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5. Exercise sheet Algebraic Geometry I

Note: all solutions have to be completely justified.

Aufgabe 1 Let X, Y be prevarieties and $f : X \to Y$ a continuous map. Show that f is a morphism of prevarieties if and only if there exists an open covering $\{U_i\}_{i\in I}$ of Y such that $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is a morphism of prevarieties for all $i \in I$.

Aufgabe 2 For $n \geq 0$, let $\pi : \mathbb{A}^{n+1}(k) \setminus V(T_0, \ldots, T_n) \to \mathbb{P}^n(k)$ be the map that sends $(x_0, \ldots, x_n)$ to the equivalence class $(x_0 : \cdots : x_n)$.
(a) Show that π is a surjective morphism of prevarieties.
(b) Show that the topology of $\mathbb{P}^n(k)$ coincides with the quotient topology induced by π, i.e. a subset U of $\mathbb{P}^n(k)$ is open if and only if $\pi^{-1}(U)$ is an open subset of $\mathbb{A}^{n+1} \setminus V(T_0, \ldots, T_n)$.

Aufgabe 3 Let K be a field. We recall that a polynomial $f \in K[T_0, \ldots, T_n]$ is called homogeneous of degree d if f is a sum of monomials of degree d.
(a) Assume that K is infinite. Show that $f \in K[T_0, \ldots, T_n]$ is homogeneous of degree d if and only if $f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n)$ for all $x_0, \ldots, x_n \in K$ and all $\lambda \in K^\times$. Does it remain true if K is not infinite?
(b) Let $\mathfrak{a} \subseteq K[T_0, \ldots, T_n]$ be an iedal. Show that the following assertions are equivalent.

(i) The ideal $\mathfrak{a}$ is generated by homogeneous elements.

(ii) For every $f \in \mathfrak{a}$ all its homogeneous components are again in $\mathfrak{a}$, where the homogeneous components of f are $f_0, \ldots, f_{\deg(f)} \in K[T_0, \ldots, T_n]$ such that $f = \sum_{d=0}^{\deg(f)} f_d$ and f_d is homogeneous of degree d for all $d = 0, \ldots, \deg(f)$.

(iii) We have $\mathfrak{a} = \bigoplus_{d\geq 0}(\mathfrak{a} \cap K[T_0, \ldots, T_n]_d)$.

An ideal satisfying these equivalent conditions is called homogeneous.
(c) Show that intersections, sums, products, and radicals of homogeneous ideals are again homogeneous.
(d) Show that a homogeneous ideal $\mathfrak{p} \subseteq K[T_0, \ldots, T_n]$ is a prime ideal if and only if $fg \in \mathfrak{p}$ implies $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$ for all homogeneous elements f and g.
(e) Show that every homogeneous ideal $\mathfrak{a} \subsetneq K[T_0, \ldots, T_n]$ is contained in the homogeneous ideal $(T_0, \ldots, T_n)$.

Aufgabe 4 (a) Let $A = (a_{i,j})_{i,j=0,\dots,n} \in GL_{n+1}(k)$ be an invertible $(n+1) \times (n+1)$-matrix. Show that the linear map $k^{n+1} \to k^{n+1}$ associated to A maps one-dimensional subspaces to one-dimensional subspaces and induces a map $\varphi_A : \mathbb{P}^n(k) \to \mathbb{P}^n(k)$, given by $(x_0 : \cdots : x_n) \mapsto (\sum_{i=0}^n a_{0,i}x_i, \dots, \sum_{i=0}^n a_{n,i}x_i)$. Show that φ_A is an isomorphism of prevarieties. Such an automorphism of $\mathbb{P}^n(k)$ is called a change of coordinates.

(b) We recall that for a homogeneous ideal $\mathfrak{a} \subseteq k[T_0, \dots, T_n]$ the corresponding closed subset in $\mathbb{P}^n(k)$ is $V_+(\mathfrak{a}) = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n(k) : f(x_0, \dots, x_n) = 0 \ \forall f \in \mathfrak{a}\}$. Let L_1, L_2 be two disjoint lines in $\mathbb{P}^3(k)$ and $Z = L_1 \cup L_2$. Show that there exists a change of coordinates such that $L_1 = V_+(T_0, T_1)$ and $L_2 = V_+(T_2, T_3)$. Determine the homogeneous radical ideal $\mathfrak{a} \subseteq k[T_0, T_1, T_2, T_3]$ such that $Z = V_+(\mathfrak{a})$. For the definition of linear subspaces and lines in the projective space see page 31, §1.23 of U. Görtz-T. Wedhorn *Algebraic Geometry I*.