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**8. Exercise sheet Algebraic Geometry I**

All solutions have to be completely justified.

**Aufgabe 1** For each of the rings $A$ listed below describe the topological space $\mathrm{Spec}(A)$ and its structure sheaf. Namely, determine the points of $\mathrm{Spec}(A)$, the sections of the structure sheaf over all the open subsets of $\mathrm{Spec}(A)$ and all the stalks.
$A = \mathbb{C}[T]/(T)$, $A = \mathbb{C}[T]/(T^2)$, $A = \mathbb{C}[T]/(T(T-1))$, $A = \mathbb{C}[T]/(T^2(T-1))$, $A = \mathbb{R}[T]/(T^2+1)$, $A = \mathbb{C}[T]/(T^2+1)$.

**Aufgabe 2** Let $X$ be a topological space and $\mathcal{C}_X$ the sheaf defined by $\mathcal{C}_X(U) := \{s : U \to \mathbb{R},\ \text{continuous}\}$ for every open subset $U$ of $X$. Show that $(X, \mathcal{C}_X)$ is a locally ringed space. Let $f : X \to Y$ be a continuous map of topological spaces. Let $f^\# : \mathcal{C}_Y \to f_*\mathcal{C}_X$ be the morphism of sheaves on $Y$ defined by

$$f^\#_V : \mathcal{C}_Y(V) \to \mathcal{C}_X(f^{-1}(V)), \qquad s \mapsto s \circ f,$$

for every open subset $V$ of $Y$. Prove that $(f, f^\#) : (X, \mathcal{C}_X) \to (Y, \mathcal{C}_Y)$ is a morphism of locally ringed spaces.

**Aufgabe 3** Let $(X, \mathcal{O}_X)$ be a locally ringed space and $f \in \mathcal{O}_X(X)$. Define $X_f := \{x \in X : f(x) \neq 0\}$. Show that $X_f$ is an open subset of $X$. What is $X_f$ if $X$ is an affine scheme?

**Aufgabe 4** Let $A$ be a ring and $X = \mathrm{Spec}(A)$. For every open subset $U$ of $X$, define $\widetilde{\mathcal{O}}_X(U)$ to be the set of all functions $s : U \to \bigsqcup_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ such that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$, and such that for every $\mathfrak{p} \in U$ there exists an open neighborhood $V \subseteq U$ of $\mathfrak{p}$ and elements $a, f \in A$ such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f \in A_{\mathfrak{q}}$. The symbol $\bigsqcup$ indicates a disjoint union.
Prove that $\widetilde{\mathcal{O}}_X$ is a sheaf of rings on $X$ isomorphic to the structure sheaf $\mathcal{O}_X$ of the affine scheme $X$.