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9. Exercise sheet Algebraic Geometry I

All solutions have to be completely justified.

Aufgabe 1 Let $(X, \mathcal{O}_X)$ be a ringed space. For any $f \in \mathcal{O}_X(X)$, let $X_f := \{x \in X : f_x \text{ is a unit in } \mathcal{O}_{X,x}\}$. Prove that X_f is an open subset of X for all $f \in \mathcal{O}_X(X)$. Then, prove that $(X, \mathcal{O}_X)$ is an affine scheme if and only if the following three conditions are satisfied:

(i) $\mathcal{O}_X(X_f) = \mathcal{O}_X(X)_f$ for every $f \in \mathcal{O}_X(X)$;

(ii) the stalk $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$;

(iii) the map of topological spaces $X \to \operatorname{Spec} \mathcal{O}_X(X)$ that sends a point $x \in X$ to the preimage of the maximal ideal of $\mathcal{O}_{X,x}$ under the natural homomorphism $\mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ is a homeomorphism.

Aufgabe 2 Let $\mathbb{A}^1_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}[T]$, and define X to be the topological space obtained as quotient of $\mathbb{A}^1_{\mathbb{C}}$ by the equivalent relation that identifies the two closed points (T) and $(T-1)$, i.e., $\mathfrak{p} \sim \mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of $\mathbb{C}[T]$ and $(T) \sim (T-1)$. X is endowed with the quotient topology. Let $\varphi : \mathbb{A}^1_{\mathbb{C}} \to X$ be the natural projection. Show that $(X, \varphi_* \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}})$ is a ringed space that satisfies condition (i) of the previous exercise, but does not satisfy condition (ii). Note also that there is no natural map $X \to \operatorname{Spec} \mathbb{C}[T]$.

Aufgabe 3 Let $\mathbb{A}^2_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}[T_1, T_2]$ and $X := \mathbb{A}^2_{\mathbb{C}} \setminus \{(T_1, T_2)\}$. Show that $\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}|_X(X) = \mathbb{C}[T_1, T_2]$, that $(X, \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}|_X)$ satisfies conditions (i) and (ii) of Aufgabe 1, and that the natural map $X \to \operatorname{Spec} \mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}}|_X(X)$ is the inclusion $X \to \mathbb{A}^2_{\mathbb{C}}$.

Aufgabe 4 Let $X = \{p, q_1, q_2\}$ be the topological space with three points whose open subsets are: $X, \{p, q_1\}, \{p, q_2\}, \{p\}$ and $\emptyset$. Let K be a field. Define a presheaf of rings $\mathcal{O}$ on X by setting: $\mathcal{O}(X) = \mathcal{O}(\{p, q_1\}) = \mathcal{O}(\{p, q_2\}) = K[T]_{(T)}$ and $\mathcal{O}(\{p\}) = K(T)$, with restriction maps induced by the inclusion $K[T]_{(T)} \subset K(T)$. Show that $\mathcal{O}$ is a sheaf of rings on X and that $(X, \mathcal{O})$ is a scheme. Prove that $(X, \mathcal{O})$ is not an affine scheme.