G$ the exponential map.

Definition D.11. The FUNDAMENTAL FIELD associated to $X \in \mathfrak{g}$ is the vector field $\tilde{X} \in \mathfrak{V}(P)$ given by

$$\widetilde{X}(p) := \frac{d}{dt}(p \exp(tX))|_{t=0} = [p \exp(tX)]_p \in T_pP.$$

Compare with the case of $\mathbb{C}^\times = G$ and $\mathbb{C} = \mathrm{Lie}\, G : X = \xi \in \mathbb{C}, 2\pi i\tilde{\xi} = Y_\xi$ (Definition 4.5).

For each point $p \in P$ the tangent vector $\widetilde{X}(p) \in T_pP$ points in the direction of the fibre $P_a\,,\ a = \pi(p)$, which is the same as to say that $\widetilde{X}(p) \in \mathrm{Ker}\, T_p\pi \subset T_pP$:

$$T_p\pi(\widetilde{X}(p)) = [\pi(p \exp(Xt))]_a = 0$$

since $\pi(p\exp(Xt)) = \pi(p) = a$ is constant.

For each $a \in M$ and $p \in P_a$ the tangent space to the fibre P_a at p agrees with the kernel of $T_p\pi$:

$$T_p\,(P_a) = \operatorname{Ker} T_p\pi.$$

We call $V := \operatorname{Ker} T\pi \subset TP$ the VERTICAL BUNDLE. The inclusion $V \subset TP$ induces the structure of a real vector bundle on V of (real) dimension $k := \dim G = \dim_{\mathbb{R}} T_pP_a$, where the projection $V \to P$ of the vector bundle V is the restriction $\tau|_V$ of the projection of the tangent bundle $\tau : TP \to P$.

Lemma D.12. *The fibres of V are isomorphic to $\mathfrak{g}$, where $\mathfrak{G}$ is considered as a real vector space. The isomorphism is given by*

$$\mathfrak{g} \to T_pP_a\ ,\ \ X \longmapsto \widetilde{X}(p)\,,$$

with $a = \pi(p)$. In particular,

$$V_p = \operatorname{Ker} T_p\pi = T_pP_a = \{\widetilde{X}(p) : X \in \mathfrak{g}\}\,,$$

Proof. For each $p \in P$ the map

$$\mathfrak{g} \to T_pP_a\ ,\ \ X \longmapsto \widetilde{X}(p)$$

is $\mathbb{R}$-linear and injective, since $\widetilde{X}(p) = 0$ means $p\exp(Xt)$ is constant, hence $X = 0$. Its image is all of T_pP_a because of $\dim_{\mathbb{R}} \mathfrak{g} = \dim_{\mathbb{R}} T_pP_a$. Hence $X \longmapsto \widetilde{X}(p)$ is an isomorphism. The other equalities have been shown in the considerations before. □

As a result, $X \mapsto \widetilde{X}(p)$ has an inverse $\sigma_p : V_p \to \mathfrak{g}$ and the induced map

$$\sigma : V \to P \times \mathfrak{g}\,,\ \sigma(v) := (\tau(v), \sigma_p(v))\,, v \in V,$$

turns out to be a diffeomorphism with $pr_1 \circ \sigma = \tau$ and $\sigma_p : V_p \to \{p\} \times \mathfrak{g}$ linear. Therefore, $\sigma : V \to P \times \mathfrak{g} \cong P \times \mathbb{R}^k$ is a vector bundle isomorphism. Altogether, we have shown

Proposition D.13. *The vertical bundle $V \subset TP$ is a trivial vector subbundle of TP of rank k over $\mathbb{R}$.*

PRINCIPAL CONNECTIONS:

Now, we introduce the concept of a connection on P. Before that, let us extend the notion of a differential 1-form on P to the vector valued case: A one form with values in $\mathbb{K}^r$ is an $\mathcal{E}(P)$-linear map $\alpha : \mathfrak{V}(P) \to \mathcal{E}(P, \mathbb{K}^r)$ and one denotes the space of these 1-forms by

$$\mathcal{A}^1(P, \mathbb{K}^r) := \operatorname{Hom}_{\mathcal{E}(P)}(\mathfrak{V}(P), \mathcal{E}(P, \mathbb{K}^r)) \cong \mathcal{A}^1(P) \otimes \mathbb{K}^r.$$

In the same way we have $\mathcal{A}^1(P, \mathfrak{g}) := \operatorname{Hom}_{\mathcal{E}(P)}(\mathfrak{V}(P), \mathcal{E}(P, \mathfrak{g}))$.

Definition D.14. A (global) CONNECTION FORM on P is a one form $\alpha \in \mathcal{A}^1(P, \mathfrak{g})$ satisfying

(I1) $\quad \alpha(\tilde{X}) = X \quad$ for all $X \in \mathfrak{g}$

(I2) $\quad \Psi_g^* \alpha = g^{-1} \alpha g \quad$ for all $g \in G$

Here, $\Psi_g : P \to P$ is the diffeomorphism induced by the right action Ψ of P: $\Psi_g(p) = pg$, $\Psi_g : p \longmapsto pg$. $g^{-1}\alpha(X)g$ is well-defined since for a matrix group $G \subset \mathrm{GL}(m, \mathbb{C})$ the Lie algebra $\mathfrak{g} = \mathrm{Lie}\, G$ is a subalgebra $\mathfrak{g} \subset \mathrm{Lie\ GL}(m, \mathbb{C}) = \mathbb{C}^{m \times m}$ of the LIe algebra $\mathbb{C}^{m \times m}$ of all $m \times m$-matrices. Hence, $g^{-1}\alpha(X)g$ is simply defined by matrix multiplication. And every $g \in G$ induces a map $\mathfrak{g} \to \mathfrak{g}$, $X \mapsto g^{-1}Xg$ (which is $\mathrm{ad}_{g^{-1}} : \mathfrak{g} \to \mathfrak{g}$).

Evidently, the conditions (I1), (I2) above agree with the line bundle case (c.f. Proposition 4.7), since $c^{-1}\alpha c = \alpha$ in the case of $c \in \mathbb{C}^\times = G$.

Given a principal fibre bundle $\pi : P \to M$ with a connection, i.e. a connection form $\alpha \in \mathcal{A}^1(P, \mathfrak{g})$ with (I1) and (I2) we obtain an associated HORIZONTAL BUNDLE $H \subset TP$ in the following way:

$$H := \mathrm{Ker}\, \alpha \subset TP,$$

with the fibres $H_p = \mathrm{Ker}\ \alpha_p = \{Y_p \in T_pP \mid \alpha_p(Y_p) = 0\}$. Since $\alpha_p|_{V_p} : V_p \to \mathfrak{g}$ is an isomorphism (by (I1) and Lemma D.12) the dimension of H_p is $n = \dim M$ (note, that the dimension of P is $n+k$). Therefore, the induced structure from TP yields on H the structure of a real vector bundle of dimension n. Since $H_p \cap V_p = \{0\}$ for all $p \in P$:, we obtain the decomposition

$$TP = H \oplus V$$

of TP into the direct sum of two real vector subbundles H, V of TP. The action Ψ_g induces an isomorphism $T_p\Psi_g : H_p \to H_{pg}$ for all $(p, g) \in P \times G$. We have shown the first half of the following.

Proposition D.15. *A connection form $\alpha \in \mathcal{A}^1(P)$ on a principal fibre bundle $P \to M$ defines the horizontal bundle $H := \mathrm{Ker}\, \alpha$ with*

(H1) $\quad TP = H \oplus V$

(H2) $\quad T_p\Psi_g(H_p) = H_{pg}$ *for all* $(p, g) \in P \times G$

Conversely, any vector subbundle $H \subset TP$ satisfying (H1) and (H2) induces a connection form $\alpha \in \mathcal{A}^1(P)$ with $H = \mathrm{Ker}\, \alpha$.

Proof. To define α using (H1) and (H2) let $v : TP \to TP$ be the projection which fibrewise is the linear projection $v_p : T_pP \to T_pP$ with $\mathrm{Ker}\, v_p = H_p$ and $\mathrm{Im}\, v_p = V_p$. v is a vector bundle homomorphism. Now, $\alpha := \sigma \circ v$ as the map

$$p \longmapsto \alpha_p = \sigma_p \circ v_p \in \mathrm{End}_{\mathbb{R}}(T_pP, \mathfrak{g}) \ , \ p \in P \,,$$

induces the one form $\alpha \in \mathcal{A}^1(P, \mathfrak{g})$ through

$$\alpha(X)(p) := \sigma_p \circ v_p(X_p) \in \mathfrak{g}\,,\ X \in \mathfrak{V}(P)\,.$$

Evidently, we have $H = \operatorname{Ker} \alpha$. It remains to show that α is a connection form. (I1) is immediate, since $\alpha(\tilde{X}) = \sigma \circ v(\tilde{X}) = \sigma(\tilde{X}) = X$ for $X \in \mathfrak{g}$. To show (I2) let $Y \in H_p$, i.e. $\alpha(Y) = \sigma(v(Y)) = 0$. Then $\Psi_g^* \alpha(Y) = \sigma \circ v \circ T\Psi_g(Y) = 0$, by (H2). Consequently $\Psi_g^* \alpha(Y) = 0 = g^{-1}\alpha(Y)g$. For $X \in \mathfrak{g}$ we know

$$T_p\Psi_g(\tilde{X}(p)) = [\Psi_g(p \exp tX)]_{pg} = [pgg^{-1} \exp tXg]_{pg} = [pg \exp(g^{-1}tXg)]_{pg} = \widetilde{g^{-1}Xg}(pg).$$

Therefore, because of $v_{pg}(\widetilde{g^{-1}Xg}(pg)) = \widetilde{g^{-1}Xg}(pg)$,

$$\Psi_g^* \alpha(\tilde{X})(p) = \sigma_{pg} \circ v_{pg}(T_p\Psi_g(\tilde{X}(p)) = \sigma_{pg} \circ v_{pg}(\widetilde{g^{-1}Xg}(pg)) = g^{-1}Xg.$$

This proves (I2). □

Exploiting the preceding proof we obtain another description of a principal connection (given by a one form α satisfying the above axioms or by a decomposition $H \oplus P$ of TP with (H1) and (H2)):

Proposition D.16. *A principal connection is also given by a vector bundle homomorphism $v : TP \to TP$ with the properties:*

(V1) $v \circ v = v$ and $\Im v = V$,

(V2) $T\Psi_g \circ v = v \circ T\Psi_g$ *for each* $g \in G$.

Proof. With the complementary vector bundle $H := \operatorname{Ker} v \subset TP$ we have $TP = H \oplus V$. And $T\Psi_g(Y) \in H_{pg}$ for $Y \in H_p$ by (V2), hence (H2).

Conversely, a decomposition $TP = H \oplus V$ with (H2) immediately yields the projection $v : TP \to TP$ onto V satisfying

$$T\Psi_g \circ v = v \circ T\Psi_g.$$

□

ln physics, P is called the space of phase factors, and α is the (global) gauge potential.

We obtain local gauge potentials by pullback: Let $s : U \to P$ be a section, i.e. smooth and $s \circ \pi = \mathrm{id}_U$ ($U \subset M$ open). Denote

$$A^s := s^*\alpha \in \mathcal{A}^1(U, \mathfrak{g}).$$

Then A^s is called a local gauge potential (given by s). How do these local gauge potentials fit together?

Proposition D.17. *Given two sections $s, s' : U \to P$ over $U \subset P$ the corresponding local gauge potentials*

$$A = s^*\alpha, A' = s'^*\alpha$$

satisfy

$$\boxed{A' = gAg^{-1} + gdg^{-1}}$$

where $g(a) \in G$ is the uniquely defined group element with $s(a) = s'(a)g(a)$.

Proof. Let $Y \in T_aM$ be given by the curve $\gamma(t)$, i.e. $Y = [\gamma]_a$. Then

$$s'^*\alpha(Y)(a) = \alpha_{s'(a)}\left(\frac{d}{dt}s' \circ \gamma(t)\Big|_{t=0}\right) \text{ and}$$

$$\frac{d}{dt}s' \circ \gamma(t)\Big|_{t=0} = \frac{d}{dt}\left(s \cdot g^{-1}\right) \circ \gamma(t)\Big|_{t=0} = \frac{d}{dt}(s \circ \gamma(t))\left(g^{-1} \circ \gamma(t)\right)\Big|_{t=0}$$

$$= \frac{d}{dt}s \circ \gamma(t)g^{-1}(a)\Big|_{t=0} + \frac{d}{dt}s(a) \cdot g^{-1} \circ \gamma(t)\Big|_{t=0}$$

With $h := g^{-1}(a) \in G$, the first term is

$$\frac{d}{dt}\Psi_h \circ s \circ \gamma(t)\Big|_{t=0} = T_p\Psi_h\left(T_a s(Y)\right), p = s(a)\,.$$

To analyze the second term $\frac{d}{dt}s(a) \cdot g^{-1} \circ \gamma(t)\big|_{t=0} = \frac{d}{dt}s'(a)g(a)g^{-1} \circ \gamma(t)\big|_{t=0}$ we observe that $X := \frac{d}{dt}g(a)g^{-1}(\gamma(t))\big|_{t=0}$ is a tangential vector $X \in T_eG$, since $g(a)g^{-1}(\gamma(0)) = e =$ identity, and can be viewed as to be a Lie algebra element $X \in \mathfrak{g} \cong T_eG$. Since $g(a)g^{-1}(\gamma(t))$ is the matrix multiplication: $X = g(a)\frac{d}{dt}g^{-1}(\gamma(t))\big|_{t=0} = g(a)T_ag^{-1}(Y)$. For every curve $\eta(t)$ in G with $\eta(0) = e$ and $[\eta]_e = X$ one has

$$\tilde{X}\left(p'\right) = \frac{d}{dt}\left(p'\eta(t)\right)\Big|_{t=0}, p' \in P$$

in particular for $\eta(t) = g(a)g^{-1}(\gamma(t))$: According to (H1), we have $\alpha_{p'}\left(\tilde{X}\left(p'\right)\right) = X$ and we conclude $(p' = s'(a))$:

1. $\alpha_{p'}\left(\frac{d}{dt}s'(a)g(a)g^{-1}(\gamma(t))\right)\big|_{t=0} = X = g(a)T_ag^{-1}(Y)$.
   Moreover, for the first term $T_p\Psi_h\left(T_as(Y)\right)$ the second condition reads

$$\alpha_{ph}\left(T_p\Psi_h\left(T_as(Y)\right) = h^{-1}\alpha_p\left(T_as(Y)\right)h,\right.$$

   and because of $p' = ph, h^{-1} = g(a)$ and $A_a(Y) = s^*\alpha(Y) = \alpha_p\left(T_as(Y)\right)$ we obtain

2. $\alpha_{p'}\left(T_p\Psi_h\left(T_as(Y)\right) = g(a)A_a(Y)g^{-1}(a)\right.$

Putting everything together we have

$$A'_a(Y) = \alpha_{p'}\left(\frac{d}{dt}s'\gamma(t)\right)_{t=0} = \alpha_{p'}\left(T_p\Psi_h\left(T_a s(Y)\right) + \frac{d}{dt}s(a)g^{-1}\gamma(t)\Big|_{t=0}\right)$$
$$= g(a)A_a(Y)g^{-1}(a) + g(a)T_aG^{-1}(Y)$$

□

As before, local trivalizations (here given by local sections) over U_j with respect to an an open cover (U_j) of M yield local gauge potentials A_j with a transition rule and vice versa. In detail:
Let $s_j : u_j \to p$ smooth sections with its corresponding trivialization $\varphi_j : P_{u_j} \to U_j \times G, \quad p \mapsto (\pi p, \hat{s}_j(p))$, where $\hat{s}_j(p) \in G$ is given by $p = s_j(\pi(p))\,\hat{s}_j(p)$. These data induce transition functions $g_{jk} \in \mathcal{E}(U_{jk}, G)$, given by $g_{jk} = \hat{s}_k \circ \hat{s}_j^{-1}$. In particular, (g_{jk}) satisfies (C), and

$$s_j = s_k g_{jk} \quad \text{on} \quad U_{jk} \neq \varnothing$$

Let $A_j := s_j^*\alpha \in \mathcal{A}^1(U_j, \mathfrak{g})$.

Proposition D.18. *Let α be a connection form on the principal fibre bundle $P \to M$. Define the local gauge potentials A_j and the transition functions g_{ik} as above (depending on the sections s_j). Then*

$$\text{(Z)} \qquad A_k = g_{jk}A_j g_{jk}^{-1} + g_{jk}dg_{jk}^{-1}.$$

Conversely, a collection (A_j) of 1-forms with (Z) defines a connection form α whose local gauge potentials are the A_j.

Note that (Z) is essentially the same as (Z) in the line bundle case above for the case $G = \mathbb{C}^\times$ since $dgg^{-1} + gdg^{-1} = 0$ in general and $g\alpha g^{-1} = \alpha$ for $g \in \mathbb{C}^\times$.

D.5 Connection on a Vector Bundle

In our proposition 4.16 we have already anticipated that there is also a general procedure in which a given connection on the principal fibre bundle $L^\times$ induces a connection on the line bundle L: One simply uses the globally given connection form $\alpha \in \mathcal{A}^1(L^\times)$ and local sections $s_j : U_j \to L$ (where $(U_j)_{j\in I}$ is an open cover) to obtain $\alpha_j := s_j^*\alpha \in \mathcal{A}^1(U_j)$ satisfying (Z) and thus defining a connection on L.

This procedure has its generalization to associated vector bundles of a principal fibre bundle $P \to M$ with connection: Let $\rho : G \to \mathrm{GL}(r, \mathbb{C})$ be a representation of the of structure group G of $P \to M$. The associated vector bundle E_ρ is

$$E_\rho = P \times_\rho \mathbb{C}^r \longrightarrow M$$

the quotient manifold of $P \times \mathbb{C}^r$ with respect to the equivalence relation

$$(p,v) \sim (p',v') \iff \text{there exits } g \in G \text{ with } (p',v') = \left(pg, \rho(\left(g^{-1}\right) v\right)$$
$$E_\rho := P \times \mathbb{C}^\rho / \sim .$$

Assume that the connection on $P \to M$ is given by a connection one form $\alpha \in \mathcal{A}^1(P)$. Let $(U_j)_{j\in I}$ be an open cover of M with sections $s_j : U_j \to P$, and define α_j by $\alpha_j := \rho_* \left(s_j^* \alpha\right) \in \mathcal{A}^1\left(U_j, \mathbb{C}^r\right)$. Here, the representation $\operatorname{Lie}\rho : \mathfrak{g} \to \mathfrak{g}\left(\mathbb{C}^r\right) = \operatorname{End}_{\mathbb{C}}\left(\mathbb{C}^r\right)$ is induced by ρ :

$$\operatorname{Lie}\rho(X) := \frac{d}{dt}\rho(\exp tX)\bigg|_{t=0} \in \mathfrak{g}\left(\mathbb{C}^r\right)$$

And it leads to the definition $\rho_*\beta(Y) := \operatorname{Lie}\rho(\beta(Y)) \in \mathbb{C}^r$ for a $\mathfrak{g}$-valued form $\beta \in \mathcal{A}^1(U,\mathfrak{g})$ and $Y \in \mathfrak{V}(U)$.

The collection (α_j) satisfies the compatibility condition

$$(\mathrm{Z_g}) \qquad \alpha_k = \tilde{g}_{jk}\alpha_j\tilde{g}_{jk}^{-1} + \tilde{g}_{jk}d\tilde{g}_{jk}^{-1},$$

where $\tilde{g}_{jk} := \rho\left(g_{jk}\right)$. These (α_j) induce (as in D.14) a covariant derivative

$$\nabla_X : \Gamma\left(U, E_\rho\right) \to \Gamma\left(U, E_\rho\right), \tag{86}$$

$U \subset M$ open, compatible with restrictions to $V \subset U$ and satisfying (K1) and (K2) with L replaced by E_ρ.

Proposition D.19. *On every line bundle over a paracompact manifold there exists a connection.*

Proof. We take an open cover $(\varphi_j)_{j\in I}$ over which a given line bundle $L \longrightarrow M$ is trivial with trivializations $\varphi_j : L_{U_j} \to Y_j \times \mathbb{C}$ and local sections $s_j(a) = \varphi_j^{-1}(a,1)$, $a \in U_j.\omega_e$ choose $\beta_j \in \mathcal{A}^1\left(U_j\right)$ and obtain the connections $\nabla_X^{(j)} f s_j := \left(L_X f + 2\pi i \beta_j(x) f\right) s_j$ on $L_{U_j}, j \in I$. Let (k_j) be a partition of unity subordinate to (U_j), i.e $k_j \in \mathcal{E}(M)$, $\operatorname{Supp} k_j \subset U_j, (k_j)$ locally finite and $\sum_{j\in I} k_j(a) = 1$ for each $a \in M$. Then

$$\nabla_X s := \sum_{i\in I} k_i \nabla_X^{(i)} s|_{U_i \cap U} \quad , s \in \Gamma(U,L)$$

defines a connection on L. □

The sum of 2 connections ∇, ∇' on $L \to M$ is is general not a connection. The difference $\nabla - \nabla'$ is a one form on M. In fact, for $X \in \mathcal{D}(M)$ and $s \in \Gamma(M,L)$ the equation

$$\left(\nabla_X - \nabla'_X\right) s = \beta(X)s$$

defines a value $\beta(X)(a) \in \mathbb{C}$ for $s(a) \neq 0$. Because of

$$\left(\nabla_X - \nabla'_X\right) fs = L_X fs + f\nabla_X s - L_X fs - f\nabla'_X s = f\left(\nabla_X - \nabla'_X\right) s$$

this value is independent of s. Since for every $a \in M$ there exists $s \in \Gamma(M, L)$ with $s(a) \neq 0$ we obtain a uniquely defined $\beta(x) \in \mathcal{E}(M)$ such that $X \mapsto \beta(X)$ is $\mathcal{E}(M)$-linear, and hence $\beta \in \mathcal{A}^1(M)$.

Proposition D.20. *Given a fixed connection ∇ on L, every other connection on L has the form*

$$\nabla' = \nabla + \beta$$

for an arbitrary $\beta \in \mathcal{A}^1(M)$. The set of connections is the affine space $\nabla_X + \mathcal{A}^1(M)$. This set can therefore be understood as an affine subspace of $\mathcal{A}^1(L^\times)$.

Proof. It only remains to check that $\nabla_X + \beta, \beta \in \mathcal{A}^1(M)$, is a connection. The last statement follows from 4.3.

Isomorphism classes of line bundles with connections are classified by cohomology, similar to the description of $\mathrm{Pic}^\infty(M) \cong H^1(M, \varepsilon^\times)$, now using (g_{ij}) and (α_i). □

E Cohomology

In this chapter, we give a short introduction to Čech cohomology and compare it with de Rham cohomology. One major objective is to show that these two cohomology theories on a manifold are equivalent.

It is helpful but not necessary to base everything in terms of sheaves. To support this aspect, we proceed in a twofold way: We first present Čech cohomology with values in an abelian group G. In particular, we compare the cases $G = \mathbb{R}, G = \mathbb{C}$ with de Rham cohomology. We then extend the situation to sheaves. As a result, we see that Čech cohomology with values in sheaves is not much more complicated or difficult than the case of Čech cohomology with values in groups, - except for the notion of a sheaf, which is a bit involved.

In the following, M will be a topological space, which we assume to be paracompact. We are mainly interested in the case of a paracompact manifold.

The quite restricted but already interesting case is the following:

E.1 Čech Cohomology with Values in an Abelian Group

Let G be a fixed abelian group. The elements of G are sometimes called coefficients in the context of cohomology with values in G.

Definition E.1. For open $U \subset M$ we define

$$\mathcal{F}(U) = \mathcal{F}(U, G) := \{g : U \to G \mid g \text{ locally constant}\} .$$

In another formulation one has $\mathcal{F}(U) = \{g : U \to G \mid g \text{ continuous}\}$ when G is endowed with the discrete topology (which is the topology where all subsets $H \subset G$ are open).

$\mathcal{F}(U)$ is an abelian group for each open subset $U \subset M$ by pointwise multiplication or addition depending of whether the composition in G is written multiplicatively or additively. We choose the additive notation in the following. To every inclusion

$$V \subset U \text{ of open subsets } U, V \subset M$$

there corresponds the natural RESTRICTION MAP

$$\rho_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V) , \; g \mapsto g|_V .$$

Lemma E.2. *$\rho_{V,U}$ is a group homomorphism and for open subsets $W, V, U \subset M$ with $W \subset V \subset U$ the identities*

$$\rho_{W,V} \circ \rho_{V,U} = \rho_{W,U} , \quad \rho_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$$

are satisfied.

This property resembles the cocycle condition (C) in Section 3.2 for cocycles $(g_{ij})_{i,j\in I}$ induced by line bundles. Here, with respect to the index set $I := \{U \subset M \mid U \text{ open}\}$.

The case $G = \mathbb{R}$ or $G = \mathbb{C}$ is particularly interesting for manifolds. It leads to de Rham cohomology in case M is a manifold. In Section 15.5 and in Section 16.3 we need the cases $G = \mathbb{Z}$ and $G = \mathbb{Z}_2$.

When the abelian group E is a vector space over $\mathbb{R}$, the groups $\mathcal{F}(U) = \mathcal{F}(U, E)$ are $\mathbb{R}$-vector spaces as well and the restrictions $\rho_{U,V}$ are linear maps.

Definition E.3. Let $\mathfrak{U} = (U_j)_{j\in I}$ be an open cover of M.

1. A basic q-SIMPLEX σ ($q \in \mathbb{N}$) is an ordered $(q+1)$ tuple $(U_{j_0}, U_{j_1}, \ldots, U_{j_q})$, $j_i \in I$, such that
$$U_{j_0\ldots j_q} := U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_q} \neq \varnothing\,.$$
$|\sigma| = U_{i_0\ldots i_q}$ is the SUPPORT of the q-simplex σ.

2. $\Sigma_b(q)$ denotes the set of such basic q-simplices. The set $\Sigma(q)$ is the collection of finite (formal sums) of q-simplices, i.e.
$$\Sigma(q) := \left\{ \sum_{\mu} n^{\mu}\sigma_{\mu} \mid \sigma_{\mu} \in \Sigma_b(q)\,,\, n^{\mu} \in \mathbb{Z}, \text{ only finitely many } n^{\mu} \neq 0 \right\}.$$
$\Sigma(q)$ is an abelian group with respect to the obvious addition, namely the so-called free abelian group $\Sigma_b(q)^{(\mathbb{Z})}$ spanned by $\Sigma_b(q)$. The finite sums in $\Sigma(q)$ are called q-simplices.

3. Let $q \in \mathbb{N}, q > 0$. The k-th PARTIAL BOUNDARY $\partial_k\sigma$ of a q-simplex $\sigma = (U_{j_0}, U_{j_1}, \ldots, U_{j_q}) \in \Sigma_b(q)$, $0 \le k \le q$, is the $(q-1)$-simplex
$$\partial_k\sigma := (U_{j_0}, \ldots, \widehat{U_{j_k}}, \ldots U_{j_q}),$$
where $\widehat{U_{j_k}}$ means, that the entry U_{j_k} has to be omitted.

4. The BOUNDARY ∂_σ of σ is
$$\partial\sigma := \sum_{k=0}^{q} (-1)^k \partial_k\sigma.$$
The map $\partial : \Sigma_b(q) \to \Sigma(q-1)$ induces a homomorphism $\partial : \Sigma(q) \to \Sigma(q-1)$ by $\partial(\sum n^{\mu}\sigma_{\mu}) := \sum n^{\mu}\partial\sigma_{\mu}$.

We describe q-simplices for low q in some detail:

In the case of $q = 0$ a 0-simplex in $\Sigma_b(0)$ is of the form $\sigma = (U_j)$ where $j \in I$ is a fixed index, and a general element $\tau \in \Sigma(0)$ is a finite sum of the form $\tau = n^\mu U_{j_\mu}$ where $n^\mu \in \mathbb{Z}$, and only finitely many of the n^μ are not zero. σ and τ have no boundary.

In the case of $q = 1$ a 1-simplex in $\Sigma_b(1)$ is of the form $\sigma = (U_i, U_j)$ or $\sigma = (U_{j_0}, U_{j_1})$, where $i, j, j_0, j_1 \in I$. The support of σ is $|\sigma| = U_i \cap U_j = U_{ij}$. The partial boundaries are $\partial_0 \sigma = (U_j)\,,\ \partial_1 \sigma = (U_i)$. And the boundary is

$$\partial\sigma = (U_j) - (U_i)\,, \quad \text{for} \;\; \sigma = (U_i, U_j)\,.$$

In the case of $q = 2$ a 2-simplex in $\Sigma_b(2)$ is of the form $\sigma = (U_i, U_j, U_k)$ where $i, j, k \in I$. The support of σ is $|\sigma| = U_i \cap U_j \cap U_k = U_{ijk}$. The partial boundaries are $\partial_0\sigma = (U_j, U_k)\,,\ \partial_1\sigma = (U_i, U_k)\,,\ \partial_2(U_i, U_j)$. And the boundary is

$$\partial\sigma = (U_j, U_k) - (U_i, U_k) + (U_i, U_j)\,, \quad \text{for} \;\; \sigma = (U_i, U_j, U_k)\,.$$

In the following we fix an abelian group G.

Definition E.4. Let G be an abelian group and consider the induced groups $\mathcal{F}(U) = \mathcal{F}(U, G)$ of locally constant functions (cf. Definition E.1) with the restriction homomorphisms $\rho_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V), V \subset U \subset M$. Moreover, let $\mathfrak{U}$ be an open cover of M.

1. A q-COCHAIN of $\mathfrak{U}$ with coefficients in $\mathcal{F}$ (or in G) is a family

   $$\eta = (\eta_\sigma)_{\sigma \in \Sigma_b(q)}, \eta_\sigma \in \mathcal{F}(|\sigma|)\,,$$

   or a map on $\Sigma_b(q)$

   $$\sigma \mapsto \eta(\sigma) = \eta_\sigma \in \mathcal{F}(|\sigma|),\ \sigma \;\text{ a q-simplex of }\; \mathfrak{U}.$$

   $C^q(\mathfrak{U}, \mathcal{F}) = C^q(\mathfrak{U}, G)$ denotes the abelian group of q-cochains with pointwise addition: $(\eta + \eta')_\sigma := \eta_\sigma + \eta'_\sigma$. For a q-simplex $\sigma = (U_{j_0}, \ldots\ldots, U_{j_k})$ a q-cochain $\eta = (\eta_\sigma)$ will often be written in the form $\eta = (\eta_{j_0,\ldots,j_k})$ with $\eta_{j_0,\ldots,j_k} := \eta_\sigma = \eta_{(U_{j_0},\ldots\ldots,U_{j_k})}$.

2. The COBOUNDARY OPERATOR $\delta = \delta^q$

   $$\delta^q : C^q(\mathfrak{U}, \mathcal{F}) \to C^{q+1}(\mathfrak{U}, \mathcal{F})$$

   is the homomorphism given by

   $$\boxed{\delta^q \eta(\sigma) := \sum_{j=0}^{q+1} (-1)^j \rho_{|\sigma|,|\partial_j\sigma|}(\eta(\partial_j\sigma)).}$$

   Here, σ is a $(q+1)$-simplex, hence $\partial_j\sigma$ is a q-simplex and $\eta(\partial_j\sigma) \in \mathcal{F}(|\partial_j\sigma|)$. This element will be restricted to $|\sigma|$ as $\rho_{|\sigma|,|\partial_j\sigma|}$ to obtain the summands $(-1)^j \rho_{|\sigma|,|\partial_j\sigma|}\eta(\partial_j\sigma)$ in the above formula defining $\delta^q\eta \in C^{q+1}(\mathfrak{U}, \mathcal{F})$.

   A q-cochain η with $\delta^q\eta = 0$ is called a q-COCYCLE.

It is easy to show that δ is a homomorphism and that $\delta^2 = 0$ i.e. $\delta^{q+1} \circ \delta^q = 0$. Hence, the following notations make sense:

$$\begin{aligned} Z^q(\mathfrak{U}, G) &:= \operatorname{Ker}\left(\delta^q : C^q(\mathfrak{U}, G) \to C^{q+1}(\mathfrak{U}, G)\right), \\ B^q(\mathfrak{U}, G) &:= \operatorname{Im}\left(\delta^{q-1} : C^{q-1}(\mathfrak{U}, G) \to C^q(\mathfrak{U}, G)\right) \\ \check{H}^q(\mathfrak{U}, G) &:= Z^q(\mathfrak{U}, G)/B^q(\mathfrak{U}, G). \end{aligned}$$

As an example, we consider the case $q = 1$. A 1-cochain $\eta \in C^1(\mathfrak{U}, G)$ is given by $\eta(\sigma) = \eta_{(U_i,U_j)} \in \mathcal{F}(U_{ij})$ written as $\eta_{ij} := \eta_{(U_i,U_j)} \in \mathcal{F}(U_{ij})$. The coboundary of η is

$$(\delta^1\eta)_{ijk} = \eta_{jk} - \eta_{ik} + \eta_{ij},$$

where the restrictions ρ of the η_{jk} etc. to U_{ijk} have been suppressed. Therefore, if $\eta_{ij} + \eta_{jk} + \eta_{ki} = 0$ for all $i, j, k \in I$, in particular $\eta_{ii} = 0$ and $\eta_{ij} = -\eta_{ji}$ the cochain is a cocycle $\eta \in Z^1(\mathfrak{U}, G)$ and determines a cohomology class $[\eta] \in \check{H}^1(\mathfrak{U}, G)$. Two such η, η' are equivalent (and hence determine the sam class) if and only if there is a 0-cochain $h = (h_j)$ with $\eta_{jk} = \eta'_{jk} + h_k - h_j$.

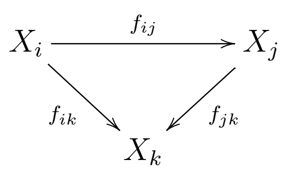
The cohomology groups $\check{H}^q(\mathfrak{U}, G)$ are interesting algebraic invariants of the topological space M. But one is interested to find invariants which do not depend on special open coverings $\mathfrak{U}$ of M. To get rid of this dependence one uses direct limits, also called inductive limits or, in the language of modern category theory, colimits.

Definition E.5.

1. Let $(I, \leq)$ be a directed set. That is, $\leq$ is a reflexive and transitive partial order on I such that every pair of elements $i, j \in I$ has an upper bound $k \in I$, $i \leq k$, $j \leq k$.

2. Let $(X_i)_{i \in I}$ be a family of structured objects, for instance groups, vector spaces, topological spaces, rings, etc.[117] We stick to the case of groups in the following. Let $f_{ij} : X_i \to X_j$ be a homomorphism for all $i \leq j$ with the following properties:

   - $f_{ii} = \mathrm{id}_{X_i}$, and
   - $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$.



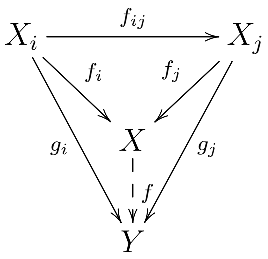
   Then the pair (X_i, f_{ij}) is called a direct system over I.

[117] or, in general, objects in a category

3. A direct limit of the direct system (X_i, f_{ij}) is a group X with homomorphisms $f_j : X_j \to X$ satisfying $f_i = f_j \circ f_{ij}$ for $i \leq j$, such that the following universal property is fulfilled: If $g_j : X_j \to Y$ are homomorphisms with $g_i = g_j \circ f_{ij}$ for $i \leq j$, then there exists a unique homomorphism $f : X \to Y$ such that

$$f \circ f_j = g_j \,.$$

The universal property is expressed by the diagram:



The direct limit of the direct system (X_i, f_{ij}) is unique up to isomorphism. It is denoted by $\varinjlim X_i$ if it exists. In the case of groups X_j it always exists and can be constructed as follows. One starts with the disjoint union X_i 's $\bigsqcup_i X_i$, and defines the following equivalence relation: For $x_i \in X_i$ and $x_j \in X_j$, $x_i \sim x_j$ if there is $k \in I$ with $i \leq k$ and $j \leq k$ for which $f_{ik}(x_i) = f_{jk}(x_j)$. Now, $\varinjlim X_i := \bigsqcup_i X_i / \sim$ and

$$f_i : X_i \to \varinjlim X_i \,,\; x_i \mapsto [x_i]$$

sending each element to its equivalence class. The algebraic operations on $\varinjlim X_i$ are defined such that these maps become homomorphisms.

In order to obtain cohomology groups, which are independent of open coverings $\mathfrak{U}$ of M, we use the concept of a direct limit in the following way: The direct set is $Co(M) := \{\mathfrak{U} \mid \mathfrak{U} \text{ open cover of } M\}$ with the following partial order relation: For two open covers $\mathfrak{U} = (U_i)_{i \in I}$ and $\mathfrak{V} = (V_k)_{k \in K}$ of M we set $\mathfrak{U} \leq \mathfrak{V}$ if for each $k \in \mathbb{K}$ there exists a $j(k) \in I$ with $V_k \subset U_{j(k)}$. Then $\mathfrak{V}$ is called a REFINEMENT of $\mathfrak{U}$ and $j : K \to I$ is a REFINEMENT MAP. The refinement map induces a homomorphism

$$\tilde{j} : C^p(\mathfrak{U}, G) \to C^p(\mathfrak{V}, G)$$

for each $q \in \mathbb{N}$ by restriction: To $\sigma = (V_{k_0}, \dots, V_{k_q})$ there corresponds $\sigma' = (U_{j(k_0)}, \dots U_{j(k_q)})$ and

$$\tilde{j}(\eta)(\sigma) := \rho|_{|\sigma|,|\sigma'|}\eta(\sigma').$$

Because of $\tilde{j}(Z^q(\mathfrak{U}, G)) \subset Z^q(\mathfrak{V}, G)$ and $\tilde{j}(B^q(\mathfrak{U}, G)) \subset B^q(\mathfrak{V}, G)$ the restriction map $\tilde{j}$ descends to a homomorphism

$$\tilde{j} := j_{\mathfrak{U}\mathfrak{V}} : H^q(\mathfrak{U}, G) \to H^q(\mathfrak{V}, G).$$

In this way we obtain a direct system of abelian groups $\left(\check{H}^q(\mathfrak{U}, G)\,,\, j_{\mathfrak{U}\mathfrak{V}}\right)$ indexed by CoM. The direct limit of this direct system is - by definition - the p-th ČECH COHOMOLOGY GROUP:

Definition E.6.

$$\check{H}^q(M,G) := \varinjlim_{\mathfrak{U}} \check{H}^q(\mathfrak{U},G).$$

is the q-th Čech Cohomology Group on M with values in the abelian group G.

Remark E.7. Fortunately, one can avoid to consider the limit in some concrete calculations. This is the case if there exist open covers $\mathfrak{U}$ of M such that

$$\check{H}^q(M,G) = \check{H}^q(\mathfrak{U},G) \text{ (independently of } \mathfrak{U}).$$

More precisely, if the canonical homomorphism $\check{H}^q(\mathfrak{U},\mathcal{F}) \to \check{H}^q(M,\mathcal{F})$ is an isomorphism. Such open covers $\mathfrak{U}$ are called Leray covers.

E.2 DeRham Cohomology

Let us now come to the comparison of $\check{H}^q(\mathfrak{U},\mathbb{R})$ (and $\check{H}^q(M,\mathbb{R})$) with the de Rham cohomology groups $H^q_{dR}(M,\mathbb{R})$. Recall:

$$H^q_{dR}(M,\mathbb{R}) := \{\alpha \in \mathcal{A}^p(M) \mid d\alpha = 0\}/\{\alpha \in \mathcal{A}^p(M) \mid \exists \beta : d\beta = \alpha\}.$$

Proposition E.8. *Let* $\mathfrak{U} = (U_i)_{i\in I}$ *be an open cover of a smooth manifold* M *such that all intersections* $U_{i_0 i_1 \dots i_p} := U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_p}$ *are empty or diffeomorphic to a convex open subset of* $\mathbb{R}^n$. *Then there exists a natural isomorphism*

$$H^q_{dR}(M,\mathbb{R}) \to \check{H}^q(\mathfrak{U},\mathbb{R}), \quad q \in \mathbb{N}.$$

In particular, $\mathfrak{U}$ *is a Leray cover, and*

$$H^q_{dR}(M,\mathbb{R}) \cong \check{H}^q(M,\mathbb{R}), \quad q \in \mathbb{N}.$$

Proof. Note that there always exist such open covers for a manifold. First of all we prove the isomorphism in the case $q = 1$ in full detail:

Let $\alpha \in \mathcal{A}^1(M)$ with $d\alpha = 0$. Then $\alpha|_{U_j}$ is exact by the lemma of Poincaré (since U_j is diffeomorphic to a convex open subset). Let $f_j \in \mathcal{E}(U_j)$ with $df_j = \alpha|_{U_j}$. Then $\eta_{ij} := (f_i - f_j)|_{U_{ij}}$ is a constant $\eta_{ij} \in \mathbb{R}$ since

$$d(f_i - f_j)|_{U_{ij}} = \alpha|_{U_{ij}} - \alpha|_{U_{ij}} = 0, \tag{87}$$

and the U_{ij} are connected.

The 1-cochain $\eta_\alpha := \eta := (\eta_{ij})$ satisfies

$$(\delta\eta)_{ijk} = \eta_{jk} - \eta_{ik} + \eta_{ij} = f_j - f_k - f_i + f_k + f_i - f_j = 0,$$

hence $\eta_\alpha = \eta$ is a cocycle $\eta \in Z^1(\mathfrak{U},\mathbb{R})$ and defines an element $[\eta_\alpha] \in \check{H}^1(\mathfrak{U},\mathbb{R})$.

To what extent is this cohomology element $[\eta]$ independent of the various choices made? Let $\alpha^* \in \mathcal{A}^1(M)$ be in the same de Rham class as α, i.e. $\alpha^* - \alpha = dg$, where $g \in \mathcal{E}(M)$. Choose $f_j^* \in \mathcal{E}(U_j)$ with $df_j^* = \alpha^*|_{U_j}$ and $\eta_{ij}^* = (f_i^* - f_j^*)|_{U_{ij}}$. By $d(f_j^* - f_j) = (\alpha^* - \alpha)|_{U_j} = dg|_{U_j}$ we can write

$$f_j^* - f_j = g + c_j$$

with suitable constants $c_j \in \mathbb{R}$. Hence,

$$\eta_{ij}^* - \eta_{ij} = c_i - c_j\,, \quad \text{i.e.} \quad \delta c = \eta^* - \eta\,,$$

when $c = (c_i)$. As a result, $\eta^* - \eta \in B^1(\mathfrak{U}, \mathbb{C})$, hence $[\eta^*] = [\eta] \in \check{H}^1(\mathfrak{U}, \mathbb{R})$, and the map

$$\Psi : H^1_{dR}(M, \mathbb{R}) \to \check{H}^1(\mathfrak{U}, \mathbb{R})\,,$$

with $\Psi(\alpha) := [\eta_\alpha]$, is well-defined. Evidently Ψ is a homomorphism. Moreover, Ψ is injective: $\Psi(\alpha) = [\eta_\alpha] = 0$ implies $\eta_{ij} = c_i - c_j$ for $c_i \in \mathcal{F}(U_i, \mathbb{R})$, $c_j \in \mathcal{F}(U_j, \mathbb{R})$, i.e. $c_i, c_j \in \mathbb{R}$. Hence, $g_j := f_j + c_j$ satisfy $dg_j = \alpha|_{U_j}$ and $g_i|_{U_{ij}} = g_j|_{U_{ij}}$ for all $i, j \in I$. Therefore, $g|_{U_j} := g_j$ defines a function $g \in \mathcal{E}(M)$ with $dg = \alpha$ which means $[\alpha] = 0$ in $H^1_{dR}(M, \mathbb{R})$.

To show surjectivity of Ψ let $[\eta] \in \check{H}^1(\mathfrak{U}, \mathbb{R})$, with $\eta = (\eta_{ij})$ and $\eta_{ij} \in \mathcal{F}(U_{ij}, \mathbb{R})$ satisfying $\delta\eta = 0$. We find a smooth, locally finite partition of unity $(h_k)_{k \in I}$ such that the support $\operatorname{supp} h_j$ of h_j is compact and $\operatorname{supp} h_j \subset U_j$. Define

$$\alpha_\eta := \sum_{i,j} \eta_{ij} h_i dh_j\,.$$

$\eta = (\eta_{ij}) \in Z^1(\mathfrak{U}, \mathbb{R})$. We can show

Claim:

$$\alpha_\eta|_{U_k} = df_k,$$

where $f_k = \sum \eta_{kj} h_j$. (All sums are finite, since $(h_k)_{k \in I}$ is locally finite.)

Using this result it is clear that the 1-form α_η is closed. Furthermore, we get

$$\begin{aligned} f_k - f_l &= \sum_j (\eta_{kj} - \eta_{lj})\, h_j \\ &= \sum_j \eta_{kl} h_j, \quad \text{because} \quad \text{of } \delta\eta = 0 \\ &= \eta_{kl} \sum_j h_j \\ &= \eta_{kl}. \end{aligned}$$

We conclude $\Psi(\alpha_\eta) =[\eta] \in \check{H}^1(\mathfrak{U}, \mathbb{R})$ and the surjectivity of Ψ is proven.

In order to prove the claim

$$\alpha_\eta|_{U_k} = df_k = d(\sum_j \eta_{kj} h_j)\,,$$

consider:

$$\begin{aligned}
\alpha_\eta &= \sum_j \sum_{i \neq k} \eta_{ij} h_i dh_j + \sum_j \eta_{kj} h_k dh_j \\
&= \sum_j \sum_{i \neq k} \eta_{ij} h_i dh_j + \sum_j \eta_{kj} \left(1 - \sum_{i \neq k} h_i\right) dh_j \\
&= \sum_j \sum_{i \neq k} \left(\eta_{ij} + \eta_{jk}\right) h_i dh_j + \sum_j \eta_{kj} dh_j \\
&= \sum_j \left(\sum_{i \neq k} \eta_{ik} h_i\right) dh_j + d\left(\sum \eta_{kj} h_j\right) \\
&= \left(\sum_{i \neq k} \eta_{ik} h_i\right) \sum_j dh_j + d\left(\sum \eta_{kj} h_j\right) \quad \text{and} \quad d\left(\sum_j h_j\right) = 0 \\
&= d\left(\sum \eta_{kj} h_j\right).
\end{aligned}$$

We have shown that $\Psi : H^1_{dR}(M, \mathbb{R}) \to \check{H}^1(\mathfrak{U}, \mathbb{R})$ is an isomorphism.

The proof extends directly to cases $q > 1$. The definition of Ψ will be done as before by descending from $\alpha \in \mathcal{A}^q(M), d\alpha = 0$, first to $\beta_j \in \mathcal{A}^{q-1}(M)$ with $d\beta_j = \alpha|_{U_j}$, then to $\gamma_{ij} \in \mathcal{A}^{q-2}(M)$, with $d\gamma_{ij} = \beta_i - \beta_j|_{U_{ij}}$ etc.

The main part of the proof is again to establish the surjectivity of Ψ with the help of a smooth partition of unity (h_j). To $\eta \in Z^q(\mathfrak{U}, \mathbb{R})$ we define:

$$\alpha_\eta := \sum \eta_{i_0 i_1 \ldots i_p} h_{i_0} dh_{i_1} \wedge \cdots \wedge dh_{i_p}$$

and see that $\Psi(\alpha_\eta) =[\eta]$. □

Remark E.9. With an open cover $\mathfrak{U}$ of a manifold M as in Proposition E.8 the same result holds for the corresponding $\mathbb{C}$-valued cohomologies: There exists a natural isomorphism

$$H^q_{dR}(M, \mathbb{C}) \to \check{H}^q(\mathfrak{U}, \mathbb{C}), \ \ \forall q \in \mathbb{N}.$$

We want to explain the definition of Ψ in the case of $q = 2$ in order to comment the integrality condition in the form we need it in Section 8.1.

Remark E.10. The natural isomorphism

$$\Psi : H^2_{dR}(M, \mathbb{C}) \to \check{H}^2(M, \mathbb{C})$$

can be constructed as follows. Choose an open cover $\mathfrak{U} = (U_j)$ as before in the last proposition so that all intersections $U_{j_0 j_1 \cdots j_n}$ are diffeomorphic to a convex open subset of $\mathbb{R}^n$ or empty.

Given a closed 2-form $\alpha \in \mathcal{A}^2(M)$, we find $\beta_j \in \mathcal{A}^1(M)$ with $d\beta_j = \alpha|_{U_j}$ and functions $f_{ij} \in \mathcal{E}(U_{ij})$ with $df_{ij} = \beta_i - \beta_j|_{U_{ij}}$. Hence,

$$\eta_{ijk} = f_{ij} + f_{jk} + f_{ki} \in \mathbb{C}$$

is constant and defines $\eta = (\eta_{ijk}) \in Z^2(\mathfrak{U}, \mathbb{C})$. The class $[\eta]{=}\Psi(\alpha)$ is independent of the choices α, β_j, f_{ij} and yields the isomorphism

$$\Psi : H^2_{dR}(M, \mathbb{C}) \to \check{H}^2(\mathfrak{U}, \mathbb{C}) = \check{H}^2(M, \mathbb{C})$$

The integrality condition can now be reformulated in a complete and satisfying way.

Definition E.11. A closed $\omega \in \mathcal{A}^2(M)$ is ENTIRE (or satisfies condition *(E)* if there exists an open cover $(U_j)_{j\in I}$ of M such that the class $[\omega] \in H^2_{dR}(M, \mathbb{C})$ contains as a Čech class $[c] \in \check{H}^2((U_j)_{j\in I}, \mathbb{C}) \cong \check{H}^2(M, \mathbb{C})$ a cocycle $c = (c_{ijk})$, with $c_{ijk} \in \mathbb{Z}$ for all $i, j, k \in \mathbb{I}$ with $U_{ijk} \neq \varnothing$.

E.3 Sheaf Cohomology

As a general structure we investigate in the following the situation in which for all open subsets $U \subset M$ of a topological space M a collection $\mathcal{F}(U)$ of functions or sections of a special type (for example locally constant or smooth, or continuous or holomorphic etc.) is given with compatibility conditions with respect to the inclusion $V \subset U$, $U, V \subset M$ open. A careful analysis of such data lead to presheaves and sheaves.

For such sheaves we give a short description is the corresponding Čech cohomology.

Definition E.12. For a topological space M we always have the category $t(M)$ of open subsets. The objects are the open subsets and the morphisms are the inclusions $U \subset V, U, V \in M$ open. A PRESHEAF of abelian groups on M is a contravariant functor

$$\mathcal{F} : t(M) \to Ab$$

from $t(M)$ into the category of abelian groups Ab.

Hence, $\mathcal{F}(U)$ is an abelian group for each $U \subset M$ open and to every inclusion $V \subset U$ there corresponds a homomorphism

$$\rho_{V,U} = \mathcal{F}(V \subset U) : \mathcal{F}(U) \to \mathcal{F}(V) \tag{88}$$

such that

$$\rho_{W,V} \circ \rho_{V,U} = \rho_{W,U} \text{ and } \rho_{U,U} = \mathrm{id}_{\mathcal{F}_U} \tag{89}$$

for open $W \subset V \subset U$.

We use the notation $\rho_{V,U}$ instead of $\mathcal{F}(V \subset U)$ (which corresponds to the use in category theory) since in many respects, these homomorphisms behave like a restriction. In many cases of interest they are in fact restrictions but not in all. We also use the notation $g|_V$ instead of $\rho_{V,U}g$ for an element $g \in \mathcal{F}(U)$.

We do not need more than the properties of a presheaf listed above in (88),(89), in particular, we can avoid using the language of category theory.

Examples E.13.

1. G an abelian group, $\mathcal{F}(U) := \{f : U \to G \mid f \text{ a map}\}$ and $\rho_{V,U}(f) = f|_V$.
2. G as before, $\mathcal{F}(U) := \{f : U \to G \mid f \text{ locally constant}\}$. This is the case we have studied in the first section of this chapter.
3. $\mathcal{F}(U) = \mathcal{C}(U) = \mathcal{C}(U, \mathbb{K})$ and ρ restriction, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$, as before.
4. Let G be a topological group and abelian. Then $\mathcal{F}(U) = \mathcal{C}(U, G)$ and ρ restriction as before defines a presheaf, generalizing the above.
5. $\mathcal{F}(U) = \mathcal{C}^\infty(U) = \mathcal{E}(U)$.
6. $\mathcal{F}(U) = \mathcal{O}(U)$ if M is a complex manifold and $\mathcal{O}(U)$ is the space of holomorphic functions on M.
7. $\mathcal{F}(U) = \Gamma(U, L)$ the $\mathcal{E}(U)$-module of differentiable sections for a line bundle over M.
8. $\mathcal{F}(U) = \Gamma_{\mathrm{hol}}(U, L)$ the $\mathcal{O}(U)$-module of holomorphic sections for a holomorphic line bundle over a complex mainfold M.
9. $\mathcal{F}(U) = \mathcal{A}^p(U), U \subset M$ open subset of a smooth manifold.
10. Similarly, for the holomorphic forms.

The notion of a presheaf of abelian groups can be extended to presheaves with values in other mathematically structered objects like vector spaces, rings, Banach spaces, etc. In most of the examples $\mathcal{F}(U)$ is a vector space, in some cases an algebra, or a module over another presheaf, for instance, $\mathcal{A}^q$ is a presheaf of $\mathcal{E}$-modules.

A presheaf $\mathcal{F}$ is a sheaf if in case of an open cover (U_j) of an open $U \subset M$, $U = \bigcup U_j$,

- the elements $f \in \mathcal{F}(U)$ are determined by its "restrictions" $f|_{U_j} = \rho_{U_j,U} f$ and
- local elements $f_j \in \mathcal{F}(U_j)$ can be glued together to obtain an $f \in \mathcal{F}(U)$ with $\rho_{U_j,U} f = f|_{U_j} = f_j$ whenever they are compatible in the following sense: $\rho_{U_{jk},U_j} f_j = \rho_{U_{jk},U_k} f_k$, $j, k \in I$.

More precisely:

Definition E.14. A presheaf $\mathcal{F}$ is a SHEAF if for all open subsets $U \subset M$ and all open covers $\mathfrak{U} = (U_j)_{j \in I}$ of U the following property is satisfied:
A collection $f_i \in \mathcal{F}(U_i), i \in I$, is of the form

$$f_i = \rho_{U_i,U}(f) \; \forall \, i \in I,$$

for a unique element $f \in \mathcal{F}(U)$ if and only if for all $i, j \in I$ the compatibility property

$$\rho_{U_i \cap U_j, U_i}(f_i) = \rho_{U_j \cap U_i, U_j}(f_j)$$

holds.

For instance, if a collection of maps

$$f_i : U_i \to X$$

into a topological space X is continuous and $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then the map

$$f(a) := f_i(a), a \in U_i$$

is a well-defined continuous map with $f|_{U_i} = f_i, i \in I$. Moreover, f is unique.

All the examples in E.13 are sheaves.

The presheaf $\mathcal{F}(U) = G, U \subset M$ open and $\rho_{V,U} = id_G : G \to G$ is in general not a sheaf.

For the cohomology of sheaves one needs the simplices introduced in the first section of this chapter.

Definition E.15. Let F be a sheaf on a topological space M and let $\mathfrak{U} = (U_j)_{j \in I}$ be an open cover of M.

1. A q-cochain of $\mathfrak{U}$ with coefficients in $\mathcal{F}$ is a map

   $$\sigma \to \eta(\sigma) \in \mathcal{F}(|\sigma|), \;\; \sigma \text{ a q-simplex of } \mathfrak{U},$$

   $C^q(\mathfrak{U}, F)$ is the abelian group of cochains.

2. The COBOUNDARY OPERATOR

   $$\delta = \delta^q : C^q(\mathfrak{U}, \mathcal{F}) \to C^{q+1}(\mathfrak{U}, \mathcal{F}),$$

   is

   $$\eta \mapsto \delta\eta, \delta\eta(\sigma) = \sum_{j=0}^{q+1} (-1)^j \rho_{|\sigma|,|\partial_j\sigma|} \eta(\partial_j \sigma).$$

   It is easy to show that $\delta^2 = 0$ and that δ is a homomorphism.

3. We define:

$$Z^q(\mathfrak{U},\mathcal{F}) = \operatorname{Ker}\left(\delta^q : C^q(\mathfrak{U},\mathcal{F}) \to C^{q+1}(\mathfrak{U},\mathcal{F})\right) ,$$
$$B^q(\mathfrak{U},\mathcal{F}) = \operatorname{Im}\left(\delta^{q-1} : C^{q-1}(\mathfrak{U},\mathcal{F}) \to C^q(\mathfrak{U},\mathcal{F})\right) ,$$
$$\check{H}^q(\mathfrak{U},\mathcal{F}) = Z^q(\mathfrak{U},\mathcal{F})/B^q(\mathfrak{U},\mathcal{F}) .$$

Example E.16. Let L be a line bundle over the manifold M which is given by transition functions $g_{ij} \in \mathcal{E}^\times(U_{ij})$ with respect to an open cover $\mathfrak{U} = (U_j)$ of M. As usual, we abbreviate for a q-cochain η the value $\eta(\sigma)$, $\sigma = (U_{j_0}, \dots, U_{j_q})$, by $\eta_{j_0,\dots,j_q}$. In particular $g(U_i, U_j) = g_{ij}$. Then the transition functions form a 1-cochain $g = (g_{ij})$ with values in the sheaf $\mathcal{E}^times$. Since g satisfies the cocycle condition, $g \in Z^1(\mathfrak{U},\mathcal{E}^\times)$ is a Čech cocycle. Hence it determines a Čech cohomology class $[g] \in \check{H}^1(\mathfrak{U},\mathcal{E}^\times)$.

Now, let $\mathfrak{U}$ an open cover, where all intersections U_j are contractible. Then every line bundle L can be described by suitable transition functions on U_{ij}. Recall, that $\mathrm{Pic}_{\mathsf{diff}}(M)$ is the abelian group of equivalence classes of line bundles on M. Each such equivalence class $[L]$ is determined by a collection $g = (g_{ij})$ of suitable transition functions which define a Čech class $[g] \in \check{H}^1(\mathfrak{U},\mathcal{E}^\times)$. We thus have established an isomorphism

$$\mathrm{Pic}_{\mathsf{diff}}(M) \to \check{H}^1(\mathfrak{U},\mathcal{E}^\times).$$

Finally, if $\mathfrak{V} = (V_k)_{k\in K}$ is a refinement of $\mathfrak{U}$ with refinement map $j : K \to I$, then we again have an induced homomorphism

$$\tilde{j} : \check{H}^q(\mathfrak{U},\mathcal{F}) \to \check{H}^q(\mathfrak{V},\mathcal{F}).$$

The corresponding direct limit gives the cohomology.

Definition E.17. The q-th Čech cohomology group on M with values in the sheaf $\mathcal{F}$ on M is defined by

$$\check{H}^q(M,\mathcal{F}) = \varinjlim_{\mathfrak{U}} \check{H}^q(\mathfrak{V},\mathcal{F})$$

Summary: The main objective of the chapter is to prove the equivalence of deRham cohomology and Čech cohomology with values in $\mathbb{R}$ resp. in $\mathbb{C}$. The mechanism of defining Čech cohomology can be transferred to sheaf cohomology without great effort. The need of sheaves (instead of presheaves) is not apparent so far. In the applications one considers short exact sequences

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

of sheaves and the corresponding cohomologies which only behave well for sheaves.

F Quantum Mechanics

In this chapter we present the main principles of Quantum Mechanics in form of four postulates and explain the mathematical framework needed for the formulation of these postulates. These postulates have been chosen in accordance with the principles of Quantum Mechanics which are described in the literature on the foundation of Quantum Mechanics. They comprise much more than is needed for these notes on Geometric Quantization. But they manifest the geometric nature of Quantum Mechanics. And, of course, when Geometric Quantization is presented in these notes on a mathematical basis it is reasonable and helpful to have a mathematical formulation of Quantum Mechanics at hand.

We restrict this chapter to the formulation of the mathematical models of Quantum Mechamics and do not discuss, for example, the measurement process in Quantum Mechanics or any interpretation.

The mathematics of Quantum Mechanics is advanced and requires a profound understanding of self-adjoint operators in a Hilbert space. Therefore, after the formulation of the postulates in the first section of the chapter, Section F.1, we provide in Section F.2 a short exposition of the theory of self-adjoint operators in a Hilbert space including examples and the Spectral Theorem.

In many sources about the foundation of Quantum Mechanics more postulates are required, for instance the representation of the canonical commutation relations (CCR). Such a postulate can be viewed as to be a special example of a quantum mechanical system along the postulates 1-4. We investigate the CCR and the closely related Stone-von Neumann Theorem in Section F.3.

F.1 Four Postulates of Quantum Mechanics

Definition F.1. A QUANTUM MECHANICAL SYSTEM is a pair $(\mathbb{H}, H)$, which satisfies the following four postulates.

Postulate 1. *The* STATES *of the system are the complex lines through the origin of* $\mathbb{H}$*, where* $\mathbb{H}$ *is a* COMPLEX SEPARABLE HILBERT SPACE. *In other words, the state space is the projective Hilbert space* $\mathbb{P}(\mathbb{H})$.

Remarks and Explanations F.2 (to the first postulate)**.**

1° Hilbert Space: A complex Hilbert space $\mathbb{H}$ is a complex vector space together with a Hermitian scalar product $\langle \cdot, \cdot \rangle$ or Hermitian metric, such that $\mathbb{H}$ is complete with respect to the norm induced by $\langle \cdot, \cdot \rangle$. More explicitly, the Hermitian scalar product is a map:

$$\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{C}, \tag{90}$$

with the following properties, which hold for all $\phi, \psi, \theta \in \mathbb{H}, \lambda \in \mathbb{C}$:

- $\langle \phi + \psi, \theta \rangle = \langle \phi, \theta \rangle + \langle \psi, \theta \rangle$ and $\langle \phi, \psi + \theta \rangle = \langle \phi, \psi \rangle + \langle \phi, \theta \rangle$, $\langle \lambda\phi, \psi \rangle = \overline{\lambda} \langle \phi, \psi \rangle$, $\langle \phi, \lambda\psi \rangle = \lambda \langle \phi, \psi \rangle$. In other words, this means that $\langle \cdot, \cdot \rangle$ is $\mathbb{R}$-bilinear, complex linear in the second and complex antilinear in the first entry.
- $\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}$,
- $\langle \phi, \phi \rangle > 0$, if $\phi \neq 0$.

A scalar product defines a norm on $\mathbb{H}$ through $||\phi|| := \sqrt{\langle \phi, \phi \rangle}$. The space $\mathbb{H}$ is called complete with respect to this norm, if every Cauchy sequence (ϕ_n) in $\mathbb{H}$ converges in $\mathbb{H}$.

The homomorphisms between Hilbert spaces are the linear maps $T : \mathbb{H} \to \mathbb{H}'$ which respect the norm, i.e. for which $\|T\phi\|=\|\phi\|$ for all $\phi \in \mathbb{H}$. Equivalently, a linear map $T : \mathbb{H} \to \mathbb{H}$ is a Hilbert space homomorphism, whenever T leaves the scalar product invariant, i.e. $\langle T\phi, T\psi \rangle = \langle \phi, \psi \rangle$ for all $\phi, \psi \in \mathbb{H}$. Such a homomorphism of Hilbert spaces is injective but in general not surjective.[118] The surjective homomorphisms of Hilbert spaces are called Unitary Operators. Thus, the unitary operators are the isomorphisms of the theory.

More generally, when Hilbert spaces are viewed as special Banach spaces, one considers the bounded operators $T : \mathbb{H} \to \mathbb{H}'$, i.e. the linear maps T from $\mathbb{H}$ to $\mathbb{H}'$ with finite operator norm

$$\|T\| := \sup\{\|T\phi\| \mid \|\phi\| = 1\} < \infty \,.$$

The operator norm equips the complex vector space $\mathrm{B}(\mathbb{H}, \mathbb{H}')$ of bounded operators from $\mathbb{H}$ to $\mathbb{H}'$ with the structure of a Banach space. $\mathrm{B}(\mathbb{H})$ denotes the Banach space of bounded operators from $\mathbb{H}$ to $\mathbb{H}$.

Bounded operators are continuous linear maps with respect to the natural norm topology. The norm topology is the metric topology induced by the metric $d(\phi, \psi) = \|\phi - \psi\|$ on $\mathbb{H}$: A subset $U \subset \mathbb{H}$ is open in the metric topology if and only if for every $\phi \in U$ there exists an $r > 0$ such that the ball

$$B(\phi, r) := \{\psi \in \mathbb{H} \mid \|\phi - \psi\| < r\}$$

of radius r around ϕ is contained in U. Notice, that continuous linear maps $\mathbb{H} \to \mathbb{H}'$ are bounded.

2° The Examples: The most familiar complex Hilbert spaces are the finite dimensional complex Hilbert spaces, which are also called Unitary Spaces. Typical examples are the number spaces $\mathbb{H} = \mathbb{C}^n$ with the scalar product:

$$\langle z, w \rangle = \sum_{i=1}^{n} \overline{z}^i w^i.$$

[118] In the finite dimensional case an injective and linear map is always surjective. Hence, it is an isomorphism of vector spaces.

for $z = (z^1, \dots, z^n), w = (w^1, \dots, w^n) \in \mathbb{C}^n$.

Note, that each n-dimensional Hilbert $\mathbb{H}$ is unitarily equivalent to the above unitary space $\mathbb{C}^n$, where $n = \dim_{\mathbb{C}} \mathbb{H}$: There exists a unitary operator $T : \mathbb{H} \to \mathbb{C}^n$. The existence of T is equivalent to the existence of an orthonormal basis (e_j) of $\mathbb{H}$: $\langle e_j, e_k \rangle = \delta_{jk}$.

A typical infinite dimensional complex Hilbert space is the space ℓ^2 of square summable complex sequences

$$\ell^2 := \{(z^j)_{j\in\mathbb{N}} \mid \sum \bar{z}^j z^j < \infty\}$$

with the scalar product $\langle z, w \rangle := \sum \bar{z}^j w^j$. ℓ^2 is complete: Any Cauchy sequence convergences coordinatewise, and the corresponding limit is the limit of the sequence in norm. ℓ^2 is separable, since - with the basis elements $e_k := (\delta_k^j)_{j\in\mathbb{N}}$ - the countable set $D = \{\sum_{k=0}^{k=n} q^k e_k \mid q^k \in \mathbb{Q}\,, n \in \mathbb{N}\}$ is dense in ℓ^2.

However, the most important examples of Hilbert spaces for quantum mechanics are certain function spaces. Namely, the space of square integrable functions on the configuration space. For example, for an open $Q \subset \mathbb{R}^n$

$$L^2(Q) := \left\{\phi : Q \to \mathbb{C} \mid \phi \ \text{ measurable and } \ \int_Q |\phi(q)|^2 dq < \infty \right\},$$

where in this situation dq stands for the Lebesgue integral[119]. The scalar product for $\mathbb{H} = L^2(Q)$ is given by $\langle \phi, \psi \rangle := \int_Q \overline{\phi}(q)\psi(q)dq$. In this approach to define $L^2(Q)$ one needs to identify those functions, which differ only on a set of measure zero.

Without using the Lebesgue integral, one can construct the Hilbert space $L^2(Q)$ in the following way: One defines

$$R^2(Q) := \left\{\phi : Q \to \mathbb{C} \mid \phi \ \text{ is continuous and } \ \int_Q |\phi(q)|^2 dq < \infty \right\},$$

where now $\int h(q)dq$ is the Riemann integral for continuous functions h on Q. The scalar product on $R^2(Q)$ will be obtained in the same way as before,

$$\langle \phi, \psi \rangle := \int_Q \overline{\phi}(q)\psi(q)dq\,,$$

with the only difference that the integration is Riemann integration, instead of Lebesgue integration. This scalar product determines a norm by $\|\phi\| := \sqrt{\langle \phi, \phi \rangle}$ on $R^2(Q)$. And the final Hilbert space is the abstract completion $\widehat{R^2(Q)}$ of the space $R^2(Q)$ with respect to the norm. Since the scalar product $\langle \cdot, \cdot \rangle : R^2(Q) \times R^2(Q) \to \mathbb{C}$ is continuous with

[119] More precisely, instead of functions $\phi : Q \to \mathbb{C}$ one has to take equivalence classes of measurable functions ϕ which only differ on a set of measure zero.

respect to the topology given by the norm, it can be continued to the completion. Altogether, $\widehat{R^2(Q)} = \mathbb{H} \cong L^2(Q)$ is a Hilbert space, unitarily equivalent to $L^2(Q)$.

Note, that it is possible to start with a smaller space than $R^2(Q)$, for example with $E := R^2(Q) \cap \mathcal{E}(Q,\mathbb{C})$ or with the space $E = \mathcal{E}_0(Q)$ of smooth functions on Q with compact support in Q. Scalar product and norm can be defined on E as before. The completion process encompasses $R^2(Q)$ and leads to the same completion: $\widehat{E} = \widehat{R^2(Q)} \cong L^2(Q)$. Note that E can be viewed as to be a subspace of $L^2(Q)$ where the scalar product on $L^(Q)$ given by Lebesgue integration is, when restricted to E, the scalar product given by Riemann integration. As a result, the space $\widehat{E}$ can be understood as the closure $\overline{E}$ in $L^2(Q)$, i.e. $\widehat{U} = \overline{E} = L^2(Q)$

One can show that the examples ℓ^2 and $L^2(Q)$ are unitarily equivalent. In fact, every infinite dimensional and separable Hilbert space is unitarily equivalent to ℓ^2 since there exists an orthonormal (Hilbert space) basis (e_j) of $\mathbb{H}$, i.e. $\langle e_j, e_k\rangle = \delta_{jk}$ and each $\phi \in \mathbb{H}$ has a unique expression as a sum $\phi = \sum \langle e_j, \phi\rangle e_j$[120].

Moreover, any Hilbert space is unitarily equivalent to a suitable $L^2(X,\mu) = L^2(X)$[121] where (X,Σ,μ) is a measure space with σ-algebra Σ and measure μ. As before,

$$L^2(X,\mu) := \left\{ \phi : X \to \mathbb{C} \mid \phi \ \text{ measurable and } \ \int_X |\phi(x)|^2 d\mu(x) < \infty \right\},$$

with the scalar product $\langle \phi, \psi\rangle := \int_X \overline{\phi}(x)\psi(x)d\mu(x)$.

For instance, for $\mathbb{H} = \mathbb{C}^n$ let (X,μ) be the measure space with finite $X := \{1,\dots,n\}$ and $\mu(\{x\}) := 1, x \in X$. This yields the n-dimensional Hilbert space $L^2(X,\mu) \cong \mathbb{C}^n$.

The Hilbert space ℓ^2 can be defined as $L^2(\mathbb{N},\mu)$ with $\mu(\{x\}) = 1, x \in \mathbb{N}$.

Moreover, for an uncountable set X and $\mu(\{x\}) = 1, x \in X$ one obtains the non-separable Hilbert space $\ell^2(X) = L^2(X,\mu)$.

3° State Space: As required in Postulate 1, the space of states is the projective Hilbert space $\mathbb{P}(\mathbb{H}) = \mathbb{P}\mathbb{H}$. To obtain this projective space we consider on $\mathbb{H} \setminus \{0\}$ the equivalence relation

$$z \sim w \iff \exists \lambda \in \mathbb{C} \ \text{ with } \ z = \lambda w .$$

The equivalence classes are the complex lines in $\mathbb{H}$ through the origin. For an element $z \in \mathbb{H}$, $z \neq 0$, we denote its equivalence class by $\gamma(z)$ or $[z]$. In case of $\mathbb{H} = \ell^2$ (or $\mathbb{C}^n$) we can denote the class $[z]$ also by its homogeneous coordinates $[z^0 : z^1 : \dots : z_j : \dots]$. The quotient space

$$\mathbb{P}(\mathbb{H}) = \mathbb{P}\mathbb{H} := \mathbb{H}/\sim \ = \{\gamma(z) \mid z \in \mathbb{H} \setminus \{0\}\},$$

[120] There is no constructive way to define an orthonormal basis, one has to use the axiom of choice. It is quite remarkable, the the converse is true, as well: When it is possible to find an orthonormal basis in every separable Hilbert space, then the axiom of choice is valid.

[121] The measure μ will often be omitted if it is clear from the context which measure is meant.

is the PROJECTIVE HILBERT SPACE. $\mathbb{PH}$ is the space of states of a quantum mechanical system according the Postulate 1. $\mathbb{PH}$ is determined by the canonical map $\gamma : \mathbb{H}\setminus\{0\} \to \mathbb{PH}$ and inherits its structures by that projection map.

For instance, $\mathbb{PH}$ is endowed with a natural topology, the QUOTIENT TOPOLOGY (c.f. Section A.4), for which a subset $V \subset \mathbb{P}(\mathbb{H})$ is open if and only if $\gamma^{-1}(V)$ is open in $\mathbb{H}$. By definition, γ is continuous. Moreover, γ is an open map and it is holomorphic with respect to the holomorphic quotient structure given by the holomorphic charts (c.f. Examples A.30).

4° Example: Particle. An example is the description of a non-relativistic particle in Quantum Mechanics. Let $Q \subset \mathbb{R}^3$ be an open subset of $\mathbb{R}^3$. $\mathbb{H} = L^2(Q, dq)$ is the Hilbert space of square integrable functions on Q. The elements ϕ of $\mathbb{H}$ are called WAVE FUNCTIONS and the equivalence classes

$$a = \gamma(\phi) \in \mathbb{PH}\,,\ \phi \in \mathbb{H} \setminus \{0\}\,,$$

are the states of the system. In case of $||\phi|| = 1$ the quantity $\int_B |\phi(q)|^2 dq$ for a measurable subset $B \subset Q$ has the interpretation of the probability that the particle in state $a = \gamma(\phi)$ is contained in B.

5° Pseudometric: In contrast to classical mechanics, where we have the phase space equipped with a symplectic form ω which together with a classical Hamiltonian H determines the dynamics of the theory, here we have as state space the projective space $\mathbb{PH}$ with the structures induced by the canonical map γ. Part of the dynamics[122] of the quantum theory is given by the "pseudometric" $c : \mathbb{PH} \times \mathbb{PH} \to \mathbb{R}$ induced by the scalar product $\langle \cdot, \cdot \rangle$: The transition probability between $\phi, \psi \in \mathbb{H}_1 := \{\phi \in \mathbb{H} \mid ||\phi|| = 1\}$, $\phi \neq \psi$, is

$$t(\phi, \psi) := \left| \left\langle \frac{\phi}{||\phi||}, \frac{\psi}{||\psi||} \right\rangle \right|^2 .$$

And for $a = \gamma(\phi), b = \gamma(\psi)$ the pseudometric is

$$c(a, b) := \sqrt{t(\phi, \psi)}\,.$$

c is independent of the choice of representatives ϕ, ψ and thus well-defined. The pseudometric is invariant under symmetry transformations of the system, the maps induced by unitary or antiunitary operators in $\mathbb{H}$.

Note, that the pseudometric c determines the natural topology on the state space $\mathbb{PH}$. By elementary geometry we see that for two vectors $\phi, \psi \in \mathbb{H}$ of unit length the quantity $c([\phi], [\psi])$ is nothing else than $\cos\alpha$, where α can be understood as to be the angle between ϕ and ψ. Therefore, for each $r > 0$ there exists h , $0 < h < 1$ such that the "pseudodisc"

$$D_1(\phi, h) := \{\psi \in \mathbb{H}_1 \mid c([\phi], [\psi]) > h\}$$

[122] The dynamics of $(\mathbb{H}, H)$ is determined by H, see Postulate 3 below.

is contained is the ball

$$B_1(\phi, r) = \{\psi \in \mathbb{H}_1 \mid ||\phi - \psi|| < r\}\,.$$

Conversely, for each h, $0 < h < 1$ there is an $r > 1$ with $B_1(\phi, r) \subset D_1(\phi, h)$. Hence, the collection $(D_1(\phi, h))_{h<1,\phi\in\mathbb{H}_1}$ of subsets of $\mathbb{H}_1$ induces on $\mathbb{H}_1$ the natural norm topology. As a consequence, with the notation $D(a, h) := \gamma(D_1(\phi, h)) = \{b \in \mathbb{PH} \mid c(a, b) > h\}$ the collection

$$(D(a, h))_{a\in\mathbb{PH},h<1}$$

generates the natural (quotient) topology on $\mathbb{PH}$ as a basis of open subsets.

Moreover, the pseudometric obtains its significance in the fourth postulate: the quantity $t(a, b) = t(\phi, \psi) \in [0, 1]$ for two states $a = [\phi], b = [\psi] \in \mathbb{PH}$ is the transition probability from a to b. Namely, when the system is initially in the state a, after a measurement the probability of the system being in the state b is t(a,b).

Postulate 2. *Every observable of the system is represented by a self-adjoint operator acting on the Hilbert space $\mathbb{H}$. As a consequence, the set of all possible observables is:*

$$\mathcal{SA} := \{T : D(T) \to \mathbb{H} \mid D(T) \subset \mathbb{H} \text{ and } T : D(T) \to \mathbb{H} \text{ self-adjoint}\}\,.$$

Remarks and Explanations F.3 (to the second postulate)**.**

1° Self-Adjointness: An operator T is a $\mathbb{C}$-linear map $T : D(T) \to \mathbb{H}$ from a linear subspace $D(T) \subset \mathbb{H}$ into $\mathbb{H}$ which is called the domain of T.

The ADJOINT T^* of a linear operator T with dense domain $D(T)$ is defined as follows: The domain of T^* is

$$D(T^*) := \{\psi \in \mathbb{H} \mid \text{ there exists } \xi \in \mathbb{H} : \langle \xi, \phi \rangle = \langle \psi, T\phi \rangle \text{ for all } \phi \in D(T)\}\,.$$

For $\psi \in D(T^*)$ the ξ with $\langle \xi, \phi \rangle = \langle \psi, T\phi \rangle$ for $\phi \in D(T)$ is unique (since $D(T)$ is dense), and $T^*(\psi) := \xi$.

T is a SELF-ADJOINT OPERATOR if $D(T)$ is dense and the adjoint T^* of T agrees with T, i.e. $D(T) = D(T^*)$ and $T\phi = T^*\phi$ for all $\phi \in D(T) = D(T^*)$.

T is SYMMETRIC, if for all $\phi, \psi \in D(T) : \langle T\phi, \psi \rangle = \langle \phi, T\psi \rangle$. Hence a self-adjoint operator is symmetric. The converse does not hold, in general.

Details and results about self-adjoint operators will be explained in the next section, we only point out the following:

2° Finite Dimensional Hilbert Space: In case of $\mathbb{H} \cong \mathbb{C}^n$ with the usual scalar product it is easy to see: Every dense linear subspace D of $\mathbb{H}$ is all of $\mathbb{H}$: $D = \mathbb{H}$, and every $\mathbb{C}$-linear map $T : \mathbb{H} \to \mathbb{H}$ is automatically continuous and therefore also closed. T is self-adjoint, if the matrix $M = M_T$ which represents T with respect to an orthonormal basis of $\mathbb{H}$ satisfies $\overline{M}^{\top} = M$. One also says T (or M) is symmetric with respect to the hermitian scalar product.

3° Infinite Dimensional Hilbert Space: In case of an infinite dimensional Hilbert space $\mathbb{H}$ any self-adjoint operator T with $D(T) = \mathbb{H}$ is bounded (equivalently: continuous) since T is closed (c.f. Definition F.7) and closed operators T with domain $D(T) = \mathbb{H}$ are bounded (see Proposition F.10). However, there exist self-adjoint operators, which cannot be defined for all points of the Hilbert space $\mathbb{H}$, i.e. $D(T) \neq \mathbb{H}$, the so called unbounded operators. For Quantum Mechanics most of the important operators are unbounded and self-adjoint.

4° Multiplication operator: Let $v : \mathbb{R}^n \to \mathbb{C}$ a continuous function, and let $\mathbb{H} = L^2(\mathbb{R}^n, d\lambda)$[123] be the Hilbert space of square integrable complex functions on $\mathbb{R}^n$. Consider the multiplication operator

$$M_v =: D(M) \to \mathbb{H}\,,\ \phi \mapsto v\phi\,,$$

defined on

$$D(M) := \{\phi \in \mathbb{H} \mid \int_{\mathbb{R}^n} |v(q)\phi(q)|^2 d\lambda(q) < \infty\}\,.$$

$D(M)$ is dense in $\mathbb{H}$ since all bounded ϕ with bounded support are contained in $D(M)$. So the adjoint exists.

Now, for all $\psi \in \mathbb{H}$ and all $\phi \in D(M)$ the equality

$$\langle \overline{v}\psi, \phi\rangle = \int_{\mathbb{R}^n} \overline{\overline{v}\psi} d\lambda = \int_{\mathbb{R}^n} \bar{\psi} v\phi d\lambda = \langle \psi, v\phi\rangle$$

holds. As a consequence $D(M^*) = D(M)$ and $M^*\phi = \overline{v}\phi$.

We conclude: $M = M_v$ is self-adjoint if and only if $v = \overline{v}$, i.e. v is real-valued. M is bounded if and only if v is essentially bounded, i.e. there is a subset $N \subset \mathbb{R}^n$ of measure 0 such that $\sup\{|v(q)| \mid q \in \mathbb{R}^n \setminus N\} < \infty$.

The example has a straightforward generalization to general measure spaces (Ω, Σ, μ) instead of $\mathbb{R}^n$, see Example F.22 below.

5° Position: The example of a quantum mechanical system of a non-relativistic particle on the real axis. Here, we set $\mathbb{H} := L^2(\mathbb{R})$. A typical observable is the position operator Q with

$$D(Q) := \left\{\phi \in \mathbb{H} \mid \int |q\phi(q)|^2 dq < \infty\right\}\,,$$

and $Q\phi(q) := q\phi(q)$ for $q \in \mathbb{R}$ and $\phi \in D(Q)$. Q is self-adjoint according to the preceding example with $v(q) = q$. However, Q is not bounded: The sequence $\phi_n :=$

[123] λ the Lebesgue measure

$\chi_{[n,n+1]}$ of indicator functions of the interval $[n, n+1]$ is bounded by $1 : ||\phi_n|| = 1$, but $||Q\phi_n|| \geq n$.

6° Momentum Another unbounded observable is the momentum operator P in $\mathbb{H} = \mathrm{L}^2(\mathbb{R})$ with

$$D(P) := \{\phi \in \mathbb{H} \mid \exists D\phi \in \mathbb{H} : \langle D\phi, \psi \rangle = -\langle \phi, \psi' \rangle \;\; \forall \psi \in \mathcal{E}(\mathbb{R}) \,, \, \psi \text{ compactly supported}\}$$

and $P\phi := -iD\phi$ for $\phi \in D(P)$. Here $\psi' = \dot{\psi}$ is the usual derivative of the differentiable function ψ, while $D\phi$ is the "weak derivative" of $\phi \in D(P)$.

The notion of a self-adjoint operator is fundamental for the mathematical modeling of quantum mechanical systems. Several deep results about self-adjoint operators are needed in a mathematical treatment of Quantum Mechanics. Therefore, we include a separate section on self-adjoint operators (see Section F.2), in which elementary properties of self-adjoint operators are described and important results as e.g. the Theorem of Stone and the Spectral Theorem are explained.

Postulate 3. *H is an observable and determines the dynamics of the quantum mechanical system $(\mathbb{H}, H)$ in the following sense: Let $p_0 \in \mathbb{P}(\mathbb{H})$ be a state of the quantum mechanical system with a representative (state vector) $\phi_0 \in \mathbb{H}$, $||\phi_0|| = 1$. Then the time evolution $\phi(t)$ of the state vector ϕ_0 is given by:*

$$\frac{d}{dt}\phi(t) = \dot{\phi}(t) = -iH(\phi(t)) \,, \; t \in \mathbb{R} \,, \tag{91}$$

with the initial condition $\phi(0) = \phi_0$. This means that the time evolution $p(t)$ of the state p_0 is represented by the unique solution $\phi(t)$ of the above equation with $\phi(0) = \phi_0$, i.e. $p(t) := [\phi(t)]$ with $p(0) = p_0$.

Remarks and Explanations F.4 (to the third postulate)**.**

1° Schrödinger Equation: The above equation (91) is called the Schrödinger equation and H is called the HAMILTONIAN ("Hamiltonoperator") of the quantum mechanical system $(\mathbb{H}, H)$.

2° Unitary Group: In order to better understand the time evolution of a quantum mechanical state we want to formulate Stone's Theorem which describes a strong relation between self-adjoint operators and unitary operators in a Hilbert space $\mathbb{H}$.

Recall that a unitary operator U is a bijective $\mathbb{C}$-linear map $U : \mathbb{H} \to \mathbb{H}$ which leaves the scalar product invariant, that is: $\langle U\phi, U\psi \rangle = \langle \phi, \psi \rangle$ for all $\phi, \psi \in \mathbb{H}$. A unitary operator U is automatically bounded since $||U\phi|| = \langle U\phi, U\phi \rangle = ||\phi||$. In particular, the operator norm of U is $||U|| = \sup\{||Uf|| : ||f|| = 1\} = 1$ and U is continuous.

The inverse U^{-1} of a unitary operator U exists and is again a unitary operator. Finally, the composition $U \circ V : \mathbb{H} \to \mathbb{H}$ of two unitary operators U, V in $\mathbb{H}$ is also a unitary operator. As a consequence, the set of unitary operators in $\mathbb{H}$ form a group, called the UNITARY GROUP, which is denoted by $\mathrm{U}(\mathbb{H})$.

A ONE-PARAMETER GROUP OF UNITARY OPERATORS in $\mathbb{H}$ is given by an $\mathbb{R}$-indexed family of unitary operators $(U_s)_{s\in\mathbb{R}}$, which can be described by an action Φ of $\mathbb{R}$ on $\mathbb{H}$, i.e.

$$\Phi : \mathbb{R} \times \mathbb{H} \to \mathbb{H}, U_s f := \Phi(s, f), \quad (s, f) \in \mathbb{R} \times \mathbb{H}$$

satisfying

1. $U_s \in \mathrm{U}(\mathbb{H})$ for all $s \in \mathbb{R}$,
2. $U_s \circ U_t = U_{s+t}$, for $s, t \in \mathbb{R}$
3. for all $f \in \mathbb{H}$, the map $\mathbb{R} \to \mathbb{H}$, $s \to U_s(f)$, is continuous.

In another terminology, $\mathbb{R} \to \mathrm{U}(\mathbb{H})$, $s \mapsto U_s$, is a representation, i.e. a continuous group homomorphism, where $U(\mathbb{H})$ is endowed with the strong topology.

Theorem F.5 (Theorem of Stone)**.** *To any one-parameter group of unitary operators* (U_s) *there corresponds an* INFINITESIMAL GENERATOR A*, which is the operator defined by:*

$$D(A) := \{\phi \in \mathbb{H} \mid \lim_{s\to 0} \frac{1}{s}(U_s\phi - \phi) \ \text{exists}\},$$
$$A(\phi) := i\,(\lim_{s\to 0} \frac{1}{s}(U_s\phi - \phi)), \phi \in D(A)\,.$$

Then

1. *the infinitesimal generator of a one-parameter group of unitary transformations is self-adjoint,*
2. *to each self-adjoint operator* A *on* $\mathbb{H}$ *there is a corresponding one-parameter group of unitary operators* (U_s)*, whose infinitesimal generator is* A*. We denote this as* $U_s = e^{-isA}$ *(in accordance with the functional calculus of self-adjoint operators, see below).*

Therefore, the observables defined in Postulate 2 are in one-to-one correspondence with one-parameter groups of unitary operators.

We conclude that the Hamiltonian H induces the one parameter group $U_t = e^{-itH}$ and the solution of the Schrödinger equation (91) is $\phi(t) = U_t\phi$. for $\phi(0) = \phi_0$. Or, in another form

$$\boxed{\phi(t) = e^{-itH}\phi\,, \ t \in \mathbb{R}\,.}$$

Postulate 4. *Let $[\phi] \in \mathbb{P}(\mathbb{H})$ be a state of the quantum mechanical system with representative $\phi \in \mathbb{H}$, $||\phi|| = 1$, and let T be an observable with its corresponding spectral family $(E_\lambda)_{\lambda\in\mathbb{R}}$. Then, for an open interval $J =]a,b[$, the probability that an eigenvalue of the observable T in the state $[\phi]$ is contained in the interval J is given by the formula*

$$p(\phi, T, J) := ||E(J)\phi||^2 = \langle \phi, E(J)\phi \rangle = \int_a^b d(||E_\lambda \phi||^2)\,.$$

The concept of a spectral family of projection operators corresponding to a self-adjoint operator will be explained in the next section.

F.2 Self-Adjoint Operators

This section is a short introduction to self-adjoint operators and the spectral theorem. Although we do not need details about self-adjoint operators in the Lecture Notes, the notion and the results of self-adjoint operators are necessary for the mathematical formulation of quantum mechanics as is apparent from the preceding section on the postulates of Quantum Mechanics. Moreover, we need a certain familiarity with self-adjoint operators in the next section.

The material presented in this section can be found in any book on linear operators in Hilbert spaces, e.g. in [Wei12] or [Hal13].

As before, in this section $\mathbb{H}$ denotes a separable Hilbert space. A Linear Operator in $\mathbb{H}$ – often called Operator – is a linear map A defined on a linear subspace $D(A)$ of $\mathbb{H}$ with values in $\mathbb{H}$: $A : D(A) \to \mathbb{H}$. $D(A)$ will be called the domain of the operator A and $\mathrm{Im}\,(A) = A(D(A))$ will be called its range[124].

It is important to realize, that in the case of an operator B with $D(A) \subset D(B)$ and $B|_{D(A)} = A$ the two operators are regarded as to be different operators when $D(A) \neq D(B)$. But B will be called an Extension of A.

Definition F.6. A linear operator A in the Hilbert space $\mathbb{H}$ will be called Densely Defined if the linear subspace $D(A)$ is dense in $\mathbb{H}$.

For a densely defined operator A the Adjoint Operator A^* of A is defined as follows: The domain if A^* is

$$D(A^*) := \{\phi \in \mathbb{H} \mid \exists\, \chi \in \mathbb{H} : \langle \phi, A\psi \rangle = \langle \chi, \psi \rangle \ \text{ for all } \ \psi \in D(A)\,\}$$

[124]In the Hilbert space literature the range of an operator A is denoted by $R(A)$. We prefer $\mathrm{Im}\,(A)$ since we use this notation in the linear algebra context throughout the Lecture Notes.

and the value $A^*\phi \in \mathbb{H}$ is defined by $\langle \phi, A\psi \rangle = \langle A^*\phi, \psi \rangle$ for all $\psi \in D(A)$. $A^*\phi$ is well-defined since $D(A)$ is dense.

Finally, a densely defined operator A is called

1. SELF-ADJOINT if A and A^* agree, i.e. $D(A) = D(A^*)$ and $A\phi = A^*\phi$ for all $\phi \in D(A)$ (this is a reformulation of the definition in Remark 1° of F.3).
2. SYMMETRIC, if $\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle$ for all $\phi, \psi \in D(A)$.

For a symmetric operator A the domain of $D(A^*)$ contains $D(A)$ and the two operators agree on the domain $D(A)$ of A: $A^*|_{D(A)} = A$. Thus, A^* is an extension of A: $A \subset A^*$. As a consequence, in order that an operator A is self-adjoint, it is necessary that A is SYMMETRIC.

Definition F.7. An operator A in $\mathbb{H}$ is said to be

1. CLOSED if the graph $\Gamma(A)$ of A, defined as $\Gamma(A) := \{(\phi, A\phi) \in \mathbb{H} \times \mathbb{H} \mid \phi \in D(A)\}$, is closed in $\mathbb{H} \times \mathbb{H}$.
2. CLOSABLE if the closure $\overline{\Gamma(A)} \subset H \times H$ in $H \times H$ is the graph of an operator B. In that case B is called the closure of A and will be denoted by $\overline{A} := B$.
3. ESSENTIALLY SELF-ADJOINT if A is closable and the closure $\overline{A}$ satisfies $\overline{A} = A^*$, i.e. is self-adjoint.

In particular, for a closable operator A, $\overline{\Gamma(A)} = \Gamma(\overline{A})$ and $\overline{A}$ is a closed operator. The following assertions are easy to prove.

Proposition F.8. *Let A be a densely defined operator. Than the adjoint A^* of A is closed. Moreover, the double adjoint $A^{**} = (A^*)^*$ exists if A is closable, and in that case $\overline{A}^* = A^*$ and $\overline{A} = A^{**}$.*

In particular, a self-adjoint operator A is closed and fulfills $A = A^{**}$.

By definition, an operator A is closed if for all sequences (ϕ_n) in $D(A)$ for which (ϕ_n) and $(A\phi_n)$ converge in $\mathbb{H}$ the following holds:

$$\lim \phi_n \in D(A)\,, \quad \text{and} \quad \lim A\phi_n = A(\lim \phi_n)\,.$$

This looks very much like a continuity condition, but a closed operator need not be continuous as the following example shows.

Example F.9. In the Hilbert space $\mathbb{H} = \ell^2 = \{(z_j) \in \mathbb{C}^{\mathbb{N}} \mid \sum_1^\infty |z_j|^2 < \infty\}$ of square summable sequences the operator $A(z_j) := (jz_j)$ with domain $D(A) = \{(z_j) \mid \sum j^2|z_j|^2 < \infty\}$ is densely defined. Let $\phi = (\frac{1}{j}) \in \mathbb{H}$ and define $\phi_n \in \mathbb{H}$ by $(\phi_n)_j := \frac{1}{j}$ for $j = 1, 2, \ldots, n$ and $(\phi_n)_j := 0$ for $j > n$. Then $\phi_n \to \phi$ but $A\phi_n$ does not converge in $\mathbb{H}$, since $\|A\phi_n\|^2 = n \to \infty$. Hence, A is not continuous. However, A is closed. When $\psi_n = (z_j^{(n)}) \to (v_j) =: \psi$ and $A\psi_n \to (w_j) =: \phi$, for each fixed $j \in \mathbb{N}$ the coordinates converge:

$$\lim_n z_j^{(n)} = v_j \ , \ \lim_n j z_j^{(n)} = w_j \, .$$

Hence, $jv_j = w_j$. Therefore, $\sum j^2 |v_j|^2 < \infty$ which means that $\psi \in D(A)$ and $A\psi = (jv_j) = (w_j) = \phi$.

By the way, A is self-adjoint. Notice, that in Remark 4° of F.3 a general example of an unbounded and closed (and self-adjoint) operator, the multiplication operator, is presented.

A continuous (or equivalently bounded) operator $A : \mathbb{H} \to \mathbb{H}$ ($D(A) = \mathbb{H}$) is always closed. We cite the following well-known result:

Assertion F.10 (Closed Graph Theorem)**.** *An operator A with $D(A) = \mathbb{H}$ is continuous if and only if A is closed.*

Spectrum of an Operator

An eigenvalue of the operator A in $\mathbb{H}$ is a complex number $\lambda \in \mathbb{C}$ such that there exists a $\phi \in \mathbb{H}$, $\phi \neq 0$, satisfying $A\phi = \lambda\phi$ i.e. such that $\lambda I - A = \lambda - A$[125] is not injective. ϕ is called eigenvector and Ker $(\lambda - A) = \mathrm{N}(\lambda - A)$[126] is the eigenspace of λ.

Proposition F.11. *Every eigenvalue λ of a symmetric operator is real, i.e. $\lambda \in \mathbb{R}$, and eigenvectors of different eigenvalues are orthogonal to each other. Moreover, for each $\lambda \in \mathbb{C} \backslash \mathbb{R}$ the operator $\lambda - A$ is injective and the inverse $(\lambda - A)^{-1} : \mathrm{Im}(\lambda - A) \to \mathbb{H}$ is a bounded operator with $\|(\lambda - A)^{-1}\| \leq |\Im\lambda|^{-1}$.*

Proof. Let λ be an eigenvalue of A with $A\phi = \lambda\phi$, $\phi \neq 0$. Then

$$(\bar{\lambda} - \lambda) \|\phi\| = (\bar{\lambda} - \lambda) \langle \phi, \phi \rangle = \langle \lambda\phi, \phi \rangle - \langle \phi, \lambda\phi \rangle = \langle A\phi, \phi \rangle - \langle \phi, A\phi \rangle = 0 \, ,$$

hence $\lambda = \bar{\lambda}$.
For $A\phi_j = \lambda_j \phi_j$, $j = 1, 2$, the following holds:

$$(\lambda_1 - \lambda_2) \langle \phi_1, \phi_2 \rangle = \langle \lambda_1 \phi_1, \phi_2 \rangle - \langle \phi_1, \lambda \phi_2 \rangle = \langle A\phi_1, \phi_2 \rangle - \langle \phi_1, A\phi_2 \rangle = 0$$

As a consequence, $\langle \phi_1, \phi_2 \rangle = 0$ if $\lambda_1 \neq \lambda_2$.
For $\lambda = \xi + i\eta$ and $\phi \in D(A)$ the symmetry of A implies

$$\langle (\xi - A)\phi, i\eta\phi \rangle + \langle i\eta\phi, (\xi - A)\phi \rangle = 0 \, ,$$

[125] I denotes the identity operator $\mathbb{H} \to \mathbb{H}, I(\phi) = \phi$, and the symbol I will often be omitted in the notations like $\lambda - A$ for $\lambda I - A$.

[126] In the Hilbert space literature the kernel of a linear operator is mostly denoted by $N(A)$.

and we obtain

$$\|(\lambda - A)\phi\|^2 = \|(\xi - A)\phi + i\eta\phi\|^2 = \|(\xi - A)\phi\|^2 + \eta^2 \|\phi\|^2 \geq \eta^2 \|\phi\|^2 \ ,$$

When $\lambda \in \mathbb{C} \setminus \mathbb{R}$, i.e. $\Im\lambda = \eta \neq 0$, this implies that $\lambda - A$ has to be injective. Moreover, for $\psi = (\lambda - A)\phi$, $\phi \in D(A)$:

$$\left\|(\lambda - A)^{-1}\psi\right\| = \|\phi\| \leq |\Im\lambda|^{-1} \|(\lambda - A)\phi\| = |\Im\lambda|^{-1} \|\psi\| \ ,$$

which is the inequality we intended to prove. □

Definition F.12. Let A be an arbitrary operator A in $\mathbb{H}$. complex number $\lambda \in \mathbb{C}$ is in the RESOLVENT SET $\rho(A)$ if $\lambda - A : D(A) \to \mathbb{H}$ is bejective and the RESOLVENT OPERATOR $R(\lambda, A)$ (for A at λ)

$$R(\lambda, A) := (\lambda - A)^{-1}$$

is a bounded operator $R(\lambda, A) : \mathbb{H} \to \mathbb{H}$.

The complement $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is the SPECTRUM of A:

In most cases A will be assumed to be closed. Since for an operator A which is not closed, the operators $\lambda - A$ and $R(\lambda, A)$ (in case $\lambda - A$ is injective) will not be closed. Therefore, $\rho(A) = \emptyset$. For a closed operator A, the resolvent set has the slightly simpler description $\rho(A) = \{\lambda \in \mathbb{C} \mid \lambda - A : D(A) \to \mathbb{H} \text{ is bijective}\}$ according to the closed graph theorem F.10.

Assertion F.13. *For a closed operator A the resolvent set $\rho(A)$ is an open subset of $\mathbb{C}$ and the spectrum $\sigma(A)$ is closed. Moreover, when $\lambda_0 \in \rho(A)$ the open disc $D = \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_0| < \|R(\lambda_0, A)\|^{-1}\}$ is contained in $\rho(A)$ and*

$$R(\lambda, A) = \sum_0^\infty (\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1} \ ,$$

where the series converges in norm and uniformly on compact subsets of the disc. In particular, $R(\,\cdot\,, A) : \rho(A) \to \mathrm{B}(\mathbb{H})$ is continuous and holomorphic.

Observation F.14. The spectrum $\sigma(A)$ of an operator is divided into the following subsets:

$\sigma_p(A) := \{\lambda \in \sigma(A) \mid \lambda - A \text{ is not injective}\}$
(POINT SPECTRUM)

$\sigma_c(A) := \{\lambda \in \sigma(A) \mid \lambda - A \text{ is injective, } \overline{\mathrm{Im}(\lambda - A)} = \mathbb{H}, \text{ and } R_\lambda(A)$
$\text{is not bounded}\}$
(CONTINUOUS SPECTRUM)

$\sigma_r(A) := \{\lambda \in \sigma(A) \mid \lambda - A \text{ is injective and } \overline{\mathrm{Im}(\lambda I - A)} \neq \mathbb{H}\}$
(RESIDUAL SPECTRUM)

$\sigma_p(A), \sigma_c(A), \sigma_r(A)$ are pairwise disjoint and $\sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A) = \sigma(A)$.

Note, that $\lambda \in \sigma_p(A)$ if and only if $A = \lambda\phi$ has a nontrivial solution ϕ, i.e. if λ is an eigenvalue of A.

It is easy to see, that a bounded operator A with $D(A) = \mathbb{H}$ has bounded spectrum $\sigma(A) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|A\|\}$ and $\sigma(A) \neq \emptyset$. If the operator A s not bounded it can happen that $\sigma(A) = \mathbb{C}$, even if A is densely defined, but also that $\sigma(A) = \emptyset$.

We have deduced that a self-adjoint operator has to be densely defined, symmetric and closed. What property is missing? The crucial missing property can be expressed using the spectrum as the following result shows.

Proposition F.15. *Let A be an operator on a Hilbert space $\mathbb{H}$ which is closed and symmetric. Then*

1. *The index $d_\lambda(A) := \dim(\mathrm{Im}(\lambda - A))^\perp$*
   (a) *is constant throughout the open upper half-plane.*
   (b) *is constant throughout the open lower half-plane.*
2. *For the spectrum $\sigma(A)$ one of the following alternatives holds true*
   (a) *$\sigma(A)$ is the closed upper half-plane,*
   (b) *$\sigma(A)$ is the closed lower half-plane,*
   (c) *$\sigma(A)$ is the entire plane,*
   (d) *$\sigma(A)$ is a subset of the real axis.*
3. *A is self-adjoint if and only if the indices $d_\lambda(A)$ are zero for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*
4. *A is self-adjoint if and only if $\sigma(A) \subset \mathbb{R}$ i.e. the case 2.(d) holds.*

In particular: $\sigma(A) \subset \mathbb{R}$ if and only if A is essentially self-adjoint. When A is self-adjoint: $\sigma_r(A) = \emptyset$.

The indices $d_+(A) := d_i(A)\,,\, d_-(A) := d_{-i}(A)$ are called the deficiency indices. $d_\pm(A)$ can be expressed as follows

$$d_\pm(A) = \dim\{\phi \in \mathbb{H} \mid A^*\phi = \pm i\phi\}\,,$$

since $\mathrm{Ker}\,(\bar{\lambda} - A^*) = (\mathrm{Im}(\lambda - A))^\perp$.

Corollary F.16. *A symmetric operator A is self-adjoint if it satisfies $d_+(A) = d_-(A) = 0$. Moreover, A is essentially self-adjoint if $d_+(A) = d_-(A) = 0$, and it has a self-adjont extension if $d_+(A) = d_-(A)$.*

We present three illustrative examples: Let $\mathbb{H} = \mathrm{L}^2(I)$ with $I \subset \mathbb{R}$ a closed interval in $\mathbb{R}$. I is of the form $I = \mathbb{R}, I = [a, \infty[, T =]\infty, b]$ or $I = [a, b]$ for $a, b \in \mathbb{R}$.

Example F.17. The multiplication operator $M = M_v$ for a measurable function $v : I \to \mathbb{R}$ (position operator in the appropriate context and with $v(x) = x$) is given by

$$D(M) := \{\phi \in \mathbb{H} \mid \int_I |v(x)|^2 |\phi(x)|^2 < \infty\},$$

$$\phi \mapsto M\phi := v\phi\,,\ phi \in D(M).$$

We have $d_\pm = 0$ since $v\phi = \pm i\phi$ is satisfied only for $\phi = 0$. Hence M is self-adjoint. This example is related to the example in Remark 4° of F.3.

Example F.18. The differentiation operator P (momentum operator in the appropriate context)

$$\phi \mapsto P\phi := -i\phi'\,,$$

with $\phi \in D(P)$, where $D(P) := \mathbb{C}_0^\infty(I)$ is the space of infinitely often differentiable functions on I with compact support in the interior of I. Of course, P is symmetric.

In case of $I = \mathbb{R}$ the indices are $d_\pm = 0$ and P is essentially self-adjoint. This is the example in Remark 5° of F.3.

In case of $I = [a, \infty[$ we have $d_+(P) = 1$ and $d_-(P) = 01$, since $P(\phi) = -i\phi' = i\phi$ for $\phi = e^{-x}$ and since $P(\phi) = -i\phi$, i-e. $\phi' = \phi$ has no non-trivial solution $\phi \in L^2([a, \infty[)$. Thus there is no self-adjoint extension. The case $]\infty, b]$ is analogous.

In the case of $I = [a, b]$, $d_\pm(P) = 1$. A class of self-adjoint extensions can be described by boundary conditions. For $t \in]0, 1]$ set

$$D(P_t) := \{\phi \in \mathbb{H} \mid \phi \text{ absolutely continuous with } \phi' \in \mathbb{H}\,,\ \phi(a) = \phi(b)\exp(2\pi i t)\}\,.$$

Then $P_t\phi = -i\phi\,,\ \phi \in D(P_t)$, is a self-adjoint extension of P.

Example F.19. The Laplace operator $H := -\Delta\,,\ \phi \mapsto -\phi''$, is symmetric on $D = \mathcal{C}_0^\infty(I) \subset \mathbb{H}$.

In case of $I = \mathbb{R}$ the indices are $d_\pm = 0$ and H is essentially self-adjoint.

In case of $I = [a, \infty[$ we have $d_\pm(H) = 1$. The self-adjoint extensions of H are determined by boundary conditions of the form $\phi(a)\cos\theta + \phi'(a)\sin\theta = 0$. The case $]\infty, b]$ is analogous.

In the case of $I = [a, b]$, $d_\pm(H) = 2$. The self-adjoint extensions are determined by boundary conditions of the form $\phi(a) = \phi(b) = 0$, or $\phi'(a) = \phi(b)' = 0$, or $\phi(a) = \phi(b)$ and $\phi'(a) = \phi(b)'$.

Spectral Theory for Self-Adjoint Operators

One of the most beautiful and efficient results for self-adjoint operators is the spectral theorem.

In order to motivate the spectral theorem let us have a look at the finite dimensional case. A symmetric operator A in a finite dimensional complex Hilbert space $\mathbb{H}$ is already self-adjoint. Symmetry can be described by the matrix which represent A: Let (e_j) be an orthonormal basis of $\mathbb{H}$. An operator A is symmetric if and only if its matrix (a^j_k) with respect to the basis (e_j) is Hermitian, i.e. if $(a^j_k) = (\bar{a}^j_k)^\top$. From Linear Algebra we know:

Theorem F.20 (Spectral Theorem in Finite Dimension)**.** *Every symmetric operator A in an n-dimensional Hilbert space $\mathbb{H}$ has diagonal form*

$$A\phi = \sum_1^n \lambda_j \langle e_j, \phi \rangle \, e_j \, ,$$

where (e_j) is a suitable orthonornal basis and $\lambda_j \in \mathbb{R}$.

The λ_j , $j = 1, \ldots, n$, are the eigenvalues of A and constitute the spectrum: $\sigma(A) = \{\lambda_j\}$. The eigenspaces are the subspaces $\operatorname{Ker}(\lambda_j - A)$.

Example F.21. As a generalization we consider a sequence $(\lambda_j)_{j\in\mathbb{N}}$ of complex numbers $\lambda_j \in \mathbb{C}$ and define in $\mathbb{H} = \ell^2$ the operator $A\phi := (\lambda_j \phi_j)$, $\phi = (\phi_j) \in D(A)$, where $D(A) := \{\phi \in \ell^2 \mid \sum |\lambda_j \phi_j|^2 < \infty\}$. $D(A)$ is dense and the adjoint A^* is given through the sequence $(\overline{\lambda}_j)$ with $D(A^*) = D(A)$. Hence, A is self-adjoint if and only if $\lambda_j = \overline{\lambda}_j$ for all $j \in \mathbb{N}$. The operator A can be expressed as

$$A\phi = \sum_1^n \lambda_j \langle e_j, \phi \rangle \, e_j \, ,$$

where (e_k) is the orthonormal basis $e_k := (\delta_{jk})_{j\in\mathbb{N}}$.

Now, the spectral theorem for a self-adjoint and compact operator[127] A in a Hilbert space $\mathbb{H}$ says that T is essentially of the form just described: There exist a unitary operator $U : \mathbb{H} \to \ell^2$ and a sequence (λ_j) of real numbers such that $T = U^{-1}AU$. In that case (λ_j) has at most 0 as an accumulation point and the eigenspaces $\operatorname{Ker}(\lambda_j - A)$ are finite dimensional for $\lambda_j \neq 0$.

Note, that the example A is a multiplication operator: ℓ^2 is $L^2(\mathbb{N})$ for the measure $\mu(e_j) = 1$ and with $v(j) = \lambda_j$ the operator A has the form $A\phi = v\phi$.

In the following we need a generalization of the example in Remark 4° of F.3 which includes the example just described.

Example F.22 (General Multiplication Operator)**.** Let (Ω, Σ, μ) a measure space and let $v : \Omega \to \mathbb{C}$ be a measurable function. The multiplication operator $M = M_v$ in $\mathbb{H} := L^2(\Omega, \mu)$ is defined by

$$\begin{aligned} M\phi &:= v\phi \, , \; \phi \in D(M) \, , \\ D(M) &:= \{\phi \in \mathbb{H} \mid \int_\Omega |v(x)\phi(x)|^2 d\mu(x) < \infty\} \, . \end{aligned}$$

[127] T is compact if the image $T(B(0,1))$ of the unit ball has a compact closure in $\mathbb{H}$.

It can be proven that $D(M)$ is dense in $\mathbb{H}$. Hence, the adjoint exists. As in the example in Remark 4° of F.3 one shows that $M_v^* = M_{\overline{v}}$. Therefore, M_v is self-adjoint if and only if $v = \overline{v}$, i.e. v is real-valued. M_v is bounded if and only if v is essentially bounded,i.e. there is a subset $N \subset \Omega$ of measure 0 such that $\sup\{|v(q)| \mid q \in \Omega \setminus N\} < \infty$..

Theorem F.23 (Spectral Theorem – Multiplication Form)**.** *Let* $(A, D(A))$ *be a self-adjoint operator on a separable Hilbert space* $\mathbb{H}$*. Then there exists a* (Σ-finite*) measure space* (Ω, Σ, μ)*, a measurable function* $v : \Omega \to \mathbb{R}$ *and a unitary operator* $U : \mathbb{H} \to L^2(\Omega, \mu)$ *with*

- $\phi \in D(A) \iff U\phi \in D(M_v)$,
- $UAU^{-1} = M_v$ *on* $D(M_v)$.

$$\begin{array}{ccccc} \mathbb{H} & \supset & D(A) & \xrightarrow{A} & \mathbb{H} \\ {\scriptstyle U}\downarrow & & \uparrow{\scriptstyle U^{-1}} & & {\scriptstyle U}\downarrow \\ \mathbb{H} & \supset & D(M) & \xrightarrow{M} & \mathbb{H} \end{array}$$

Example F.24 (Hamiltonian of a Free Particle)**.** In classical mechanics the dynamics of the free particle in $Q = \mathbb{R}^n$ is determined by the Hamiltonian $H : M = TQ \to \mathbb{R}$, the energy:

$$H(q,p) = \frac{1}{2m} \sum (p_j)^2 \,.$$

We set $m = 1$ in the following. The quantum mechanical counterpart is the Hamiltonian

$$\hat{H} = \frac{1}{2} \sum_1^n \left(-i \frac{\partial}{\partial p_j} \right)^2 ,$$

in $\mathbb{H} = L^2(\mathbb{R}^n)$. We obtain $\hat{H}$ from canonical quantization in elementary Quantum Mechanics. The Hamiltonian $\hat{H}$ is essentially the negative of the Laplacian: $\hat{H} = -\frac{1}{2}\Delta$.

In order to describe an equivalent multiplication operator we use the Fourier transform $\mathcal{F}$ given as

$$\mathcal{F}(\phi)(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \phi(y) e^{i\langle x,y \rangle} dy \,,\, \phi \in \mathcal{E}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \,,$$

defining a unitary map $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. For differentiable $\phi \in L^2(\mathbb{R}^n)$ partial differentiation $-i\frac{\partial}{\partial x^k}$ and multiplication by x^k is interchanged by $\mathcal{F}$ in the folloewing way:

$$-i \frac{\partial \phi}{\partial x^k} = \mathcal{F}^{-1} x^k \mathcal{F} \phi = \left(\mathcal{F}^{-1} M_{x^k} \mathcal{F} \right) \phi \,.$$

Therefore,

$$-\Delta = \mathcal{F}^{-1} M_v \mathcal{F} \,,\, M_v = \mathcal{F}(-\Delta)\mathcal{F}^{-1} \,, \tag{92}$$

with $v = ||x||^2$ and we have a concrete example of a self-adjoint operator being unitarily equivalent to a multiplication operator. Note, that we have neglected to describe the domain and check the self-adjointness of $-\Delta$. But this can be done now using (92).

We also can determine the spectrum of $-\Delta$ as an application of the subsequent lemma: $\sigma(-\Delta) = [\,0, \infty\,[$.

Lemma F.25. *The spectrum of the multiplication operator M_v is the essential range of v defined as*

$$\mathrm{essrg}(v) := \{\lambda \in \mathbb{C} \mid \text{ for all } \varepsilon > 0 : \mu(\{\omega \in \Omega \mid |v(\omega) - \lambda| < \varepsilon\}) > 0\} \ :$$

$\sigma(M_v) = \mathrm{essrg}(v)$.

Observation F.26. Note that the spectral theorem allows one to introduce a Functional Calculus for self-adjoint operators: If A is a self-adjoint operator in $\mathbb{H}$ with $UAU^{-1} = M_v$ (according to Theorem F.23) and $f : \mathbb{R} \to \mathbb{C}$ a measurable function. Then an operator $f(A)$ can be defined in the following way:

$$D(f(A)) := \{\phi \in \mathbb{H} \mid (f \circ v)U\phi \in \mathrm{L}^2(\Omega, \Sigma, \mu)\}\,,$$

$$f(A)\phi := U^{-1}M_{f\circ v}U\phi\,, \quad \text{for } \phi \in D(f(A))\,.$$

We come back to the functional calculus in Observation F.37 after having introduced spectral families in order to formulate the spectral theorem in the spectral measure form.

To motivate the spectral theorem in the spectral measure form let us go back again to the finite dimensional case. Let A be a self-adjoint (i.e. symmetric) operator in the n-dimensional Hilbert space $\mathbb{H}$. Then the eigenspaces of A yield a decomposition of $\mathbb{H}$: Let $P_1, P_2, \ldots, P_k$ be the orthogonal projections $P_j : \mathbb{H} \to \mathbb{H}$ onto the pairwise orthogonal eigenspaces of A corresponding to the eigenvalues $\lambda_q, \ldots, \lambda_k$. Then

$$A = \sum_1^k \lambda_j P_j\,.$$

Example F.27. A more general situation is the decomposition of a general separable $\mathbb{H}$ into closed subspaces $\mathbb{H}_j$ with projections $P_j : \mathbb{H} \to \mathbb{H}$ with $\mathrm{Im}\, P_j = \mathbb{H}_j$ and $\mathbb{H} = \bigoplus \mathbb{H}_j$. Given a sequence $(\lambda_j)_{j\in\mathbb{N}}$ of pairwise different real numbers the definition

$$A\phi := \sum \lambda_j P_j\phi\,,\ \phi \in D(A) := \{\phi \mid \sum \lambda_j P_j \phi \ \text{ converges}\}$$

yields a self-adjoint operator described by projections.

The sum can be given by a suitable integral as well. This approach can be generalized to the concept of a spectral family, or spectral resolution of the identity. For the formulation of the concept we need some elementary results on orthogonal projections.

Recall that a projection (more precisely an orthogonal projection) in the Hilbert space $\mathbb{H}$ is a bounded operator $P : \mathbb{H} \to \mathbb{H}$ with $P \circ P = P$. Then $Q := I - P$ is again a projection and $I = P + Q$. As a consequence, $\operatorname{Ker} P$ and $\operatorname{Im} P$ are closed subspaces with $\mathbb{H} = \operatorname{Ker} P \oplus \operatorname{Im} P$. A projection is self-adjoint. Each closed subspace $V \subset \mathbb{H}$ has a unique orthogonal complement W and defines a unique projection P with $V = \operatorname{Im} P$ and $W = \operatorname{Ker} P$.

For a second projection Q we define $P \geq Q$ if and only if $\operatorname{Ker} P \subset \operatorname{Ker} Q$ which is equivalent to $\operatorname{Im} Q \subset \operatorname{Im} P$. In case of $P \geq Q$ the equality $Q = P \circ Q = Q \circ P$ holds and the difference $P - Q$ is again a projection, namely the projection onto the subspace $\operatorname{Im} P \cap \operatorname{Ker} Q$. The following result is easy to show.

Proposition F.28. *Let (P_k) be a increasing sequence of projections. Then the projection P induced by the closed subspace*

$$\operatorname{Im} P = \overline{\bigcup_0^\infty \operatorname{Im} P_k}$$

is the pointwise limit of the P_k: $P\phi = \lim P_k \phi$. Analogously, for a decreasing sequence (P_k) of projections. In that case the limit is the projection onto

$$\operatorname{Im} P = \overline{\bigcap_0^\infty \operatorname{Im} P_k} \, .$$

Definition F.29 (Spectral Family)**.** A spectral family is a map $E : \mathbb{R} \to \mathrm{B}(\mathbb{H})$, often written in the form $(E_\lambda) = (E_\lambda)_{\lambda \in \mathbb{R}}$, with the following properties:

1. Each E_λ is a projection.
2. $E_\lambda \leq E_\mu$ for $\lambda \leq \mu$, $\lambda, \mu \in \mathbb{R}$.
3. $\lim_{\lambda \searrow -\infty} E_\lambda = 0 \, , \lim_{\lambda \nearrow \infty} E_\lambda = 1$.
4. $\lim_{\varepsilon \searrow 0} E_{\lambda + \varepsilon} = E_\lambda$ for all $\lambda \in \mathbb{R}$.

$(E_\lambda)_{\lambda \in \mathbb{R}}$ is also called spectral resolution of the identity.

The support $\operatorname{Supp}(E_\lambda)$ of (E_λ) is the interval

$$I = \operatorname{Tr}(E_\lambda) = \overline{\{\lambda \in \mathbb{R} : E_\lambda \neq 0 \text{ or } E_\lambda \neq 1\}}.$$

E_λ is called to have bounded support when this interval is bounded.

Note that for increasing (resp. decreasing) sequences of real numbers (λ_k) the corresponding projections (E_{λ_k}) is increasing (resp. decreasing), so that Proposition F.28 is applicable. The convergence in 3. and 4. is meant is the sense of this proposition, it is pointwise convergence.

Example F.30. Let P be a projection and $a, b \in \mathbb{R}$, $a < b$. Set $E_\lambda = 0$ for $\lambda < a$, $E_\lambda = P$ for $a \leq \lambda < b$ and $E_\lambda = 1$ for $b \leq \lambda$. The spectral family has $[a, b]$ as its support. The spectral theorem in finite dimension leads to a similar spectral family with finitely many projections P_k

Example F.31. The spectral family for the Example F.27 and generalizing Example F.21 is the following

$$E_\lambda = \sum_{\lambda_j \leq \lambda} \lambda_j P_j \, .$$

Example F.32. As in the Example F.22, let (Ω, Σ, μ) a measure space and let $v : \Omega \to \mathbb{C}$ be a measurable function defining the multiplication operator $M = M_v$ in $\mathbb{H} := L^2(\Omega, \mu)$, $M\phi = v\phi$, $\phi \in D(M)$. Assume v to be real-valued. Denote $S(\lambda) := \{\omega \in \Omega \mid v(\omega) \leq \lambda\}$ and let $\chi_{S(\lambda)} : \Omega \to \{0, 1\}$ be the corresponding characteristic function of $S(\lambda)$. Then

$$E_\lambda \phi := \chi_{S(\lambda)} \phi \, , \, \phi \in \mathbb{H}$$

is a spectral family. (It is the spectral family of M, c.f. Example F.35.)

Recall the Riemann-Stieltjes integral: Let $w : \mathbb{R} \to \mathbb{R}$ be weight function which we assume to be increasing and continuous from the right. The Riemann-Stieltjes integral $\int f dw$ for continuous functions $f : \mathbb{R} \to \mathbb{R}$ is defined in essentially the same manner as the Riemann integral:

$$\int_a^b f(\lambda) dw(\lambda) \, , \, a < b \, ,$$

is the limit of sums

$$\sum_1^n f(t_j)(w(t_j) - w(t_{j-1})) \, ,$$

with $a = t_0 < t_1 > \ldots < t_n = b$ where the length $\sup(t_j - t_{j-1})$ tends to zero.

In order to introduce the concept of integrating a spectral family (E_λ) we use the weights

$$w_\phi(\lambda) := \langle \phi, E_\lambda \phi \rangle \, ,$$

w_ϕ is decreasing, continuous from the right and bounded. w_ϕ determines a measure $w_\phi d\lambda = dw_\phi$ on $\mathbb{R}$ and so we obtain a well-defined integral

$$\int_{\mathbb{R}} f(\lambda) w_\phi d\lambda = \int_{\mathbb{R}} f(\lambda dw_\phi(\lambda) \, .$$

(Without measure theory this integral can also be expressed by the (improper) Riemann-Stieltjes integral for continuous f.)

This integral is also denoted by

$$\int f(\lambda) \langle \phi, E_\lambda \phi \rangle d\lambda = \int f(\lambda) d\langle \phi, E_\lambda \phi \rangle = \int f(\lambda) \langle \phi, dE_\lambda \phi \rangle = \int f(\lambda) dE_\lambda \phi \, .$$

Proposition F.33 (Integrating a Spectral Family)**.** *Let $(E_\lambda)_{\lambda\in\mathbb{R}}$ be a spectral family. For each measurable function $f:\mathbb{R}\to\mathbb{R}$ one obtains the operator $\hat{E}(f)$ in $\mathbb{H}$ in the following way:*

$$D(\hat{E}(f)) := \{\phi\in\mathbb{H} \mid \int_{\mathbb{R}} |f|^2\langle\phi, E_\lambda\phi\rangle d\lambda < \infty\}\,,$$
$$\hat{E}(f)\phi := \int_{\mathbb{R}} f(\lambda)\langle\phi, E_\lambda\phi\rangle d\lambda\,,\ \ \phi\in D(\hat{E}(f))\,.$$

$\hat{E}(f)$ is self-adjoint and we write

$$\hat{E}(f) = \int f(\lambda)dE_\lambda = \int f dE_\lambda\,.$$

$\hat{E}(f)$ is bounded whenever f is bounded.

Example F.34. Applied to the Examples F.27, F.21 and F.31 let $f:\mathbb{R}\to\mathbb{R}$. Then

$$\hat{E}(f) = \sum_j f(\lambda_j)P_j\,.$$

Example F.35. In the Example F.32 let $f:\mathbb{R}\to\mathbb{R}$ a step function, i.e. $f(\lambda) = \sum_1^m c_j\chi_{I_j}$ with finitely many pairwise disjoint intervals. The integral is

$$\hat{E}(f)\phi(\omega) = \left(\int f(\lambda)dE_\lambda\phi\right)(\omega) = \sum_1^m c_j(v(\omega))\phi(\omega) = f\circ v(\omega)\phi(omega)\,,$$

for $\phi\in\mathbb{H}$, so $\hat{E}(f)\phi)(F\circ v)\phi$. As a consequence, for any measurable f

$$D(\hat{E}(f)) = \{\phi\in\mathbb{H} \mid (f\circ v)\phi\in\mathbb{H}\}$$
$$\hat{E}(f)\phi = (f\circ v)\phi\,.$$

In particular, with $f = \mathrm{id}_{\mathbb{R}}$:

$$\hat{E}(\mathrm{id}) = \int \lambda dE_\lambda = M_v\,.$$

As a generalization we have:

Theorem F.36 (Spectral Theorem – Spectral Measure Form)**.** *To each self-adjoint operator $A\in\mathcal{SA}$, there corresponds a unique spectral family $(E_\lambda)_{\lambda\in\mathbb{R}}$ such that $A = \hat{E}(\mathrm{id}_{\mathbb{R}})$, i.e.*

$$\boxed{A = \int \lambda dE_\lambda.}$$

The spectral family is given by

$$\langle\psi, (E_b - E_a)\phi\rangle = \lim_{\delta\searrow 0}\lim_{\varepsilon\searrow 0}\frac{1}{2\pi i}\int_{a+\delta}^{b+\varepsilon}\langle\psi, \left(R(\lambda - i\varepsilon A) - R(\lambda + i\varepsilon A)\right)\phi\rangle\,.$$

With the aid of the Spectral Theorem – Multiplication Form F.23 this version of the spectral theorem can be proven using the preceding Example F.35.

Observation F.37. The functional calculus introduced in Observation F.26 can now be reformulated. If E_λ is the spectral family of a self-adjoint operator A, then for measurable functions $f : \mathbb{R} \to \mathbb{R}$ the corresponding self-adjoint operator is

$$\boxed{f(A) = \int f(\lambda) dE_\lambda .}$$

For each spectral family a self-adjoint operator is defined as $A := \int \lambda dE_\lambda$. The corresponding one-parameter group of unitary operators is, in accordance with Stone's Theorem, but now using functional calculus the family $U_s = \int e^{-is\lambda} dE_\lambda$ of unitary operators.

A final remark concerning the use of the term "spectral measure form": A spectral family induces an abstract projection valued measure on $\mathbb{R}$. In fact for interval $J := [a, b[$ one can define $p(J) := E_b - E_a$ to obtain a map on the set $\mathcal{B}(\mathbb{R})$ of Borel subsets of $\mathbb{R}$ whose values are projections: $p : \mathcal{B}(\mathbb{R}) \to \mathrm{B}(\mathbb{H})$. This is the so called projection valued measure induced by the spectral family. Vice versa, any spectral family can be induced by a spectral measure $E_\lambda := p(] - \infty, \lambda])$.

F.3 Canonical Commutation Relations

Definition F.38. The Canonical Commutation Relations (CCR) or Heisenberg commutation relations are

$$[P_j, P_k] = 0\,,\ [Q_j, Q_k] = 0\,,\ [P_j, Q_k] = -i\delta_{jk}\,, \quad \text{for} \quad 1 \le j, k \le n\,,$$

for a set $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ of operators or elements of a Lie algebra.

In Quantum Mechanics one is interested to realize the CCR by linear operators in a Hilbert space $\mathbb{H}$. In fact, in some treatises on the foundation of Quantum Mechanics a realization of the CCR is part of the postulates, for example in the following form:

Postulate *For a quantum mechanical system in which cartesian coordinates q_j with corresponding momenta p_j, $j = 1, \dots, n$, are represented by self-adjoint operators Q_j and P_j in a Hilbert space $\mathbb{H}$ the following realization of the CCR has to be satisfied*

$$[P_j, P_k] = 0\,,\ [Q_j, Q_k] = 0\,,\ [P_j, Q_k] = -i\delta_{jk}\mathrm{id}_{\mathbb{H}}\,, \quad \textit{for} \quad 1 \le j, k \le n\,.$$

Observation F.39. We have seen a realization of the CCR at several places in these lecture notes, for instance in the context of prequantization in (35) or later in the case of using natural polarizations in the simple case (see Examples 10.12). We recall a special case, the so-called SCHRÖDINGER REPRESENTATION: The Hilbert space is $\mathbb{H} = L^2(\mathbb{R}^n, \lambda)$ ($\lambda = dq$ Lebesgue measure), and the Q_j, P_j are the unbounded self-adjoint operators in $\mathbb{H}$ with

$$Q_j(\phi) := q_j\phi\,,$$
$$P_j(\phi) : -i\frac{\partial}{\partial q^j}\phi\,,$$

$j = 1, \dots, n$, where the $\phi \in \mathbb{H}$ are in the respective domains $D(P_j)$, $D(Q_j) \subset \mathbb{H}$.

Note, that the Schrödinger representation is a representation of the Heisenberg algebra $\mathfrak{hs}_n$ (see Example C.14) by self-adjoint operators.

Are there simpler realizations of the CCR?

It is easy to see that for any realization of the CCR the Hilbert space $\mathbb{H}$ has to be infinite dimensional. The identity $PQ - QP = -i \cdot \mathrm{id}_{\mathbb{H}}$ (CCR with $n = 1!$) is not possible for linear maps in a d-dimensional Hilbert space $\mathbb{H} \neq \{0\}$, since the trace of a commutator is zero while the trace of $\mathrm{id}_{\mathbb{H}}$ is $d \neq 0$.

Furthermore, $PQ - QP = -i \cdot \mathrm{id}_{\mathbb{H}}$ (with self-adjoint P, Q) can only hold for unbounded operators. To see this, we need a formula for iterated commutators: We define $P^{(m)}(Q)$ by recursion, $P^{(0)}(Q) := Q$ and $P^{(m+1)}(Q) := [P, P^{(m)}(Q)]$, and obtain the following formula by induction.

Lemma F.40.

$$\frac{1}{m!}P^{(m)}(Q) = \sum_{k=0}^{m} \frac{1}{k!}P^kQ\frac{(-1)^{m-k}}{(m-k)!}P^{m-k}\,.$$

Proof. The formula holds for $m = 0$. The induction step $m \to m + 1$:

$$\frac{1}{(m+1)!}P^{(m+1)}(Q) = \frac{1}{(m+1)!}[P, P^{(m)}(Q)] = \frac{1}{m+1}\left(\frac{1}{m!}P^{(m)}(Q) - \frac{1}{m!}P^{(m)}(Q)P\right)$$

By the induction hypothesis:

$$
\begin{aligned}
&= \frac{1}{m+1}\left(P\sum_{k=0}^{m}\frac{1}{k!}P^kQ\frac{(-1)^{m-k}}{(m-k)!}P^{m-k} - \sum_{k=0}^{m}\frac{1}{k!}P^kQ\frac{(-1)^{m-k}}{(m-k)!}P^{m-k}P\right)\\
&= \frac{1}{m+1}\left(\sum_{k=1}^{m+1}\frac{1}{(k-1)!}P^kQ\frac{(-1)^{m-k+1}}{(m-k)+1!}P^{m-k+1} - \sum_{k=0}^{m}\frac{1}{k!}P^kQ\frac{(-1)^{m-k}}{(m-k)!}P^{m-k+1}\right)\\
&= \frac{1}{m+1}\left(\frac{1}{m!}P^{m+1}Q\right)\\
&+ \frac{1}{m+1}\left(\sum_{k=1}^{m}\frac{1}{(k-1)!}P^kQ\frac{(-1)^{m-k+1}}{(m-k)+1!}P^{m-k+1} - \sum_{k=1}^{m}\frac{1}{k!}P^kQ\frac{(-1)^{m-k}}{(m-k)!}P^{m-k+1}\right)\\
&- \frac{1}{m+1}\left(Q\frac{(-1)^m}{(m)!}P^{m+1}\right)\\
&= \frac{1}{(m+1!)}P^{m+1}Q + \frac{1}{m+1}\left(\sum_{k=1}^{m}\frac{k}{k!}P^kQ\frac{(-1)^{m+1-k}}{(m+1-k)!}P^{m+1-k}\right.\\
&+ \left.\sum_{k=1}^{m}\frac{1}{k!}P^kQ\frac{(-1)^{m+1-k}(m+1-k)}{(m+1-k)!}P^{m+1-k}\right) + \frac{(-1)^{m+1}}{(m+1)!}QP^{m+1}\\
&= \frac{1}{(m+1!)}P^{m+1}Q + \frac{m+1}{m+1}\sum_{k=1}^{m}\left(\frac{1}{k!}P^kQ\frac{(-1)^{m+1-k}}{(m+1-k)!}P^{m+1-k}\right) + \frac{(-1)^{m+1}}{(m+1)!}QP^{m+1}\\
&= \sum_{k=0}^{m+1}\left(\frac{1}{k!}P^kQ\frac{(-1)^{m+1-k}}{(m+1-k)!}P^{m+1-k}\right)
\end{aligned}
$$

□

Proposition F.41. *Let P, Q be self-adjoint operators in the Hilbert space $\mathbb{H}$ and define $U(t) := e^{itP}$. Then*

$$U(t)QU(-t) = \sum_{m=0}^{\infty}\frac{(it)^m}{m!}P^{(m)}Q\,.$$

In particular, if $PQ - QP = -i\cdot\mathrm{id}_{\mathbb{H}}$, it follows that $U(t)QU(-t) = Q + t\cdot\mathrm{id}_{\mathbb{H}}$.

Proof. Formally, we have

$$U(t)QU(-t) = \sum_k\frac{(it)^k}{k!}P^kQ\sum_n\frac{(-it)^n}{n!}P^n\,,$$

and using Lemma F.40

$$\sum\frac{(it)^k}{k!}P^kQ\sum\frac{(-it)^n}{n!}P^n = \sum_m\sum_{k+n=m}\frac{(it)^k}{k!}P^kQ\frac{(-it)^n}{n!}P^n = \sum_m\frac{(it)^m}{m!}P^{(m)}(Q)$$

we obtain the first result. (The formal calculation can be justified by applying it to vectors ϕ in the range of a spectral projection of Q.) If now $[P, Q] = -i$ holds, the brackets $P^{(m)}Q$ vanish for $m > 1$ and the sum

$$\sum_m \frac{(it)^m}{m!} P^{(m)}(Q)$$

reduces to $Q + (it)P^{(1)}Q = Q + t$. □

Corollary F.42. *Self-adjoint operators P, Q on a Hilbert space $\mathbb{H}$ with $PQ - QQ = -i$ are unbounded with spectrum $\sigma(Q) = \sigma(P) = \mathbb{R}$.*

Proof. If Q were bounded then $\lambda - Q$ would be invertible as a bounded operator for $|\lambda| > \|Q\|$. Since $Q + t$ is unitarily equivalent to Q for all real t the operator Q would have empty spectrum contradicting the fact that self-adjoint operators have non-empty spectrum. For Q as an unbounded operator each $t \in \mathbb{R}$ turns out to be in the spectrum of Q, since $Q + t$ is unitarily equivalent to $Q + s$ for all $s \in \mathbb{R}$. $\sigma(P) = \mathbb{R}$ by using the symmetry $(P, Q) \mapsto (Q, -P)$. □

We now introduce the so-called Weyl relations.

Proposition F.43. *Let P, Q be self-adjoint operators in the Hilbert space $\mathbb{H}$ satisfying $PQ - QP = -i \cdot \mathrm{id}_{\mathbb{H}}$, and define $U(t) := e^{itP}$, $V(s) := e^{isQ}$. Then*

$$U(t)V(s) = e^{ist}V(s)U(t)\,.$$

Proof. Generalizing Lemma F.40 one gets $e^{itP}f(Q)e^{-itP} = f(Q+t)$ for any measurable function on $\mathbb{R}$ using the functional calculus or some induction formulas as above. As a result

$$e^{itP}e^{+isQ}e^{-itP} = e^{+is(Q+t)} = e^{ist}e^{+isQ}\,.$$

□

This is the "integrated" version of the CCR in case of $n = 1$. In order to formulate the Theorem of Stone-von Neumann we need some definitions:

Definition F.44. A pair of strongly continuous unitary groups $(U(t))$ and $(V(s))$ on a Hilbert space $\mathbb{H}$ is called to represent the Weyl relations, if $U(t)V(s) = e^{ist}V(s)U(t)$ holds for all $s, t \in \mathbb{R}$.

The representation is called Irreducible, if there is no non-trivial closed linear subspace $\mathbb{H}_0 \subset \mathbb{H}$ such that $U(t)\mathbb{H}_0 \subset \mathbb{H}_0$, $V(s)\mathbb{H}_0 \subset \mathbb{H}_0$ for all $s, t \in \mathbb{R}$.

Two representations $(U(t), V(t))$ on $\mathbb{H}$ and $(U'(t), V'(s))$ on another Hilbert space $\mathbb{H}'$ are called unitarily equivalent if there exists a unitary map $\Phi : \mathbb{H} \to \mathbb{H}'$ with

$U(t) = \Phi^{-1}U'(t)\Phi$ and $V(s) = \Phi^{-1}V'(s)\Phi$, i.e. if the following diagram is commutative:

$$\begin{CD} \mathbb{H} @>{U(t)}>{V(s)}> \mathbb{H} \\ @V{\Phi}VV @AA{\Phi^{-1}}A \\ \mathbb{H}' @>{U'(t)}>{V'(s)}> \mathbb{H}' \end{CD}$$

Remark F.45. A representation of the Weyl relations by $U(t), V(s)$ is essentially a projective unitary representation of the abelian group $\mathbb{R}^2$:

$$W' : \mathbb{R}^2 \to \mathrm{U}(\mathbb{P}) \,, \ (p, q) \mapsto [U(p)V(q)] \,,$$

where $[U] = \hat{\gamma}(U) \in \mathrm{U}(\mathbb{P})$ denotes the unitary projective operator determined by the unitary operator $U \in \mathrm{U}(\mathbb{H})$, $\mathbb{P} = \mathbb{P}(\mathbb{H})$: $[U](\gamma(\phi)) = \gamma(U(\phi))$. Recall, that a projective unitary representation can be lifted to a suitable central extension of the group (see Remark C.11). In our situation we can use the central extension

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{HS}_1 \stackrel{\pi}{\longrightarrow} \mathbb{R}^2 \longrightarrow 0 \,.$$

and obtain

$$\begin{CD} 0 @>>> \mathbb{R} @>>> \mathrm{HS}_1 @>{\pi}>> \mathbb{R}^2 @>>> 0 \\ @. @V{\mathrm{id}}VV @V{W}VV @V{W'}VV \\ 0 @>>> \mathbb{R} @>>> \mathrm{U}(\mathbb{H}) @>{\hat{\gamma}}>> \mathrm{U}(\mathbb{P}) @>>> 1 \end{CD}$$

The definition of W in case of the Weyl relations is

$$W(p, q, s) := e^{is}e^{-\frac{1}{2}ipq}U(p)V(q) \,, \ (p, q, s) \in \mathrm{HS}_1 = \mathbb{R}^2 \times \mathbb{R} \,.$$

(See (70) for a proof that this defines a homomorphism.) Of course, for any parameter $\lambda \in \mathbb{R}^\times$ the following definition yields another unitary representation W_λ with $W' \circ \pi = \hat{\gamma} \circ W_\lambda$:

$$W_\lambda(p, q, s) := e^{i\lambda s}e^{-\frac{1}{2}i\lambda pq}U(p)V(q) \,, \ (p, q, s) \in \mathrm{HS}_1 = \mathbb{R}^2 \times \mathbb{R} \,.$$

Given $\lambda \neq \lambda'$ the representations W_λ and $W_{\lambda'}$ are not equivalent.

Proposition F.46. *The Schrödinger representation of the Weyl relations on the Hilbert space $L^2(\mathbb{R})$ given by $U(t)\psi(q) = \psi(q + t)$ and $V(s)\psi(q) = e^{isq}\psi(q)$ is irreducible.*

Proof. Let $\mathbb{H}_0 \neq \{0\}$ be an invariant sub-Hilbert space of $\mathbb{H}$ and $\phi \in \mathbb{H}_0$ $\phi \neq 0$. Let ψ be an arbitrary element of the orthogonal complement $\mathbb{H}_1$ of $\mathbb{H}_0$. Since $\mathbb{H}_1$ is also invariant, we have $U(t)\psi, V(s)U(t)\psi \in \mathbb{H}_1$. Hence, $\langle \phi, V(s)U(t)\psi \rangle = 0$, i.e.

$$\int_{\mathbb{R}} \bar{\phi}(q)e^{isq}\psi(q + t)dq = 0.$$

Since the Fourier transformation

$$\mathcal{F} : \mathbb{H} \to \mathbb{H}\,,\; g \mapsto \mathcal{F}(g)(s) := c \int_{\mathbb{R}} e^{isq} g(q) ds\,,$$

(c some constant) is bijective, the functions $\bar{\phi}(q)\psi(q+t)$ vanish for all $t \in \mathbb{R}$. We conclude $\psi = 0$, and the orthogonal complement $\mathbb{H}_1$ of $\mathbb{H}_0$ is zero, hence $\mathbb{H} = \mathbb{H}_0$. □

Theorem F.47 (Stone-von Neumann)**.** *Any irreducible representation of the Weyl relations is unitarily equivalent to the Schrödinger representation.*

Proofs can be found, for example, in in [Hal13] or [Spe20]. It is interesting that Hermann Weyl was the first to formulate the result in his book on group theory and quantum mechanics in 1928 [Wey31], but he did not provide a proof. Marshall Stone shortly after the appearance of the book pointed out that the result needs a proof and presented in 1930 the unitary map yielding the unitary equivalence. He was not providing a proof that his unitary map really does the job. A thorough proof was given later by John von Neumann in 1931. This and other interesting aspects of the Stone-von Neumann Theorem can be found in the article of J. Rosenberg, see [Ros04].

The corresponding result for n degrees of freedom is of the same nature, and has essentially the same proof. Transforming a representation of the Weyl relations

$$U_k(t)V_j(s) = e^{\delta_{jk}ist} V_j(s)U_k(t)\,,\; j,k = 1,\ldots,n\,,$$

besides $U_k(t)U_j(t') = U_j(t')U_k(t)$ as well as $V_k(s)V_j(s') = V_j(s')V_k(s)$, into a unitary representation W_λ of the Heisenberg Lie group HS_n as in Remark F.45, the result can be described in the following way.

Theorem F.48 (Stone-von Neumann)**.** *Any irreducible unitary representation W of the Heisenberg group HS_n with $W(0,s) = e^{i\lambda s}\mathrm{id}_{\mathbb{H}}$ is unitarily equivalent to the Schrödinger representation W_λ.*

This is a remarkable result. In general, when investigating Lie groups as symmetry groups in physics, geometry or number theory there appear plenty of nonequivalent unitary and irreducible representations with finite and with infinite dimensional Hilbert spaces. For applications and also to obtain a good overview, one is quite content to give a complete list of irreducible unitary representations for a given Lie group (or, more general, for a topological group). The Heisenberg group is different. There is only one irreducible unitary representation up to unitary equivalence and it is infinite dimensional!

Summary:

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Sign Conventions

Let (M, ω) be a symplectic manifold with Poisson bracket $\{\,,\}$, and let $F, G \in \mathcal{E}(M)$ be classical observables, i.e. differentiable functions.

In the first table we list the possibilities for the signs and assign the attributes "A" and "B".

In the second table it is reported how the signs are used in the literature which we cite in these lecture notes.

Table 1: Possible sign conventions

		A	B
Hamiltonian vector field	X_H by	$i_{X_H}\omega = dH$	$i_{X_H}\omega = -dH$
Poisson Bracket	$\{F, G\} =$	$\omega(X_F, X_G)$	$-\omega(X_F, X_G)$
Representation	$[X_F, X_G] =$	$-X_{\omega(X_F, X_G)}$	$X_{\omega(X_F, X_G)}$
Lie Derivative $L_{X_F}G =$	$X_F G =$	$-\{F, G\}$	$\{F, G\}$

Table 2: Usage

Source	citation	X_H	$\{\ ,\ \}$	$\Longrightarrow$	$\pm X_{\{F,G\}}$	$\pm X_F G$
This course, Abraham-Marsden	[AM78]	A	A	$\Longrightarrow$	A	A
Sniaticky, Puta	[Sni80],[Put93]	B	B	$\Longrightarrow$	A	B
Woodhouse, Liberman-Merle	[Woo80],[LM87]	B	A	$\Longrightarrow$	B	B
Brylinski	[Bry93]	A	B	$\Longrightarrow$	B	A

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