LMU Munich — Prof. Dr. P. Pickl
Winter term 2016/17 — A. Schaal

Exercises on Mathematical Statistical Physics II
Sheet 4

Problem 1 (Bogoliubov's Hamiltonian)
In second quantization the Schrödinger Hamiltonian on a torus of unit volume reads

$$H = \sum k^2 a_k^* a_k + N^{-1} \sum_{k,j,l,m} \widehat{V}(k,l,m,n) a_k^* a_l^* a_m a_n$$

where the $\widehat{V}(k,l,m,n)$ is given by

$$\widehat{V}(k,l,m,n) = \int d^3x\, d^3y \exp(ikx + ily - imx - imy) V(x-y)$$

for a spherically symmetric V. The operators a_k and $a_k^\star$ are the annihilation- and creation-operators for a particle with momentum k.
Assuming that the wave function is close to a coherent state w.r.t. the zero mode (i.e. Ψ_0 an eigenstate of a_0 with eigenvalue N), the creation and annihilation operator of the zero mode can in good approximation be replaced by $\sqrt{N}$. Further one can argue on a heuristic level that - after this replacement - terms of order higher than 2 in creation and annihilation operators can be neglected. The resulting Hamiltonian H_{Bog} is known as Bogoliubov's Hamiltonian. Show that - up to a real constant - it can be written in the form

$$H_{\mathrm{Bog}} = \sum_k \left(2\widehat{V}(k,0,k,0) + k^2\right) a_k^* a_k + \widehat{V}(k,-k,0,0) \left(a_k^* a_{-k}^* + a_k a_{-k}\right)$$

Let $\tilde{H}$ be the Hamiltonian defined in the lecture. Give a heuristic argument that the time evolution of Ψ_0 w.r.t. H_{Bog} and $\tilde{H}$ are similar. Why (and in which sense) is the assumption of a coherent state similar to a product state in this setting?

Problem 2 (Trace Norm Convergence of Density Matrices)
Let $\tilde{H}$ again be defined as in the lecture. Show that

$$\lim_{N\to\infty} \|\mu^\psi(x,y) - \tilde{\mu}^\psi(x,y)\|_{\mathrm{tr}} = 0$$

where $\mu^\psi(x,y)$ and $\tilde{\mu}^\psi(x,y)$ are the reduced density matrices of systems with Hamiltonian H resp. $\tilde{H}$. What does this result mean? Give error estimates, make the estimates as strong as possible.

Problem 3 (Large Volume - Smooth Weight Function)
Let Λ be the volume and ρ the density of a N-body system ("$1 \ll \Lambda \ll \rho$"). Consider the weight function $m = 1 - exp\left(-\frac{k}{\rho}\right)$. Show that $|d_t\alpha| \leq \alpha + \frac{C}{N}$ for large Λ, ρ with $\alpha = \langle \psi, \widehat{m}\psi \rangle$ and $N = \rho\Lambda$.

The solutions to these exercises will be discussed on Friday, 25.11.2016.