LMU Munich
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Exercises on Mathematical Statistical Physics II
Sheet 3

Problem 1 (Convergence in Operator Norm)
Let $\alpha^\varphi(\psi)$ and μ^ψ be defined as in the lecture, i.e. $\alpha^\varphi(\psi) := \langle\psi, q^\varphi\psi\rangle$ resp. $\mu^\psi(x,y) = \int \psi_N(x, x_1, x_2, ..., x_N)\psi_N^*(y, x_1, x_2, ..., x_N)dx^2dx^3...dx^N$ (with ψ, φ, q^φ defined as in problem 2 on sheet 2).
In the lecture, it was shown that

$$\lim_{N\to\infty} \alpha^\varphi(\psi) = 0 \Longleftrightarrow \lim_{N\to\infty} \|\mu^\psi(x,y) - |\phi\rangle\langle\phi|\|_{\mathrm{tr}} = 0$$

where $\|\cdot\|_{\mathrm{tr}}$ denotes the trace norm. Show that also

$$\lim_{N\to\infty} \alpha^\varphi(\psi) = 0 \Longleftrightarrow \lim_{N\to\infty} \|\mu^\psi\|_{\mathrm{op}} = |\varphi\rangle\langle\varphi|$$

holds with $\|\mu^\varphi\|_{\mathrm{op}}$ denoting the operator norm of μ^φ.

Problem 2 (Markov's Inequality and the Weak Law of Large Numbers)
Let $X = (x_1, \dots, x_N)$ be a random vector. Assume that the x_i are identically and independently distributed with probability density $\rho \in L^1(\mathbb{R}^6, \mathbb{R}_0^+)$ and that the length N of the vector is variable. In other words, for any $N \in \mathbb{N}$ the probability to find X in a certain subset $A \subset \mathcal{B}$ of the Borel set of $\mathbb{R}^{6N}$ is given by

$$\int_A \prod_{j=1}^N \rho(x_j) d^{6N}x \ .$$

Assume further that $\int x^4\rho(x)d^6x := C < \infty$.
Prove the validity of the weak law of large numbers theorem with the following error estimates:

$$\forall\, \varepsilon > 0 : \quad \mathbb{P}\left(\left|\overline{X}_N - \mathbb{E}(\overline{X}_N)\right| \geq \epsilon\right) \leq \frac{K}{\varepsilon^4 N^2} \ .$$

Here $\overline{X}_N$ is the mean of the first N positions: $\overline{X}_N = \frac{1}{N}\sum_{j=1}^N x_j$.
Give an explicit expression for K in terms of C.
Hint: Use Markov's inequality for the 4^{th} moment, i.e.

$$\mathbb{P}(|Y| \geq \varepsilon) \leq \frac{\mathbb{E}(Y^4)}{\epsilon^4} \ .$$

Note that this estimate is better than the usual estimate one gets using Chebychev's inequality. Under which assumptions can one use Markov's inequality with a higher moment to get even better estimates?
Explain the meaning of the weak law of large numbers in words.

Problem 3 (Weight Operators)
Let

$$P_k := \left(\prod_{j=1}^{k} q_j^{\varphi} \prod_{j=k+1}^{N} p_j^{\varphi} \right)_{\text{sym}}$$

be defined as in the lecture. Furthermore, let

$$\widehat{m} := \sum_{k=1}^{N} m(k) P_k \tag{1}$$

be the corresponding weight operator to any weight function $m(k)$.
Show that for any j, k

a) $[P_j, P_k] = [P_k, q_j^{\varphi}] = 0$

b) $P_j P_k = \delta_{jk} P_j$

c) $[\hat{m}, P_k] = [\widehat{m}, q_j^{\varphi}] = 0$.

d) Show furthermore that (1) is an homomorphism between the algebra of weight functions and the algebra of weight operators, i.e. that the following holds for any two weight functions m and n and a constant C:

   (a) $\widehat{mn} = \widehat{m}\widehat{n}$
   (b) $\widehat{Cm} = C\widehat{m}$
   (c) $\widehat{m+n} = \widehat{m} + \widehat{n}$.

Problem 4
Let $\alpha^{\varphi} := \langle \psi, \widehat{m}\psi \rangle$ with the weight $m = \left(\frac{k}{N}\right)^x$ for any $x \in \mathbb{N}$. Prove that $|d_t \alpha^{\varphi}| \leq \alpha + \left(\frac{C}{N}\right)^x$ for some constant C.

The solutions to these exercises will be discussed on Friday, 18.11.2016.