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6. Exercise sheet Algebraic Geometry I

Note: all solutions have to be completely justified.

Aufgabe 1 Let $A = \mathbb{Z}[T]$ the polynomial ring in one variable over $\mathbb{Z}$, and $X = \mathrm{Spec}(A)$.
(a) Determine all points in X.
(b) Determine all closed points in X.
(c) Is X irreducible?
(d) For any prime $p \in \mathbb{Z}$ determine the closure of the point $p\mathbb{Z}[T]$.

Aufgabe 2 Let $\varphi : A \to B$ be a homomorphism of rings and ${}^a\varphi : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ the associated morphism of spectra. Show that
(a) ${}^a\varphi$ is dominant if and only if all elements in $\ker(\varphi)$ are nilpotent.
(b) if φ is surjective, then ${}^a\varphi$ is a homeomorphism of $\mathrm{Spec}\, B$ onto a closed subset of $\mathrm{Spec}\, A$.
(c) if S is a multiplicative subset of A, $B = S^{-1}A$, and $\varphi : A \to S^{-1}A$ is the canonical morphism, then ${}^a\varphi$ is a homeomorphism onto the subspace of $\mathrm{Spec}\, A$ consisting of prime ideals $\mathfrak{p}$ of A such that $\mathfrak{p} \cap S = \emptyset$.

Aufgabe 3 Let A be a ring and $X = \mathrm{Spec}\, A$.
(a) Show that every irreducible subset of X contains at most one generic point.
(b) Show that every irreducible closed subset of X contains a generic point.
(c) Show that every irreducible open subset of X contains a generic point.
(d) Let A be a principal ideal domain with infinitely many maximal ideals (for example $A = \mathbb{Z}$ or $A = k[T]$). Show that any subset of X that consists of infinitely many closed points is irreducible, but does not contain a generic point.

Aufgabe 4 Let X be a topological space.
(a) Let $\mathcal{F}$ be a sheaf on X. Show that $\mathcal{F}(\emptyset)$ consists of only one element.
(b) Let A be a nonzero abelian group (i.e. A coontains at least two distinct elements). We define $\mathcal{F}$ by $\mathcal{F}(U) = A$ for all open subsets U of X and $\mathrm{res}_U^V = \mathrm{id}_A$ for all inclusions $U \subseteq V$ of open subsets of X. Show that $\mathcal{F}$ is a presheaf on X, but not a sheaf. Let now $\mathcal{F}'$ be defined by $\mathcal{F}'(U) = A$ for all nonempty open subsets U of X, $\mathcal{F}(\emptyset) = 0$, $\mathrm{res}_U^V = \mathrm{id}_A$ for all inclusions $U \subseteq V$ of nonempty open subsets of X, and $\mathrm{res}_\emptyset^U$ the zero map $A \to 0$. Is $\mathcal{F}'$ a sheaf on X?
(c) Assume that $X = \{x_1, \ldots, x_n\}$ is a finite set, and let $\mathcal{F}$ be a sheaf on X. Show that $\mathcal{F}(X) \simeq \prod_{i=1}^n \mathcal{F}(\{x_i\})$.