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7. Exercise sheet Algebraic Geometry I

Remarks.
All solutions have to be completely justified.
The parts marked with * are more difficult.

Aufgabe 1 Let X be a topological space, $\mathcal{F}$ a sheaf on X and let $s, t \in \mathcal{F}(U)$ be two sections of $\mathcal{F}$ over an open subset $U \subseteq X$. Show that the set of $x \in U$ such that $s_x = t_x$ is open in U.

Aufgabe 2 Let X be a topological space and define

$$\mathcal{F}(U) = \{f : U \to \mathbb{R} \text{ continuous}; f(U) \subset \mathbb{R} \text{ bounded}\}$$

for all $U \subseteq X$ open.
(a) Show that $\mathcal{F}$ is a presheaf (the restriction maps being the usual restrictions of functions), but it is not a sheaf in general.
(b)* Assume that X is locally compact, i.e. for every point $x \in X$ there is an open subset $U \subseteq X$ containing x such that the closure of U is compact. Describe the sheaf $\tilde{\mathcal{F}}$ associated to $\mathcal{F}$.

Aufgabe 3 Let $f : X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$ a sheaf on X. Show that the data $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$ for every open subset $U \subseteq Y$, together with the restriction maps induced by the restriction maps of $\mathcal{F}$, define a sheaf $f_*\mathcal{F}$ on Y.

Aufgabe 4 Let X be a topological space, $x \in X$ a point and A an abelian group. Define a sheaf $i_x(A)$ on X as follows: for every open subset $U \subseteq X$ let $i_x(A)(U) = A$ if $x \in U$, $i_x(A)(U) = 0$ otherwise. $i_x(A)$ is called the skyscraper sheaf associated to x and A.
(a) Verify that $i_x(A)$ is a sheaf of abelian groups on X. Verify that the stalks of $i_x(A)$ are: $(i_x(A))_y = A$ if y is in the closure $\overline{\{x\}}$ of $\{x\}$, $(i_x(A))_y = 0$ if $y \notin \overline{\{x\}}$.
(b) Let $i : \overline{\{x\}} \to X$ be the inclusion and $\mathcal{F}$ the constant sheaf associated to A on $\overline{\{x\}}$, i.e. the sheaf associated to the constant presheaf $\{U \mapsto A : U \subseteq \overline{\{x\}} \text{ open}\}$. Show that $i_*(\mathcal{F}) = i_x(A)$.